# Appendix A
# Uniqueness of Viscosity Solutions

**Abstract** In this Appendix, we give a detailed proof of uniqueness of viscosity solutions to the HJB equation for the impulse control problem.

## A.1 Notation

**Definition A.1** A *modulus* is a continuous nondecreasing function $\omega : \mathbb{R} \to \mathbb{R}$ such that $\omega(0) = 0$.

**Definition A.2** Let $\mathscr{D}$ be a subset of a normed space. Function $f : \mathscr{D} \to \mathbb{R}$ is *uniformly continuous* on $\mathscr{D}$, if there exists a modulus $\omega_f(t)$ such that

$$|f(x) - f(y)| \leq \omega_f(\|x - y\|), \quad \forall x, y \in \mathscr{D}.$$

The set of uniformly continuous functions bounded from below by a constant on $\mathscr{D}$ is denoted by $UC_{bb}(\mathscr{D})$.

It is always possible to select a modulus of sublinear growth, i.e., there exists a constant $C_f$ such that

$$|f(x) - f(y)| \leq \omega_f(\|x - y\|) \leq C_f(1 + \|x - y\|), \quad \forall x, y \in \mathscr{D}.$$

If function $f$ is bounded, its supremum is denoted by $M_f$.

If $\omega_f(t) = L_f t$, then function $f$ is Lipschitz continuous with constant $L_f$:

$$|f(x) - f(y)| \leq L_f \|x - y\|, \quad \forall x, y \in \mathscr{D}.$$

The set of Lipschitz-continuous functions bounded from below by a constant on $\mathscr{D}$ is denoted by $Lip_{bb}(\mathscr{D})$.



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## A.2 Bounded Dynamics

Consider the following Hamilton–Jacobi equation for function $V(t, x) : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}^n$:

$$\begin{aligned} \max\{H_1, H_2\} = 0, \quad t \in [t_0, t_1], \quad x \in \mathbb{R}^n, \\ H_1 = -V_t - \langle V_x, f(t, x)\rangle, \quad H_2 = \left\| B^T(t) V_x \right\| - 1, \\ V(t_1, x) = \boldsymbol{\varphi}(x) = \inf_{h \in \mathbb{R}^m} \{\varphi(x + B(t_1)h) + \|h\|\}. \end{aligned} \tag{A.1}$$

We make the following assumptions.

**Assumption A.1** Function $\boldsymbol{\varphi}(\cdot) \geq 0$.

**Assumption A.2** Function $\boldsymbol{\varphi}(\cdot)$ is uniformly continuous on $\mathbb{R}^n$.

**Assumption A.3** Function $f(t, x)$ is Lipschitz continuous on $[t_0, t_1] \times \mathbb{R}^n$ and satisfies

$$|\langle x, f(t, x)\rangle| \leq C_f \|x\|.$$

**Assumption A.4** Function $B(t)$ is Lipschitz continuous on $[t_0, t_1]$ with constant $L_B$.

Assumption A.1 is equivalent to $\varphi(x) \geq 0$. Assumption A.2 holds if $\varphi(\cdot)$ is uniformly continuous, but this is not a necessary condition.

Example of Assumption A.3:

$$f(t, x) = A(t)x + f_0(t, x)$$

where $A^T(t) = -A(t)$ and $f_0(t, x)$ is bounded and Lipschitz continuous.

**Theorem A.1** *Let Assumptions A.1, A.2, A.3, A.4 be satisfied. If functions $V$ and $W$ from $UC_{bb}([t_0, t_1] \times \mathbb{R}^n)$ are viscosity subsolution and supersolution to (A.1), respectively, then $V \leq W$.*

*Proof* Suppose that the opposite holds, i.e., there is a point $(\bar{t}, \bar{x}) \in [t_0, t_1) \times \mathbb{R}^n$ such that

$$V(\bar{t}, \bar{x}) - W(\bar{t}, \bar{x}) \geq 2\delta > 0.$$

Then there exists $\gamma \in (0, 1)$ such that

$$\gamma V(\bar{t}, \bar{x}) - W(\bar{t}, \bar{x}) \geq \delta > 0.$$

Denote by $\omega(t)$, the common modulus of continuity of functions $V$, $W$, and $\boldsymbol{\varphi}$.

**Lemma A.1** *The following estimate holds:*

$$\gamma V(t, x) - W(s, y) \leq \omega(t_1 - t) + \omega(t_1 - s) + \omega(\|x - y\|),$$

*or in simplified form, there exists constant $C_1$ such that*

$$\gamma V(t,x) - W(s,y) \leq C_1 (1 + \|x-y\|).$$

*Proof* Since $V(t_1,x) = W(t_1,x) = \boldsymbol{\varphi}(x) \geq 0$, we have

$$\begin{aligned}
&\gamma V(t,x) - W(s,y)\\
=&\gamma(V(t,x) - V(t_1,x)) + (W(t_1,y) - W(s,y)) + \gamma(V(t_1,x) - W(t_1,y)) - (1-\gamma)W(t_1,y)\\
&\leq \gamma\omega(t_1 - t) + \omega(t_1 - s) + \gamma\omega(\|x-y\|).
\end{aligned}$$

Define an auxiliary function

$$\Phi(t,x,s,y) = \gamma V(t,x) - W(s,y) - \frac{(t-s)^2 + \|x-y\|^2}{\varepsilon} - \alpha\left(\|x\|^2 + \|y\|^2\right) + \sigma(s+t).$$

Here, $\varepsilon \in (0,1)$ and $\alpha > 0$ are sufficiently small. Parameter $\sigma = \delta/4(1+|t_0|+|t_1|) > 0$ is such that $|\sigma t| \leq \delta/4$ for all $t \in [t_0,t_1]$.

By Lemma A.1, $\Phi \to -\infty$ when $\|x\|, \|y\| \to \infty$ and it attains its maximum value at some point $(t^*,x^*,s^*,y^*)$.

**Lemma A.2** *There exists constant $C_2$ (independent of $\varepsilon$ and $\alpha$) such that*

$$(t^* - s^*)^2 + \left\|x^* - y^*\right\|^2 \leq C_2\varepsilon\,\omega\left(\sqrt{\varepsilon}\right).$$

*Proof* Denote

$$\Delta^2 = (t^* - s^*)^2 + \left\|x^* - y^*\right\|^2.$$

Since $(t^*,x^*,s^*,y^*)$ is a maximum point, we have

$$\Phi(t^*,x^*,t^*,x^*) + \Phi(s^*,y^*,s^*,y^*) \leq 2\Phi(t^*,x^*,s^*,y^*),$$

which gives

$$2\frac{\Delta^2}{\varepsilon} \leq \gamma(V(t^*,x^*) - V(s^*,y^*)) + W(t^*,x^*) - W(s^*,y^*) \leq 2\omega(\Delta).$$

Then

$$\Delta^2 \leq \varepsilon\omega(\Delta) \leq C\varepsilon(1+\Delta) \leq C\varepsilon + \frac{C^2\varepsilon^2 + \Delta^2}{2} \quad \Rightarrow \quad \Delta^2 \leq C\varepsilon.$$

Finally, $\Delta^2 \leq \varepsilon\omega\left(\sqrt{C\varepsilon}\right) \leq C_2\varepsilon\omega\left(\sqrt{\varepsilon}\right)$.

**Lemma A.3** *There exists constant $C_3$ (independent of $\varepsilon$ and $\alpha$) such that*

$$\sqrt{\alpha}\left\|x^*\right\| \leq C_3, \quad \sqrt{\alpha}\left\|y^*\right\| \leq C_3.$$

*Proof* Again using that $(t^*, x^*, s^*, y^*)$ is a maximum, we can write

$$\Phi(t_1, 0, t_1, 0) \le \Phi(t^*, x^*, s^*, y^*)$$

which is

$$-(1-\gamma)\varphi(0) + \sigma(2t_1 - s^* - t^*) \le \gamma V(t^*, x^*) - W(s^*, y^*) - \frac{\|x^* - y^*\|^2}{\varepsilon} - \alpha\left(\|x^*\|^2 + \|y^*\|^2\right).$$

Left-hand side is bounded by some constant $C$. Using Lemmas A.1 and A.2, we get

$$\alpha\left(\|x^*\|^2 + \|y^*\|^2\right) \le C + \gamma V(t^*, x^*) - W(s^*, y^*) \le C + C_1\left(1 + \|x^* - y^*\|\right) \le C_3^2.$$

**Lemma A.4** *If $\varepsilon$ is sufficiently small, then $t^*, s^* < t_1$.*

*Proof* Suppose that $t^* = t_1$.

$$\begin{aligned}
&\Phi(t^*, x^*, s^*, y^*) = \Phi(t_1, x^*, s^*, y^*)\\
=&\gamma V(t_1, x^*) - W(s^*, y^*) - \frac{(t_1 - s^*)^2 + \|x^* - y^*\|^2}{\varepsilon} - \alpha\left(\|x^*\|^2 + \|y^*\|^2\right) + \sigma(s^* + t^*)\\
&\ge \Phi(\bar t, \bar x, \bar t, \bar x) = \gamma V(\bar t, \bar x) - W(\bar t, \bar x) - 2\alpha\|\bar x\|^2 + 2\sigma \bar t \ge \delta/2
\end{aligned}$$

if $\alpha \le \alpha_0 = \delta/4\|\bar x\|^2$. Therefore, $\gamma V(t_1, x^*) - W(s^*, y^*) \ge \delta/4$. By Lemma A.1,

$$\omega(t_1 - s^*) + \omega\left(\|x^* - y^*\|\right) \ge \gamma V(t_1, x^*) - W(s^*, y^*) \ge \delta/4.$$

We have arrived at a contradiction, since left-hand side goes to zero with $\varepsilon \to 0$. Thus $t^* < t_1$. A proof of $s^* < t_1$ is similar.

We choose specific test functions

$$\phi(t, x) = W(s^*, y^*) + \frac{\|x - y^*\|^2 + |t - s^*|^2}{\varepsilon} + \alpha\left(\|x\|^2 + \|y^*\|^2\right) - \sigma(t + s^*),$$
$$\psi(s, y) = \gamma V(t^*, x^*) - \frac{\|x^* - y\|^2 + |t^* - s|^2}{\varepsilon} - \alpha\left(\|x^*\|^2 + \|y\|^2\right) + \sigma(t^* + s).$$

Their derivatives are

$$\phi_t(t^*, x^*) = 2\frac{t^* - s^*}{\varepsilon} - \sigma, \quad \psi_s(s^*, y^*) = 2\frac{t^* - s^*}{\varepsilon} + \sigma,$$
$$\phi_x(t^*, x^*) = 2\frac{x^* - y^*}{\varepsilon} + 2\alpha x^*, \quad \psi_y(s^*, y^*) = 2\frac{x^* - y^*}{\varepsilon} - 2\alpha y^*.$$

We have

$$\Phi(t, x, s^*, y^*) = \gamma V(t, x) - \phi(t, x), \quad \Phi(t^*, x^*, s, y) = \psi(s, y) - W(s, y). \tag{A.2}$$

Therefore $\gamma V - \phi$ attains its maximum at $(t^*, x^*)$ and $W - \psi$ attains its minimum at $(s^*, y^*)$. Since $V$ is a viscosity subsolution, test function $\phi/\gamma$ satisfies at point $(t^*, x^*)$

$$\begin{cases} \phi_t + \langle \phi_x, f(t^*, x^*) \rangle \geq 0, \\ \left\| B^T(t^*)\phi_x \right\| \leq \gamma. \end{cases} \tag{A.3}$$

$W$ is a viscosity supersolution, and thus $\psi$ satisfies at point $(s^*, y^*)$

$$\left[ \begin{array}{l} \psi_s + \left\langle \psi_y, f(s^*, y^*) \right\rangle \leq 0, \\ \left\| B^T(s^*)\psi_y \right\| \geq 1. \end{array} \right. \tag{A.4}$$

We show that neither of the latter two conditions can be satisfied.

In the **first case**,

$$\phi_s - \phi_t + \left\langle \psi_y, f(s^*, y*) \right\rangle - \left\langle \phi_x, f(t^*, x^*) \right\rangle \leq 0.$$

We have $\phi_s - \phi_t = 2\sigma$ and

$$\begin{aligned} &\left\langle \phi_x, f(t^*, x^*) \right\rangle - \left\langle \psi_y, f(s^*, y*) \right\rangle \\ &\quad = \frac{2}{\varepsilon} \left\langle x^* - y^*, f(x*) - f(y^*) \right\rangle + 2\alpha \left\langle x^*, f(x^*) \right\rangle + 2\alpha \left\langle y^*, f(y^*) \right\rangle \\ &\leq \frac{2}{\varepsilon} L_f \left\| x^* - y^* \right\|^2 + 2\alpha C_f \left( \left\| x^* \right\| + \left\| y^* \right\| \right) \leq 2C_2 \omega \left( \sqrt{\varepsilon} \right) + 4\sqrt{\alpha} C_3 C_f \underset{\varepsilon, \alpha \to 0}{\longrightarrow} 0. \end{aligned}$$

Thus for sufficiently small $\varepsilon$, $\alpha$, we have $2\sigma \leq 0$ which contradicts the fact that $\sigma > 0$.

In the **second case** $\left\| B^T(s^*)\psi_y \right\| \geq 1$. But at the same time $\left\| B^T(t^*)\phi_x \right\| \leq \gamma < 1$. We have

$$\begin{aligned} 0 < 1 - \gamma &\leq \left\| B^T(s^*)\psi_y \right\| - \left\| B^T(t^*)\phi_x \right\| \leq \left\| B^T(s^*)\psi_y - B^T(t^*)\phi_x \right\| \\ &\leq \frac{2}{\varepsilon} \left\| B^T(t^*) - B^T(s^*) \right\| \left\| x^* - y^* \right\| + 2\alpha \left\| B^T(t^*)x^* \right\| + 2\alpha \left\| B^T(s^*)y^* \right\| \\ &\qquad \leq 2L_B C_2 \omega(\sqrt{\varepsilon}) + 4\sqrt{\alpha} C_3 M_B \underset{\varepsilon, \alpha \to 0}{\longrightarrow} 0. \end{aligned}$$

A contradiction since left-hand side is a positive constant.

Thus neither of the two cases may take place, and we have arrived at a contradiction, which proves that $V \leq W$.

## A.3 Unbounded Solutions

*Example A.1* Requiring solutions to be uniformly continuous effectively means that they are bounded, as well as the terminal function. Indeed, consider a linear case with

$$[t_0, t_1] = [0, 1], \quad \boldsymbol{\varphi}(x) = \|x\|, \quad f(t, x) = 0, \quad B(t) = (2 - t)I.$$

Then the value function

$$V(t, x) = \frac{\|x\|}{2 - t}$$

is not uniformly continuous.

In order to allow for unbounded solutions, we introduce a change of dependent variable $V = h(\hat{V})$ given by function $h(r)$ such that

$$h \in C^1(\mathscr{I}), \quad h'(0) > 0, \quad h(\mathscr{I}) = \mathbb{R}, \quad \mathscr{I} = (p, q), \quad -\infty \le p < q \le +\infty.$$

The HJB equation (A.1) then rewrites as

$$\begin{gathered} \max\{H_1, H_2\} = 0, \quad t \in [t_0, t_1], \quad x \in \mathbb{R}^n, \\ H_1 = -\hat{V}_t - \left\langle \hat{V}_x, f(t, x) \right\rangle, \quad H_2 = \left\| B^T(t)\hat{V}_x \right\| - 1/h_r\left(\hat{V}\right), \\ \hat{V}(t_1, x) = \hat{\boldsymbol{\varphi}}(x) = h^{-1}\left(\boldsymbol{\varphi}(x)\right). \end{gathered} \tag{A.5}$$

It is straightforward to check that $V$ is a subsolution (supersolution) to (A.1) if and only if $\hat{V}$ is a subsolution (supersolution) to (A.5).

**Theorem A.2** *Suppose that*

1. *functions $\hat{\boldsymbol{\varphi}}$, $f$, $B$ satisfy Assumptions A.1, A.2, A.3 and A.4;*
2. $0 < q < \infty$;
3. *functions $\hat{V}$ and $\hat{W}$ from $UC([t_0, t_1] \times \mathbb{R}^n)$ take values in $(p_0, q) \subseteq (p, q)$;*
4. *$h_r(r)$ is nondecreasing on $(p_0, q)$;*
5. *$\hat{V}$ and $\hat{W}$ are viscosity subsolution and supersolution to (A.5), respectively.*

*Then $\hat{V} \le \hat{W}$.*

*Proof* The proof is similar to Theorem A.1. Relations (A.3) and (A.4) take the form

$$\begin{cases} \phi_t + \langle \phi_x, f(t^*, x^*) \rangle \ge 0, \\ \left\| B^T(t^*)\phi_x \right\| \le \gamma / h_r(V); \end{cases} \quad \text{and} \quad \left[ \begin{array}{l} \psi_s + \left\langle \psi_y, f(s^*, y^*) \right\rangle \le 0, \\ \left\| B^T(s^*)\psi_y \right\| \ge 1/h_r(W). \end{array} \right.$$

For sufficiently small $\alpha$ and $\sigma$, we have $\gamma q > \gamma \hat{V}(t^*, x^*) > \hat{W}(s^*, y^*)$. It then follows from conditions of the theorem that

$$\frac{1}{h_r(\hat{W})} - \frac{\gamma}{h_r(\hat{V})} > \frac{1-\gamma}{h_r(\hat{W})} > \frac{1-\gamma}{h_r(\gamma q)} = \text{const} > 0.$$

This inequality is used instead of $1 - \gamma > 0$ to prove that the second case is not possible.

**Assumption A.5** There exists a constant $C_h$ such that

$$\left(h^{-1}\right)_r(r) \leq \frac{C_h}{1+r^2}.$$

This assumptions holds for a particular transformation function

$$V = \tan(\hat{V} - \hat{V}_0), \quad \hat{V} = \arctan V + \hat{V}_0.$$

**Assumption A.6** There exist constants $C_1, C_2$ such that

$$V(t,x) \geq C_1 \|x\|^{1/2} + C_2.$$

**Assumption A.7** Function $V(t,x)$ satisfies

$$|V(t,x) - V(t,y)| \leq \omega\left(\|x-y\|\right), \quad |V(t,x) - V(s,x)| \leq (1 + \|x\|)\omega(|t-s|).$$

**Lemma A.5** *Suppose that*

1. *function $h_r(r)$ satisfies Assumption A.5 and is nondecreasing for $r \geq \hat{V}_0$, where $\hat{V}(t,x) \geq \hat{V}_0$;*
2. *function $V(t,x)$ satisfies Assumptions A.6, A.7.*

*Then function $\hat{V} = h^{-1}(V)$ is uniformly continuous on $[t_0, t_1] \times \mathbb{R}^n$.*

*Proof* Note that $\hat{V}_0 > -\infty$ due to Assumption A.6. We have from our assumptions

$$|W(t,x) - W(t,y)| = \left|h^{-1}(V(t,x)) - h^{-1}(V(t,y))\right| = \frac{1}{h_r(r^*)}|V(t,x) - V(t,y)| \leq \frac{\omega\left(\|x-y\|\right)}{h_r(p_0)}.$$

Suppose that $V_1 = V(t,x) \geq V_2 = V(s,x)$. Then

$$|W(t,x) - W(s,x)| = \left|h^{-1}(V_1) - h^{-1}(V_2)\right| = \frac{1}{h_r(r^*)}|V_1 - V_2| \leq \frac{1}{h_r(V_2)}|V_1 - V_2| \leq C_h \frac{V_1 - V_2}{1+V_2^2} \leq C\omega(|t-s|).$$

## A.4 Unbounded Dynamics

Now we relax Assumption A.3 to allow for arbitrary linear dynamics.

**Assumption A.8** Function $f(t,x) = A(t)x + f_0(t,x)$, where $A(t)$ is a continuous matrix function and $f_0(t,x)$ is a Lipschitz-continuous vector function on $[t_0, t_1] \times \mathbb{R}^n$ satisfying

$$|\langle x, f_0(t,x)\rangle| \le C_f \|x\|.$$

We introduce a change of variables $\hat{x} = X(t_1, t)x$, where $X(t,s)$ is the fundamental matrix corresponding to linear system with matrix $A(t)$:

$$\partial X(t,s)/\partial t = A(t)X(t,s), \quad X(s,s) = I.$$

Then, Hamilton–Jacobi–Bellman equation (A.5) takes the form

$$\begin{gathered}
\max\{H_1, H_2\} = 0, \quad t \in [t_0, t_1], \quad x \in \mathbb{R}^n, \\
H_1 = -\hat{V}_t - \left\langle \hat{V}_{\hat{x}}, f_0(t,x)\right\rangle, \quad H_2 = \left\| \hat{B}^T(t)\hat{V}_{\hat{x}} \right\| - 1/h_r\left(\hat{V}\right), \\
\hat{V}(t_1, x) = \hat{\boldsymbol{\varphi}}(x).
\end{gathered} \tag{A.6}$$

Here matrix function $\hat{B}(t) = X(t_1, t)B(t)$ is Lipschitz continuous.

Since mapping $(t,x) \to (t, X(t_1,t)x)$ is a diffeomorphism, thus if $\hat{V}(t,x)$ is a sub- or supersolution to (A.5), then the transformed function $\hat{V}(t,\hat{x})$ is a sub- or supersolution to (A.6). This mapping also preserves Assumptions A.6 and A.7; the verification is straightforward, e.g.,

$$\begin{aligned}
&\left|V(t, X(t,t_1)\hat{x}) - V(s, X(s,t_1)\hat{x})\right| \\
&\quad \le \left|V(t, X(t,t_1)\hat{x}) - V(s, X(t,t_1)\hat{x})\right| + \left|V(s, X(t,t_1)\hat{x}) - V(s, X(s,t_1)\hat{x})\right| \\
&\le (1 + \left\|X(t,t_1)\hat{x}\right\|)\omega(|t-s|) + \omega(\left\|X(t,t_1)\hat{x} - X(s,t_1)\hat{x}\right\|) \le C(1 + \left\|\hat{x}\right\|)\omega(|t-s|)
\end{aligned}$$

since $X$ is bounded and Lipschitz continuous.

We have arrived at the final result.

**Theorem A.3** *Suppose that*

1. *Assumptions A.1, A.2, A.4, and A.8 are satisfied;*
2. *functions $V$ and $W$ satisfy Assumptions A.6 and A.7;*
3. *functions $V$ and $W$ are viscosity subsolution and supersolution to* (A.1)*, respectively.*

*Then $V \le W$.*

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