Appendix
Implementation and Proofs

A.1 Computing the Partial Derivatives of the Inertia Matrix

To compute the Coriolis matrix we need the partial derivatives of the terms in the inertia matrix which are found by the partial derivatives of the Adjoint and Jacobian matrices. The partial derivatives of the inertia matrix $M(q_1, \ldots, q_n)$ with respect to $q_1, \ldots, q_n$ are computed by

$$\begin{aligned}\frac{\partial M(q_1, \ldots, q_n)}{\partial q_k} &= \sum_{i=k}^{n} \left(\begin{bmatrix} H^\top \\ J_i^\top \end{bmatrix} \left[\frac{\partial^\top \mathrm{Ad}_{g_{ib}}}{\partial q_k} I_i \,\mathrm{Ad}_{g_{ib}} + \mathrm{Ad}_{g_{ib}}^\top I_i \frac{\partial \,\mathrm{Ad}_{g_{ib}}}{\partial q_k} \right] \begin{bmatrix} H & J_i \end{bmatrix} \right) \\ &\quad + \sum_{i=k+1}^{n} \begin{bmatrix} 0 & H^\top \mathrm{Ad}_{g_{ib}}^\top I_i \,\mathrm{Ad}_{g_{ib}} \frac{\partial J_i}{\partial q_k} \\ \frac{\partial^\top J_i}{\partial q_k} \mathrm{Ad}_{g_{ib}}^\top I_i \,\mathrm{Ad}_{g_{ib}} H & \frac{\partial^\top J_i}{\partial q_k} \mathrm{Ad}_{g_{ib}}^\top I_i \,\mathrm{Ad}_{g_{ib}} J_i + J_i^\top \mathrm{Ad}_{g_{ib}}^\top I_i \,\mathrm{Ad}_{g_{ib}} \frac{\partial J_i}{\partial q_k} \end{bmatrix}.\end{aligned} \tag{A.1}$$

The Coriolis matrix is thus found by the partial derivatives of the Adjoint and Jacobian matrices with respect to each joint variable q_k.

A.1.1 Computing the Partial Derivatives of $\mathrm{Ad}_{g_{ij}}$

The main computational burden when computing the Coriolis matrix is on the computation of the partial derivatives of M with respect to q for which we need the partial derivatives of the Adjoint matrices, also with respect to q. To compute these one can use a relatively simple relation. If we express the velocity of joint k as $V_{(k-1)k}^{(k-1)} = X_k^k \dot{q}_k$ for constant X_k^k, then the following holds (From et al. 2010):

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Proposition A.1 *The partial derivatives of the Adjoint matrix is given by*

$$\frac{\partial\,\mathrm{Ad}_{g_{ij}}}{\partial q_k} = \begin{cases} \mathrm{Ad}_{g_{i(k-1)}}\,\mathrm{ad}_{X_k^k}\,\mathrm{Ad}_{g_{(k-1)j}} & \textit{for } i < k \le j \\ -\mathrm{Ad}_{g_{i(k-1)}}\,\mathrm{ad}_{X_k^k}\,\mathrm{Ad}_{g_{(k-1)j}} & \textit{for } j < k \le i \\ 0 & \textit{otherwise} \end{cases}$$

Proof (By computing the spatial velocity) To prove this, we start by writing out the spatial velocity of frame $\mathscr{F}_k$ with respect to $\mathscr{F}_{(k-1)}$ when $i < k \le j$:

$$\hat{X}_k \dot{q}_k = \hat{V}^{(k-1)}_{(k-1)k} = \dot{g}_{(k-1)k} g^{-1}_{(k-1)k} = \frac{\partial g_{(k-1)k}}{\partial q_k} g_{k(k-1)} \dot{q}_k$$

where $\hat{X} = \left[\begin{smallmatrix} \hat{X}_\omega & X_v \\ 0 & 0 \end{smallmatrix}\right]$. If we compare the first and the last terms, we get

$$\frac{\partial R_{(k-1)k}}{\partial q_k} = \hat{X}_\omega R_{(k-1)k}, \tag{A.2}$$

$$\frac{\partial p_{(k-1)k}}{\partial q_k} = \hat{X}_\omega p_{(k-1)k} + X_v. \tag{A.3}$$

We can use this relation in the expression for the partial derivative of $\mathrm{Ad}_{g_{(k-1)k}}$:

$$\begin{aligned} \frac{\partial\,\mathrm{Ad}_{g_{(k-1)k}}}{\partial q_k} &= \begin{bmatrix} \frac{\partial R_{(k-1)k}}{\partial q_k} & \frac{\hat{p}_{(k-1)k}}{\partial q_k} R_{(k-1)k} + \hat{p}_{(k-1)k} \frac{\partial R_{(k-1)k}}{\partial q_k} \\ 0 & \frac{\partial R_{(k-1)k}}{\partial q_k} \end{bmatrix} \\ &= \begin{bmatrix} \hat{X}_\omega R_{(k-1)k} & \widehat{\hat{X}_\omega p_{(k-1)k}} R_{(k-1)k} + \hat{X}_v R_{(k-1)k} + \hat{p}_{(k-1)k} \hat{X}_\omega R_{(k-1)k} \\ 0 & \hat{X}_\omega R_{(k-1)k} \end{bmatrix} \\ &= \begin{bmatrix} \hat{X}_\omega R_{(k-1)k} & \hat{X}_\omega \hat{p}_{(k-1)k} R_{(k-1)k} + \hat{X}_v R_{(k-1)k} \\ 0 & \hat{X}_\omega R_{(k-1)k} \end{bmatrix} \\ &= \begin{bmatrix} \hat{X}_\omega & \hat{X}_v \\ 0 & \hat{X}_\omega \end{bmatrix} \begin{bmatrix} R_{(k-1)k} & \hat{p}_{(k-1)k} R_{(k-1)k} \\ 0 & R_{(k-1)k} \end{bmatrix} \\ &= \mathrm{ad}_{X_k^k}\,\mathrm{Ad}_{g_{(k-1)k}} \end{aligned} \tag{A.4}$$

where we have used that

$$\hat{X}\hat{p} = \widehat{(\hat{X}p)} + \hat{p}\hat{X}. \tag{A.5}$$

It is now straight forward to show that

$$\begin{aligned} \frac{\partial\,\mathrm{Ad}_{g_{ij}}}{\partial q_k} &= \mathrm{Ad}_{g_{i(k-1)}} \frac{\partial\,\mathrm{Ad}_{g_{(k-1)k}}}{\partial q_k} \mathrm{Ad}_{g_{kj}} \\ &= \mathrm{Ad}_{g_{i(k-1)}}\,\mathrm{ad}_{X_k^k}\,\mathrm{Ad}_{g_{(k-1)k}}\,\mathrm{Ad}_{g_{kj}} \\ &= \mathrm{Ad}_{g_{i(k-1)}}\,\mathrm{ad}_{X_k^k}\,\mathrm{Ad}_{g_{(k-1)j}}\,. \end{aligned} \tag{A.6}$$

The proof is similar for $j < k \leq i$. □

Proof (By direct computation) We can also prove Proposition A.1 by direct computation. For $i < k \leq j$ the proof is shown by

$$
\begin{aligned}
&\frac{\partial \,\mathrm{Ad}_{g_{ij}}}{\partial q_k} \\
&= \mathrm{Ad}_{g_{i(k-1)}} \frac{\partial \,\mathrm{Ad}_{g_{(k-1)k}}}{\partial q_k} \mathrm{Ad}_{g_{kj}} \\
&= \begin{bmatrix} R_{i(k-1)} & \hat{p}_{i(k-1)} R_{i(k-1)} \\ 0 & R_{i(k-1)} \end{bmatrix} \begin{bmatrix} \frac{\partial R_{(k-1)k}}{\partial q_k} & \frac{\hat{p}_{(k-1)k}}{\partial q_k} R_{(k-1)k} + \hat{p}_{(k-1)k} \frac{\partial R_{(k-1)k}}{\partial q_k} \\ 0 & \frac{\partial R_{(k-1)k}}{\partial q_k} \end{bmatrix} \\
&\quad \times \begin{bmatrix} R_{kj} & \hat{p}_{kj} R_{kj} \\ 0 & R_{kj} \end{bmatrix} \\
&= \begin{bmatrix} R_{i(k-1)} \frac{\partial R_{(k-1)k}}{\partial q_k} R_{kj} & \begin{bmatrix} R_{i(k-1)} \frac{\partial R_{(k-1)k}}{\partial q_k} \hat{p}_{kj} R_{kj} \\ + R_{i(k-1)} \frac{\hat{p}_{(k-1)k}}{\partial q_k} R_{(k-1)j} \\ + R_{i(k-1)} \hat{p}_{(k-1)k} \frac{\partial R_{(k-1)k}}{\partial q_k} R_{kj} \\ + \hat{p}_{i(k-1)} R_{i(k-1)} \frac{\partial R_{(k-1)k}}{\partial q_k} R_{kj} \end{bmatrix} \\ 0 & R_{i(k-1)} \frac{\partial R_{(k-1)k}}{\partial q_k} R_{kj} \end{bmatrix} \\
&= \begin{bmatrix} R_{i(k-1)} \hat{X}_\omega R_{(k-1)k} R_{kj} & \begin{bmatrix} R_{i(k-1)} \hat{X}_\omega R_{(k-1)k} \widehat{\hat{p}_{kj}} R_{kj} \\ + R_{i(k-1)} ((\hat{X}_\omega p_{(k-1)k}) + \hat{X}_v) R_{(k-1)j} \\ + R_{i(k-1)} \hat{p}_{(k-1)k} \hat{X}_\omega R_{(k-1)k} R_{kj} \\ + \hat{p}_{i(k-1)} R_{i(k-1)} \hat{X}_\omega R_{(k-1)k} R_{kj} \end{bmatrix} \\ 0 & R_{i(k-1)} \hat{X}_\omega R_{(k-1)k} R_{kj} \end{bmatrix} \\
&= \begin{bmatrix} R_{i(k-1)} \hat{X}_\omega R_{(k-1)j} & \begin{bmatrix} R_{i(k-1)} \hat{X}_v R_{(k-1)j} + R_{i(k-1)} \hat{X}_\omega \hat{p}_{(k-1)j} R_{(k-1)j} \\ + \hat{p}_{i(k-1)} R_{i(k-1)} \hat{X}_\omega R_{(k-1)j} \end{bmatrix} \\ 0 & R_{i(k-1)} \hat{X}_\omega R_{(k-1)j} \end{bmatrix} \\
&= \begin{bmatrix} R_{i(k-1)} \hat{X}_\omega & R_{i(k-1)} \hat{X}_v + \hat{p}_{i(k-1)} R_{i(k-1)} \hat{X}_\omega \\ 0 & R_{i(k-1)} \hat{X}_\omega \end{bmatrix} \begin{bmatrix} R_{(k-1)j} & \hat{p}_{(k-1)j} R_{(k-1)j} \\ 0 & R_{(k-1)j} \end{bmatrix} \\
&= \begin{bmatrix} R_{i(k-1)} & \hat{p}_{i(k-1)} R_{i(k-1)} \\ 0 & R_{i(k-1)} \end{bmatrix} \begin{bmatrix} \hat{X}_\omega & \hat{X}_v \\ 0 & \hat{X}_\omega \end{bmatrix} \begin{bmatrix} R_{(k-1)j} & \hat{p}_{(k-1)j} R_{(k-1)j} \\ 0 & R_{(k-1)j} \end{bmatrix} \\
&= \mathrm{Ad}_{g_{i(k-1)}} \mathrm{ad}_{X_k^k} \mathrm{Ad}_{g_{(k-1)j}}
\end{aligned}
\tag{A.7}
$$

where we have used Eq. (A.5) and

$$
\hat{p}_{(k-1)j} = (\widehat{R_{(k-1)k} p_{kj}}) + \hat{p}_{(k-1)k}. \tag{A.8}
$$

□

A.1.2 Computing the Jacobian and Its Partial Derivatives

The Jacobian $\bar{J}_i$ of link i is given by

$$J_i(q) = \begin{bmatrix} \mathrm{Ad}_{g_{0\bar{1}}} X_1^1 & \mathrm{Ad}_{g_{0\bar{2}}} X_2^2 & \mathrm{Ad}_{g_{0\bar{3}}} X_3^3 & \cdots & \mathrm{Ad}_{g_{0\bar{i}}} X_i^i & 0_{(n-i)\times 6} \end{bmatrix}. \quad \text{(A.9)}$$

When the partial derivatives of the Adjoint map are found as in the previous section we can also use these to find the partial derivatives of the Jacobian, i.e.,

$$\frac{\partial J_i}{\partial q_k} = \begin{bmatrix} 0_{(k-1)\times 6} & \frac{\partial\, \mathrm{Ad}_{g_{b\bar{k}}}}{\partial q_k} X_k^k & \frac{\partial\, \mathrm{Ad}_{g_{b(\overline{k+1})}}}{\partial q_k} X_{k+1}^{k+1} & \cdots & \frac{\partial\, \mathrm{Ad}_{g_{b\bar{i}}}}{\partial q_k} X_i^i & 0_{(n-i)\times 6} \end{bmatrix}. \quad \text{(A.10)}$$

A.1.3 Implementation

In this section we will show how to compute the Coriolis matrix for implementation purposes. We will show this for $SO(3)$ for which the Coriolis matrix is given by

$$\begin{aligned} C(Q,v) &= \sum_{k=1}^{n} \frac{\partial M}{\partial q_k} \dot{q}_k \\ &\quad - \frac{1}{2} \begin{bmatrix} \widehat{(M(q)v)_{\tilde{V}}} & 0 \\ \frac{\partial^{\mathsf{T}}}{\partial q}\left(\begin{bmatrix} M_{VV} & M_{qV}^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} \tilde{V}_{0b}^B \\ \dot{q} \end{bmatrix} \right) & \frac{\partial^{\mathsf{T}}}{\partial q}\left(\begin{bmatrix} M_{qV} & M_{qq}^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} \tilde{V}_{0b}^B \\ \dot{q} \end{bmatrix} \right) \end{bmatrix} \end{aligned} \quad \text{(A.11)}$$

where $(M(q)v)_{\tilde{V}}$ is the vector of the first three entries of the vector $M(q)v$ (corresponding to $\tilde{V}_{0b}^B = \omega_{0b}^B$).

We first define the vector

$$\big(M(q)v\big)_{\tilde{V}} = \begin{bmatrix} (M(q)v)_1 \\ (M(q)v)_2 \\ (M(q)v)_3 \end{bmatrix} = \begin{bmatrix} M_{VV} & M_{qV}^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} V_{0b}^B \\ \dot{q} \end{bmatrix} \quad \text{(A.12)}$$

which gives

$$\mathrm{ad}_{(M(q)v)_{\tilde{V}}} = \begin{bmatrix} 0 & -(M(q)v)_3 & (M(q)v)_2 \\ (M(q)v)_3 & 0 & -(M(q)v)_1 \\ -(M(q)v)_2 & (M(q)v)_1 & 0 \end{bmatrix}. \quad \text{(A.13)}$$

The lower part of the second part of (A.11) is calculated as

$$\frac{\partial^{\mathsf{T}}}{\partial q}\left(\begin{bmatrix} M_{VV} & M_{qV}^{\mathsf{T}} \end{bmatrix}\begin{bmatrix} V_{0b}^{B} \\ \dot{q} \end{bmatrix}\right)$$

$$= \begin{bmatrix} \frac{\partial (Mv)_1}{\partial q_1} & \frac{\partial (Mv)_2}{\partial q_1} & \frac{\partial (Mv)_3}{\partial q_1} \\ \frac{\partial (Mv)_1}{\partial q_2} & \frac{\partial (Mv)_2}{\partial q_2} & \frac{\partial (Mv)_3}{\partial q_2} \\ \vdots & & \vdots \\ \frac{\partial (Mv)_1}{\partial q_n} & \frac{\partial (Mv)_2}{\partial q_n} & \frac{\partial (Mv)_3}{\partial q_n} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{3+n} \frac{\partial M_{1i}}{\partial q_1} v_i & \sum_{i=1}^{3+n} \frac{\partial M_{2i}}{\partial q_1} v_i & \sum_{i=1}^{3+n} \frac{\partial M_{3i}}{\partial q_1} v_i \\ \sum_{i=1}^{3+n} \frac{\partial M_{1i}}{\partial q_2} v_i & \sum_{i=1}^{3+n} \frac{\partial M_{2i}}{\partial q_2} v_i & \sum_{i=1}^{3+n} \frac{\partial M_{3i}}{\partial q_2} v_i \\ \vdots & & \vdots \\ \sum_{i=1}^{3+n} \frac{\partial M_{1i}}{\partial q_n} v_i & \sum_{i=1}^{3+n} \frac{\partial M_{2i}}{\partial q_n} v_i & \sum_{i=1}^{3+n} \frac{\partial M_{3i}}{\partial q_n} v_i \end{bmatrix} \tag{A.14}$$

$$\frac{\partial^{\mathsf{T}}}{\partial q}\left(\begin{bmatrix} M_{qV} & M_{qq}^{\mathsf{T}} \end{bmatrix}\begin{bmatrix} V_{0b}^{B} \\ \dot{q} \end{bmatrix}\right) = \begin{bmatrix} \sum_{i=1}^{3+n} \frac{\partial M_{(m+1)i}}{\partial q_1} v_i & \cdots & \sum_{i=1}^{3+n} \frac{\partial M_{(m+n)i}}{\partial q_1} v_i \\ \sum_{i=1}^{3+n} \frac{\partial M_{(m+1)i}}{\partial q_2} v_i & \cdots & \sum_{i=1}^{3+n} \frac{\partial M_{(m+n)i}}{\partial q_2} v_i \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{3+n} \frac{\partial M_{(m+1)i}}{\partial q_n} v_i & \cdots & \sum_{i=1}^{3+n} \frac{\partial M_{(m+n)i}}{\partial q_n} v_i \end{bmatrix} \tag{A.15}$$

and is thus also given by the partial derivative of the elements of the inertia matrix. We thus only need to compute the partial derivative $\frac{\partial M(q)}{\partial q_i}$ once and use the result in the both in the first and second part of (A.11). This will simplify the computation of the dynamic equations for an arbitrary n-link mechanism mounted on a vehicle.

References

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