

# Appendix A

## Uniqueness of Viscosity Solutions

**Abstract** In this Appendix, we give a detailed proof of uniqueness of viscosity solutions to the HJB equation for the impulse control problem.

### A.1 Notation

**Definition A.1** A modulus is a continuous nondecreasing function  $\omega : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\omega(0) = 0$ .

**Definition A.2** Let  $\mathcal{D}$  be a subset of a normed space. Function  $f : \mathcal{D} \rightarrow \mathbb{R}$  is *uniformly continuous* on  $\mathcal{D}$ , if there exists a modulus  $\omega_f(t)$  such that

$$|f(x) - f(y)| \leq \omega_f(\|x - y\|), \quad \forall x, y \in \mathcal{D}.$$

The set of uniformly continuous functions bounded from below by a constant on  $\mathcal{D}$  is denoted by  $UC_{bb}(\mathcal{D})$ .

It is always possible to select a modulus of sublinear growth, i.e., there exists a constant  $C_f$  such that

$$|f(x) - f(y)| \leq \omega_f(\|x - y\|) \leq C_f(1 + \|x - y\|), \quad \forall x, y \in \mathcal{D}.$$

If function  $f$  is bounded, its supremum is denoted by  $M_f$ .

If  $\omega_f(t) = L_f t$ , then function  $f$  is Lipschitz continuous with constant  $L_f$ :

$$|f(x) - f(y)| \leq L_f \|x - y\|, \quad \forall x, y \in \mathcal{D}.$$

The set of Lipschitz-continuous functions bounded from below by a constant on  $\mathcal{D}$  is denoted by  $Lip_{bb}(\mathcal{D})$ .

## A.2 Bounded Dynamics

Consider the following Hamilton–Jacobi equation for function  $V(t, x) : [t_0, t_1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ :

$$\begin{aligned} \max\{H_1, H_2\} &= 0, \quad t \in [t_0, t_1], \quad x \in \mathbb{R}^n, \\ H_1 &= -V_t - \langle V_x, f(t, x) \rangle, \quad H_2 = \|B^T(t)V_x\| - 1, \\ V(t_1, x) &= \varphi(x) = \inf_{h \in \mathbb{R}^m} \{\varphi(x + B(t_1)h) + \|h\|\}. \end{aligned} \quad (\text{A.1})$$

We make the following assumptions.

**Assumption A.1** Function  $\varphi(\cdot) \geq 0$ .

**Assumption A.2** Function  $\varphi(\cdot)$  is uniformly continuous on  $\mathbb{R}^n$ .

**Assumption A.3** Function  $f(t, x)$  is Lipschitz continuous on  $[t_0, t_1] \times \mathbb{R}^n$  and satisfies

$$|\langle x, f(t, x) \rangle| \leq C_f \|x\|.$$

**Assumption A.4** Function  $B(t)$  is Lipschitz continuous on  $[t_0, t_1]$  with constant  $L_B$ .

Assumption A.1 is equivalent to  $\varphi(x) \geq 0$ . Assumption A.2 holds if  $\varphi(\cdot)$  is uniformly continuous, but this is not a necessary condition.

Example of Assumption A.3:

$$f(t, x) = A(t)x + f_0(t, x)$$

where  $A^T(t) = -A(t)$  and  $f_0(t, x)$  is bounded and Lipschitz continuous.

**Theorem A.1** *Let Assumptions A.1, A.2, A.3, A.4 be satisfied. If functions  $V$  and  $W$  from  $UC_{bb}([t_0, t_1] \times \mathbb{R}^n)$  are viscosity subsolution and supersolution to (A.1), respectively, then  $V \leq W$ .*

*Proof* Suppose that the opposite holds, i.e., there is a point  $(\bar{t}, \bar{x}) \in [t_0, t_1] \times \mathbb{R}^n$  such that

$$V(\bar{t}, \bar{x}) - W(\bar{t}, \bar{x}) \geq 2\delta > 0.$$

Then there exists  $\gamma \in (0, 1)$  such that

$$\gamma V(\bar{t}, \bar{x}) - W(\bar{t}, \bar{x}) \geq \delta > 0.$$

Denote by  $\omega(t)$ , the common modulus of continuity of functions  $V$ ,  $W$ , and  $\varphi$ .

**Lemma A.1** *The following estimate holds:*

$$\gamma V(t, x) - W(s, y) \leq \omega(t_1 - t) + \omega(t_1 - s) + \omega(\|x - y\|),$$

or in simplified form, there exists constant  $C_1$  such that

$$\gamma V(t, x) - W(s, y) \leq C_1 (1 + \|x - y\|).$$

*Proof* Since  $V(t_1, x) = W(t_1, x) = \varphi(x) \geq 0$ , we have

$$\begin{aligned} & \gamma V(t, x) - W(s, y) \\ &= \gamma(V(t, x) - V(t_1, x)) + (W(t_1, y) - W(s, y)) + \gamma(V(t_1, x) - W(t_1, y)) - (1 - \gamma)W(t_1, y) \\ & \leq \gamma\omega(t_1 - t) + \omega(t_1 - s) + \gamma\omega(\|x - y\|). \end{aligned}$$

Define an auxiliary function

$$\Phi(t, x, s, y) = \gamma V(t, x) - W(s, y) - \frac{(t - s)^2 + \|x - y\|^2}{\varepsilon} - \alpha (\|x\|^2 + \|y\|^2) + \sigma(s + t).$$

Here,  $\varepsilon \in (0, 1)$  and  $\alpha > 0$  are sufficiently small. Parameter  $\sigma = \delta/4(1 + |t_0| + |t_1|) > 0$  is such that  $|\sigma t| \leq \delta/4$  for all  $t \in [t_0, t_1]$ .

By Lemma A.1,  $\Phi \rightarrow -\infty$  when  $\|x\|, \|y\| \rightarrow \infty$  and it attains its maximum value at some point  $(t^*, x^*, s^*, y^*)$ .

**Lemma A.2** *There exists constant  $C_2$  (independent of  $\varepsilon$  and  $\alpha$ ) such that*

$$(t^* - s^*)^2 + \|x^* - y^*\|^2 \leq C_2 \varepsilon \omega(\sqrt{\varepsilon}).$$

*Proof* Denote

$$\Delta^2 = (t^* - s^*)^2 + \|x^* - y^*\|^2.$$

Since  $(t^*, x^*, s^*, y^*)$  is a maximum point, we have

$$\Phi(t^*, x^*, t^*, x^*) + \Phi(s^*, y^*, s^*, y^*) \leq 2\Phi(t^*, x^*, s^*, y^*),$$

which gives

$$2\frac{\Delta^2}{\varepsilon} \leq \gamma(V(t^*, x^*) - V(s^*, y^*)) + W(t^*, x^*) - W(s^*, y^*) \leq 2\omega(\Delta).$$

Then

$$\Delta^2 \leq \varepsilon\omega(\Delta) \leq C\varepsilon(1 + \Delta) \leq C\varepsilon + \frac{C^2\varepsilon^2 + \Delta^2}{2} \Rightarrow \Delta^2 \leq C\varepsilon.$$

Finally,  $\Delta^2 \leq \varepsilon\omega(\sqrt{C\varepsilon}) \leq C_2\varepsilon\omega(\sqrt{\varepsilon})$ .

**Lemma A.3** *There exists constant  $C_3$  (independent of  $\varepsilon$  and  $\alpha$ ) such that*

$$\sqrt{\alpha} \|x^*\| \leq C_3, \quad \sqrt{\alpha} \|y^*\| \leq C_3.$$

*Proof* Again using that  $(t^*, x^*, s^*, y^*)$  is a maximum, we can write

$$\Phi(t_1, 0, t_1, 0) \leq \Phi(t^*, x^*, s^*, y^*)$$

which is

$$-(1 - \gamma)\varphi(0) + \sigma(2t_1 - s^* - t^*) \leq \gamma V(t^*, x^*) - W(s^*, y^*) - \frac{\|x^* - y^*\|^2}{\varepsilon} - \alpha(\|x^*\|^2 + \|y^*\|^2).$$

Left-hand side is bounded by some constant  $C$ . Using Lemmas A.1 and A.2, we get

$$\alpha(\|x^*\|^2 + \|y^*\|^2) \leq C + \gamma V(t^*, x^*) - W(s^*, y^*) \leq C + C_1(1 + \|x^* - y^*\|) \leq C_3.$$

**Lemma A.4** *If  $\varepsilon$  is sufficiently small, then  $t^*, s^* < t_1$ .*

*Proof* Suppose that  $t^* = t_1$ .

$$\begin{aligned} \Phi(t^*, x^*, s^*, y^*) &= \Phi(t_1, x^*, s^*, y^*) \\ &= \gamma V(t_1, x^*) - W(s^*, y^*) - \frac{(t_1 - s^*)^2 + \|x^* - y^*\|^2}{\varepsilon} - \alpha(\|x^*\|^2 + \|y^*\|^2) + \sigma(s^* + t^*) \\ &\geq \gamma V(\bar{t}, \bar{x}) - W(\bar{t}, \bar{x}) - 2\alpha\|\bar{x}\|^2 + 2\sigma\bar{t} \geq \delta/2 \end{aligned}$$

if  $\alpha \leq \alpha_0 = \delta/4\|\bar{x}\|^2$ . Therefore,  $\gamma V(t_1, x^*) - W(s^*, y^*) \geq \delta/4$ . By Lemma A.1,

$$\omega(t_1 - s^*) + \omega(\|x^* - y^*\|) \geq \gamma V(t_1, x^*) - W(s^*, y^*) \geq \delta/4.$$

We have arrived at a contradiction, since left-hand side goes to zero with  $\varepsilon \rightarrow 0$ . Thus  $t^* < t_1$ . A proof of  $s^* < t_1$  is similar.

We choose specific test functions

$$\begin{aligned} \phi(t, x) &= W(s^*, y^*) + \frac{\|x - y^*\|^2 + |t - s^*|^2}{\varepsilon} + \alpha(\|x\|^2 + \|y^*\|^2) - \sigma(t + s^*), \\ \psi(s, y) &= \gamma V(t^*, x^*) - \frac{\|x^* - y\|^2 + |t^* - s|^2}{\varepsilon} - \alpha(\|x^*\|^2 + \|y\|^2) + \sigma(t^* + s). \end{aligned}$$

Their derivatives are

$$\begin{aligned} \phi_t(t^*, x^*) &= 2\frac{t^* - s^*}{\varepsilon} - \sigma, & \psi_s(s^*, y^*) &= 2\frac{t^* - s^*}{\varepsilon} + \sigma, \\ \phi_x(t^*, x^*) &= 2\frac{x^* - y^*}{\varepsilon} + 2\alpha x^*, & \psi_y(s^*, y^*) &= 2\frac{x^* - y^*}{\varepsilon} - 2\alpha y^*. \end{aligned}$$

We have

$$\Phi(t, x, s^*, y^*) = \gamma V(t, x) - \phi(t, x), \quad \Phi(t^*, x^*, s, y) = \psi(s, y) - W(s, y). \quad (\text{A.2})$$

Therefore  $\gamma V - \phi$  attains its maximum at  $(t^*, x^*)$  and  $W - \psi$  attains its minimum at  $(s^*, y^*)$ . Since  $V$  is a viscosity subsolution, test function  $\phi/\gamma$  satisfies at point  $(t^*, x^*)$

$$\begin{cases} \phi_t + \langle \phi_x, f(t^*, x^*) \rangle \geq 0, \\ \|B^T(t^*)\phi_x\| \leq \gamma. \end{cases} \quad (\text{A.3})$$

$W$  is a viscosity supersolution, and thus  $\psi$  satisfies at point  $(s^*, y^*)$

$$\begin{cases} \psi_s + \langle \psi_y, f(s^*, y^*) \rangle \leq 0, \\ \|B^T(s^*)\psi_y\| \geq 1. \end{cases} \quad (\text{A.4})$$

We show that neither of the latter two conditions can be satisfied.

In the **first case**,

$$\phi_s - \phi_t + \langle \psi_y, f(s^*, y^*) \rangle - \langle \phi_x, f(t^*, x^*) \rangle \leq 0.$$

We have  $\phi_s - \phi_t = 2\sigma$  and

$$\begin{aligned} & \langle \phi_x, f(t^*, x^*) \rangle - \langle \psi_y, f(s^*, y^*) \rangle \\ &= \frac{2}{\varepsilon} \langle x^* - y^*, f(x^*) - f(y^*) \rangle + 2\alpha \langle x^*, f(x^*) \rangle + 2\alpha \langle y^*, f(y^*) \rangle \\ &\leq \frac{2}{\varepsilon} L_f \|x^* - y^*\|^2 + 2\alpha C_f (\|x^*\| + \|y^*\|) \leq 2C_2\omega(\sqrt{\varepsilon}) + 4\sqrt{\alpha}C_3C_f \xrightarrow{\varepsilon, \alpha \rightarrow 0} 0. \end{aligned}$$

Thus for sufficiently small  $\varepsilon, \alpha$ , we have  $2\sigma \leq 0$  which contradicts the fact that  $\sigma > 0$ .

In the **second case**  $\|B^T(s^*)\psi_y\| \geq 1$ . But at the same time  $\|B^T(t^*)\phi_x\| \leq \gamma < 1$ . We have

$$\begin{aligned} 0 < 1 - \gamma &\leq \|B^T(s^*)\psi_y\| - \|B^T(t^*)\phi_x\| \leq \|B^T(s^*)\psi_y - B^T(t^*)\phi_x\| \\ &\leq \frac{2}{\varepsilon} \|B^T(t^*) - B^T(s^*)\| \|x^* - y^*\| + 2\alpha \|B^T(t^*)x^*\| + 2\alpha \|B^T(s^*)y^*\| \\ &\leq 2L_B C_2\omega(\sqrt{\varepsilon}) + 4\sqrt{\alpha}C_3M_B \xrightarrow{\varepsilon, \alpha \rightarrow 0} 0. \end{aligned}$$

A contradiction since left-hand side is a positive constant.

Thus neither of the two cases may take place, and we have arrived at a contradiction, which proves that  $V \leq W$ .

### A.3 Unbounded Solutions

*Example A.1* Requiring solutions to be uniformly continuous effectively means that they are bounded, as well as the terminal function. Indeed, consider a linear case with

$$[t_0, t_1] = [0, 1], \quad \varphi(x) = \|x\|, \quad f(t, x) = 0, \quad B(t) = (2 - t)I.$$

Then the value function

$$V(t, x) = \frac{\|x\|}{2 - t}$$

is not uniformly continuous.

In order to allow for unbounded solutions, we introduce a change of dependent variable  $V = h(\hat{V})$  given by function  $h(r)$  such that

$$h \in C^1(\mathcal{I}), \quad h'(0) > 0, \quad h(\mathcal{I}) = \mathbb{R}, \quad \mathcal{I} = (p, q), \quad -\infty \leq p < q \leq +\infty.$$

The HJB equation (A.1) then rewrites as

$$\begin{aligned} \max\{H_1, H_2\} &= 0, \quad t \in [t_0, t_1], \quad x \in \mathbb{R}^n, \\ H_1 &= -\hat{V}_t - \left\langle \hat{V}_x, f(t, x) \right\rangle, \quad H_2 = \left\| B^T(t) \hat{V}_x \right\| - 1/h_r(\hat{V}), \\ \hat{V}(t_1, x) &= \hat{\varphi}(x) = h^{-1}(\varphi(x)). \end{aligned} \tag{A.5}$$

It is straightforward to check that  $V$  is a subsolution (supersolution) to (A.1) if and only if  $\hat{V}$  is a subsolution (supersolution) to (A.5).

**Theorem A.2** *Suppose that*

1. *functions  $\hat{\varphi}$ ,  $f$ ,  $B$  satisfy Assumptions A.1, A.2, A.3 and A.4;*
2.  $0 < q < \infty$ ;
3. *functions  $\hat{V}$  and  $\hat{W}$  from  $UC([t_0, t_1] \times \mathbb{R}^n)$  take values in  $(p_0, q) \subseteq (p, q)$ ;*
4.  $h_r(r)$  *is nondecreasing on  $(p_0, q)$ ;*
5.  $\hat{V}$  *and  $\hat{W}$  are viscosity subsolution and supersolution to (A.5), respectively.*

*Then  $\hat{V} \leq \hat{W}$ .*

*Proof* The proof is similar to Theorem A.1. Relations (A.3) and (A.4) take the form

$$\begin{cases} \phi_t + \langle \phi_x, f(t^*, x^*) \rangle \geq 0, \\ \left\| B^T(t^*) \phi_x \right\| \leq \gamma/h_r(V); \end{cases} \quad \text{and} \quad \begin{cases} \psi_s + \langle \psi_y, f(s^*, y^*) \rangle \leq 0, \\ \left\| B^T(s^*) \psi_y \right\| \geq 1/h_r(W). \end{cases}$$

For sufficiently small  $\alpha$  and  $\sigma$ , we have  $\gamma q > \gamma \hat{V}(t^*, x^*) > \hat{W}(s^*, y^*)$ . It then follows from conditions of the theorem that

$$\frac{1}{h_r(\hat{W})} - \frac{\gamma}{h_r(\hat{V})} > \frac{1-\gamma}{h_r(\hat{W})} > \frac{1-\gamma}{h_r(\gamma q)} = \text{const} > 0.$$

This inequality is used instead of  $1 - \gamma > 0$  to prove that the second case is not possible.

**Assumption A.5** There exists a constant  $C_h$  such that

$$(h^{-1})_r(r) \leq \frac{C_h}{1+r^2}.$$

This assumptions holds for a particular transformation function

$$V = \tan(\hat{V} - \hat{V}_0), \quad \hat{V} = \arctan V + \hat{V}_0.$$

**Assumption A.6** There exist constants  $C_1, C_2$  such that

$$V(t, x) \geq C_1 \|x\|^{1/2} + C_2.$$

**Assumption A.7** Function  $V(t, x)$  satisfies

$$|V(t, x) - V(t, y)| \leq \omega(\|x - y\|), \quad |V(t, x) - V(s, x)| \leq (1 + \|x\|)\omega(|t - s|).$$

**Lemma A.5** *Suppose that*

1. *function  $h_r(r)$  satisfies Assumption A.5 and is nondecreasing for  $r \geq \hat{V}_0$ , where  $\hat{V}(t, x) \geq \hat{V}_0$ ;*
2. *function  $V(t, x)$  satisfies Assumptions A.6, A.7.*

*Then function  $\hat{V} = h^{-1}(V)$  is uniformly continuous on  $[t_0, t_1] \times \mathbb{R}^n$ .*

*Proof* Note that  $\hat{V}_0 > -\infty$  due to Assumption A.6. We have from our assumptions

$$\begin{aligned} |W(t, x) - W(t, y)| &= |h^{-1}(V(t, x)) - h^{-1}(V(t, y))| \\ &= \frac{1}{h_r(r^*)} |V(t, x) - V(t, y)| \leq \frac{\omega(\|x - y\|)}{h_r(p_0)}. \end{aligned}$$

Suppose that  $V_1 = V(t, x) \geq V_2 = V(s, x)$ . Then

$$\begin{aligned} |W(t, x) - W(s, x)| &= |h^{-1}(V_1) - h^{-1}(V_2)| = \frac{1}{h_r(r^*)} |V_1 - V_2| \leq \frac{1}{h_r(V_2)} |V_1 - V_2| \\ &\leq C_h \frac{V_1 - V_2}{1 + V_2^2} \leq C\omega(|t - s|). \end{aligned}$$

## A.4 Unbounded Dynamics

Now we relax Assumption A.3 to allow for arbitrary linear dynamics.

**Assumption A.8** Function  $f(t, x) = A(t)x + f_0(t, x)$ , where  $A(t)$  is a continuous matrix function and  $f_0(t, x)$  is a Lipschitz-continuous vector function on  $[t_0, t_1] \times \mathbb{R}^n$  satisfying

$$|\langle x, f_0(t, x) \rangle| \leq C_f \|x\|.$$

We introduce a change of variables  $\hat{x} = X(t_1, t)x$ , where  $X(t, s)$  is the fundamental matrix corresponding to linear system with matrix  $A(t)$ :

$$\partial X(t, s)/\partial t = A(t)X(t, s), \quad X(s, s) = I.$$

Then, Hamilton–Jacobi–Bellman equation (A.5) takes the form

$$\begin{aligned} \max\{H_1, H_2\} &= 0, \quad t \in [t_0, t_1], \quad x \in \mathbb{R}^n, \\ H_1 &= -\hat{V}_t - \left\langle \hat{V}_{\hat{x}}, f_0(t, x) \right\rangle, \quad H_2 = \left\| \hat{B}^T(t) \hat{V}_{\hat{x}} \right\| - 1/h_r(\hat{V}), \\ \hat{V}(t_1, x) &= \hat{\phi}(x). \end{aligned} \tag{A.6}$$

Here matrix function  $\hat{B}(t) = X(t_1, t)B(t)$  is Lipschitz continuous.

Since mapping  $(t, x) \rightarrow (t, X(t_1, t)x)$  is a diffeomorphism, thus if  $\hat{V}(t, x)$  is a sub- or supersolution to (A.5), then the transformed function  $\hat{V}(t, \hat{x})$  is a sub- or supersolution to (A.6). This mapping also preserves Assumptions A.6 and A.7; the verification is straightforward, e.g.,

$$\begin{aligned} &|V(t, X(t, t_1)\hat{x}) - V(s, X(s, t_1)\hat{x})| \\ &\leq |V(t, X(t, t_1)\hat{x}) - V(s, X(t, t_1)\hat{x})| + |V(s, X(t, t_1)\hat{x}) - V(s, X(s, t_1)\hat{x})| \\ &\leq (1 + \|X(t, t_1)\hat{x}\|)\omega(|t - s|) + \omega(\|X(t, t_1)\hat{x} - X(s, t_1)\hat{x}\|) \leq C(1 + \|\hat{x}\|)\omega(|t - s|) \end{aligned}$$

since  $X$  is bounded and Lipschitz continuous.

We have arrived at the final result.

**Theorem A.3** *Suppose that*

1. *Assumptions A.1, A.2, A.4, and A.8 are satisfied;*
2. *functions  $V$  and  $W$  satisfy Assumptions A.6 and A.7;*
3. *functions  $V$  and  $W$  are viscosity subsolution and supersolution to (A.1), respectively.*

*Then  $V \leq W$ .*



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