Afterword

A physics book never ends. The cause of this fact lies only partially in the finiteness of the number of pages a book may include; also, the number of exciting questions tends to grow steadily. It is like a very fine network, ramified in a rather complex fashion. Writing about one or the other problem, its solution, the development of logical concepts and conceptual ideas, very often receives echoes from unexpected corners. Resonating to an idea there are some others, and then yet some further burring questions emerge. This process never ends, so the end of a book must be set *artificially* – no other way exists.

In the present book about challenges to the concept of temperature and some related quantities, like heat and entropy, dissipation an information, equilibrium, and motion, we have brought just a few highlights to attention. In order to lay foundations for the understanding of the nature of these challenges we started with a review of the measurement and interpretation of temperature – as it has been laid down in classical thermodynamics. Already at this level it should have been clear that the temperature as a concept and as a mathematical quantity has several connections, it plays different roles in different parts of theoretical physics. It occurs as an empirical, but yet universal scale associated to thermal equilibrium. It is derived from a Lagrange multiplier used in counting for the energy conservation. Sometimes it takes the form related to a derivative of the entropy–energy function; but sometimes a more general form. Moreover it can be related to statistical parameters of the internal (originally atomic) motion; in a more general setup to a width property of unknown dynamical agents – cited as "noise" for brevity. All these facets sit on the same diamond: These roles and definitions are interrelated.

The first – long known – challenge to this picture emerges from considering systems with a sufficiently small number of degrees of freedom. In such cases the equivalence between the different statistical ensembles is not working, the finite number corrections can be appreciable. In a way in a small, or just "mesoscopic" closed system the entropy maximum with a given energy leads to equilibrium distributions resembling a spread of the inverse temperature parameter. Akin to this

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phenomenon is the idea of superstatistics, which assumes a probability distribution of different temperatures on the top of the canonical description. As a dynamical model, coupled stochastic evolution equations in a characteristic time-hierarchy approximation can explain such effects. Leaving the canonical paradigm of equilibrium distributions exponentially falling in the energy, the question arises towards a possible generalization of the Gibbsean–Boltzmannian thermodynamics. We have presented a particular approach to this by analyzing abstract composition rules and their infinite repetitions. We have put an emphasis on indicating the possibility of this type of generalizations everywhere in the discussion of the classical laws of thermodynamics.

Classical thermodynamics is particularly challenged by the motion at relativistic speed. While some aspects can be treated by exchanging the energy-momentum relation of particles only, some other – like local and causal dissipation – need a treatment in the framework of relativistic hydrodynamics. Surprisingly, a classical debate about the correct relativistic transformation formula for the absolute temperature – already started by Planck and Einstein – can also be transparently analyzed in this framework.

High acceleration offers a further challenge on thermal concepts. The Unruh effect, by the virtue of which a constant acceleration of a monochromatic wave is observed by a far static observer as a black body radiation at a temperature proportional to the acceleration – is most shocking. In relation with this an entropy can be associated to event horizons around black holes and the zeroth, first and second law of thermodynamics can be demonstrated. Moreover other horizons, namely a cosmological horizon, also behaves mathematically a similar way: a thermal looking environment is created by non-trivial spacetime structures. This concept is carried over to higher dimensional gravity theories describing – by the duality principle – plasmas of strongly interacting matter.

Finally, we have discussed the formal use of the inverse temperature in field theory, where it becomes a period length in the imaginary time direction. This sounds quite abstract and mathematical. However, it does describe the proper thermal statistics for free bosons and fermions, relating the different sign in the basic formula to the different commutation property of the fundamental field operators. Furthermore, it opens up the possibility to treat the kinetic theory of heat and temperature on the level of elementary quantum fields – possibly including quantum effects like off-shellness (uncertainty) of particle states. We have shed a little light on the basics of this. For closing this presentation we took reference to the path integral formalism and the stochastic quantization method used in field theory: an intriguing, albeit speculative analogy between a higher dimensional classical chaotic field theory and the usual quantum field theory has been outlined. This study uncovered a further possible role, which the temperature may play; it might be a factor in producing Planck's constant.

We could have written, however, much more about.[1] Without any preference let us see here a list of omitted topics: We left out the discussion of continuum thermodynamics, the problems with very slowly flowing material, like glasses or spin glasses. From the field of material science the behavior of lattice faults, the time evolution of dislocations and other deformations could have added some new aspects to our perspective on near-equilibrium and far-equilibrium physics. The statistical behavior of granular matter, different type of sandbox and avalanche models, or the phenomenon of percolation and fragmentation – both on the solid state and on the nuclear level – including jet-fragmentation as a particular type of hadronization in high-energy collisions, also could have been mentioned.

Outgrowing from statistical studies of such self-similar dynamical systems the "self-organizing criticality" and chaotic dynamics in general may have been touched upon. Also the relatively recent development on the theory of random networks and graphs with the power-law stationary distributions of node connectivity (causing the "small-world" effect) would have been a delicacy to discuss. To support the need for a novel, presumably non-extensive approach to thermodynamics, models and calculations from the field of econophysics, magnetic systems, earthquakes, climate changes or game theory and other social models would have served excellent. Finally – in relation to the field theoretical treatment of thermal phenomena – also a number of further speculations about an emerging quantum theory would have been exciting to cite.

At the end we may summarize the contemporary challenges to the concept of temperature as follows: The relevance of independent statistics expressed in factorizing marginal probabilities and the exclusive presence of short range interactions cannot be guaranteed in a number of physical phenomena. Most characteristically a too low number of degrees of freedom, or a time hierarchy between processes establishing detailed balance in some, but failing to achieve such a state in other variables, create physical situations, where a generalization of the classical thermodynamics is desired. Motion of whole bodies with relativistic speed or high acceleration also challenges the belief that the temperature would be an intuitive, easily comprehensible quantity in theoretical physics. Finally, entanglement between statistical and quantum effects in a really dynamical situation for drastic, short-time evolution offers a possibility to re-think the interpretation of temperature in a fundamental way: can it be related to the very structure of space and time? May it be behind the very existence of quantum effects?

As to the question in the title of this book: *There is a temperature*. It is an unexpectedly useful concept even in problems far from the realm of the classical thermodynamics. However, one has to be careful in relying on its classical derivation; under given circumstances a generalization is necessary. It cannot just be used the same way as in the classical context, the different physical roles, the variable called "temperature" plays, all have to be carefully considered.

[1] In fact, originally we did have some plans to include more topics.

Solutions

Problems of Chap. 2

2.1 Convert by heart (and fast) the following temperature values between the Celsius and Fahrenheit scales: 36°C, 27°C, 22°C, 100°C, 32°F, 64°F, 80°F, 71°F, 451°F.

Temperatures given in Celsius grades have to be interpreted in groups of nine, then shifted by $+2$ and re-interpreted as 16°F each. This way $36°\text{C} \to 4 \to 6 \to 96°\text{F}$ and $27°\text{C} \to 3 \to 5 \to 80°\text{F}$. 22°C is not readily dividable by 9, but it is 5 less than 27°C. Since 5°C is equivalent to about 9°F, it follows that 22°C is about $80-9=71$°F. 100°C is about 99°C meaning 11 in nonal system. Adding 2 it makes 13×16°F, which is about 160 plus three times 16 (about 50). This way 100°C makes about 210°F.

32, 64, and 80 are multiples of 16 by 2, 4, and 5 respectively. The corresponding centigrades are then multiples of 9 by 0, 2 and 3, giving 0, 18 and 27°C, respectively. 71°F is 9 less than 80, so it is about 5 less than 27 in centigrades, i.e. 22°C. Finally, 451°F is about $480-32$ meaning $30-2=28$ groups of 16. Shifting to centigrades it is $28-2=26$ groups of 9 resulting in $9 \times 26 = 260-26 = 234$°C.

2.2 Derive Kirchhoff's law from the equality of intensities of emitted and absorbed radiation between two bodies in equilibrium.

Two bodies, labeled as 1 and 2, radiate and absorb energy. Let the intensity of the energy current arriving to body 2 from body 1 be denoted by I^+, the opposite (and in equilibrium equal) intensity by I^-. The respective emissivity coefficients are ε_1 and ε_2, the absorptivity coefficients A_1 and A_2. The $1-A_i$ $(i=1,2)$ part of the incoming radiations are reflected, these currents contribute to the total infalling intensities on the respective bodies.

The intensities are hence related as

$$I^+ = \varepsilon_1 + (1-A_1)I^-,$$

and

$$I^- = \varepsilon_2 + (1 - A_2) I^+.$$

In equilibrium, $I^+ = I^- = I$ and the above system of linear equations leads to

$$\frac{\varepsilon_1}{A_1} = \frac{\varepsilon_2}{A_2} = I.$$

This proves that the ratio ε/A is independent of the material quality of the bodies equilibrating by radiation. This is the essence of Kirchhoff's law.

2.3 Using Wien's law determine the wavelengths of maximal intensity for the Sun's surface, for a light bulb, for the human body and for the cosmic microwave background.

Following Wien's law the wavelength of intensity maximum is at $\hbar\omega/k_B T = 3$ giving $\lambda = h/3k_B T \approx 2.897\,\text{mm K}$. This relates roughly $T = 1{,}000\,\text{K}$ to $\lambda = 3\,\mu\text{m}$. The visible surface of the Sun has a temperature around $T = 6{,}000\,\text{K}$, resulting in $\lambda = 500\,\text{nm}$. In fact, biologically evolved eyes on the Earth are most sensitive near to this wavelength. A common light bulb glows at the temperature $T = 1{,}500\,\text{K}$. The corresponding wavelength is $\lambda = 2{,}000\,\text{nm}$, giving more power in the infrared than in the visible spectrum. Warm blood animal bodies maintain a temperature around 36°C, about 300 K. The radiation power is maximal at $\lambda = 10\,\mu\text{m}$, well in the infrared range. This fact is utilized by "night seeing" devices. Finally, the cosmic microwave background has a temperature of around 3 K. The corresponding wavelength is $\lambda = 1\,\text{mm}$.

2.4 What is the average energy carried by a photon in thermal radiation according to Planck's law, according to Wien's law and according to a Raleigh-Jeans law cut at the maximal frequency of Wien's formula?

The differential frequency distribution of photons is described by Planck's law (and by its low- and high-frequency limits in case of Raleigh-Jeans and Wien, respectively):

$$f(\omega)\mathrm{d}\omega = \frac{\omega^2 \mathrm{d}\omega}{\mathrm{e}^{\hbar\omega/k_B T} - 1} \longrightarrow \begin{cases} \frac{k_B T}{\hbar}\omega \mathrm{d}\omega \\ \omega^2 \mathrm{e}^{-\hbar\omega/k_B T}\mathrm{d}\omega \end{cases}$$

The maximal intensity according to Wien's displacement law is at $\omega_{\max} = 3k_B T/\hbar$. The average frequency is given by

$$\bar{\omega} = \frac{\int_0^\infty \omega f(\omega)\mathrm{d}\omega}{\int_0^\infty f(\omega)\mathrm{d}\omega}.$$

Expanding Planck's formula as a geometrical series,

$$\frac{1}{e^x - 1} = \sum_{n=1}^{\infty} e^{-nx},$$

we deal with a series of integrals with powers and exponentials of ω. The generic integral looks like

$$\sum_{n=1}^{\infty} \int_0^{\infty} x^k e^{-nx} dx = \sum_{n=1}^{\infty} \frac{k!}{n^{k+1}} = k!\zeta(k+1),$$

containing Riemann's zeta function $\zeta(k)$. The average frequency by the Planck's law becomes

$$\bar{\omega} = \frac{k_B T}{\hbar} \frac{3!\zeta(4)}{2!\zeta(3)} \approx 3T \frac{k_B}{\hbar} \frac{1.0823}{1.2021}.$$

In $k_B = c = \hbar = 1$ units $\bar{\omega} \approx 2.7T$.

Wien's approximation keeps the $n = 1$ term from the series only by using the exponential Boltzmann factor instead of Planck's law. The average frequency is equal to the one where the intensity is maximal, according to his displacement law:

$$\bar{\omega}_W = \frac{k_B T}{\hbar} \frac{3!}{2!} = 3T \frac{k_B}{\hbar}.$$

Finally, the Raleigh–Jeans formula cannot be integrated up to infinite frequencies, the integrals being divergent (this is called the "ultraviolet catastrophe"). Making the cut instead at $x_{max} = \hbar\omega_{max}/k_B T = 3$ one obtains

$$\bar{\omega}_{RJ} = \frac{k_B T}{\hbar} \frac{\int_0^{x_{max}} x^2 dx}{\int_0^{x_{max}} x dx} = \frac{2}{3}\omega_{max},$$

resulting in $\bar{\omega} = 2k_B T/\hbar$.

Problems of Chap. 3

3.1 Prove the two leading orders in the Stirling formula for $\ln N!$.

Due to the recursive property of factorial, $(N+1)! = (N+1)N!$, its logarithm, $S_N = \ln N!$ satisfies

$$S_{N+1} - S_N = \ln(N+1).$$

Comparing this with the rule for $(N-1)$ and subtracting we get

$$S_{N+1}+S_{N-1}-2S_N=\ln\left(1+\frac{1}{N}\right)\approx\frac{1}{N}$$

in the large N limit. Considering now the $N\to\infty$ limit the above recursion formula can be approximated by a differential equation

$$\frac{\mathrm{d}^2S}{\mathrm{d}x^2}=\frac{1}{x}.$$

The solution of this equation with the conditions $S(0)=0$, $S'(0)=0$ is

$$S(x)=x\,(\ln x-1)\,,$$

leading to the large N estimate we were after.

3.2 Determine the occupation probabilities for three states having zero, one and two quanta of the energy ε by excluding all other states. The average energy is fixed to be $\bar{\varepsilon}$.

Following the general discussion of the canonical BG distribution the probabilities are given as

$$w_n=\frac{\mathrm{e}^{-\beta\varepsilon_n}}{\sum_n \mathrm{e}^{-\beta\varepsilon_n}}.$$

In the present case, we have $n=0,1$ and 2 states only with the energies $\varepsilon_0=0$, $\varepsilon_1=\varepsilon$ and $\varepsilon_2=2\varepsilon$. This leads to the following relation between the average energy and the temperature $k_\mathrm{B}T=1/\beta$:

$$\bar{\varepsilon}=\varepsilon\,\frac{\mathrm{e}^{-\beta\varepsilon}+2\mathrm{e}^{-2\beta\varepsilon}}{1+\mathrm{e}^{-\beta\varepsilon}+\mathrm{e}^{-2\beta\varepsilon}}.$$

The temperature can be obtained by inverting this relation.

Formally, for $\bar{\varepsilon}>\varepsilon$ the solution is at negative absolute temperature ($\beta<0$) reflecting the so called "inverse population" of energy levels. Clearly, such a value can only be reached if the double energy states are populated more than the ones with single quantum. In fact all probability ratios

$$\frac{w_{n+1}}{w_n}=\mathrm{e}^{-\beta\varepsilon},$$

are less than one for β being positive and more than one for β negative. The average energy per the energy quantum, $\bar{\varepsilon}/\varepsilon$, is exactly one at $\beta=0$, i.e. at infinite temperature.

3.3 Determine the grand-canonical equilibrium distribution for a fractionally fermionic-bosonic system: maximum the fraction qN can be in an indistinguishable state of K states.

The number of distinguishable macrostates is given by

$$W = \binom{K+qN}{N}.$$

The entropy per level function in the $K \to \infty$ limit with fixed $N/K = p$ (in $k_B = 1$ units) becomes

$$\sigma(p) = (1+qp)\ln(1+qp) - p\ln p - (1+(q-1)p)\ln(1+(q-1)p).$$

Its derivative with respect to p equals to $\ln x = (\varepsilon - \mu)/T$. We obtain

$$\frac{(1+qp)^q}{p(1+(q-1)p)^{q-1}} = x.$$

This means $(1+p)/p = x$ for $q = 1$ leading to the Bose, and $(1-p)/p = x$ for $q = 0$ leading to the Fermi distribution. It is interesting that for $q = 1/2$ not the Boltzmann distribution emerges, but

$$\frac{(1+p/2)^{1/2}}{p(1-p/2)^{-1/2}} = \frac{\sqrt{1-p^2/4}}{p} = x$$

leading to

$$p = \frac{1}{\sqrt{x^2+1/4}}.$$

The BG distribution would be $p = 1/x$, in fact for large x any of the q-on formulas leads to this result. We note that at $x = 1$ the Fermi surface is located where $\varepsilon = \mu$. The $p = 1$ value at $x = 1$ is achieved for $q^* \approx 0.5628$.

3.4 Prove the generalized Markov inequality (3.52).

The partial measure of x values satisfying the constraint $f(x) \geq t$ is given by

$$P_f(t) = \int\limits_{f(x)\geq t} \mathrm{d}\mu(x).$$

Multiplying this integral by $g(t)$ and using its monotonic rising property, under the above constraint $g(f(x)) \geq g(t)$ is fulfilled. We estimate

$$\int\limits_{f(x)\geq t} g(t)\,\mathrm{d}\mu(x) \leq \int\limits_{f(x)\geq t} g(f(x))\,\mathrm{d}\mu(x).$$

Now, since both the PDF, $\mathrm{d}\mu(x) = p(x)\mathrm{d}x$, and the test function, $g(t)$ are non-negative, the restricted integral on the right hand side is smaller or equal to the integral over all possible values of x:

$$\int_{f(x)\geq t} g(f(x))\,\mathrm{d}\mu(x) \leq \int_{x\in X} g(f(x))\,\mathrm{d}\mu(x).$$

A division by $g(t) \neq 0$ delivers the Markov inequality

$$P_f(t) \leq \frac{1}{g(t)} \int_{x\in X} g(f(x))\,\mathrm{d}\mu(x) = \frac{\langle g(f(x))\rangle}{g(t)}.$$

3.5 What is the distribution of energy differences in an ideal Maxwell–Boltzmann gas?

Using scaled variables, $x = E/k_{\mathrm{B}}T$, the MB distribution in terms of individual energies is an Euler–Gamma distribution with the power $n = 3/2$. The distribution of a given difference, $x = x_1 - x_2$, is defined with the help of Dirac's delta function:

$$P(x) = \int\mathrm{d}x_1 \int\mathrm{d}x_2\; w(x_1)\, w(x_2)\, \delta(x_1 - x_2 - x).$$

Making use of the Fourier expansion of the Dirac delta one obtains

$$P(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\,\mathrm{e}^{-\mathrm{i}kx} \left\langle \mathrm{e}^{\mathrm{i}kx_1} \right\rangle \left\langle \mathrm{e}^{-\mathrm{i}kx_2} \right\rangle.$$

The individual energy distributions, $w(x_i)$ are given by

$$w(x_i) = \frac{1}{\Gamma(n)} x_i^{n-1} \mathrm{e}^{-x_i},$$

so their characteristic function is

$$\left\langle \mathrm{e}^{\mathrm{i}kx_i} \right\rangle = \frac{1}{\Gamma(n)} \int_0^{\infty} x_i^{n-1} \mathrm{e}^{-x_i} \mathrm{e}^{\mathrm{i}kx_i}\,\mathrm{d}x_i = \frac{1}{(ik-1)^n}.$$

The distribution of differences is hence given by the Fourier integral

$$P(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\,\mathrm{e}^{-\mathrm{i}kx} \frac{1}{(1+k^2)^n}.$$

The result of this integration for $n = 3/2$ is

$$P(x) = \frac{1}{\pi}\,|x|\; K_1(|x|),$$

with the K_1 Bessel function (Fig. 1). As it can be inspected on the figure below, the most probable is to find two molecules with the same energy in an MB gas. The 50% level of the integrated probability is for energy differences less than $0.941 k_B T$, indicated by the vertical dotted lines.

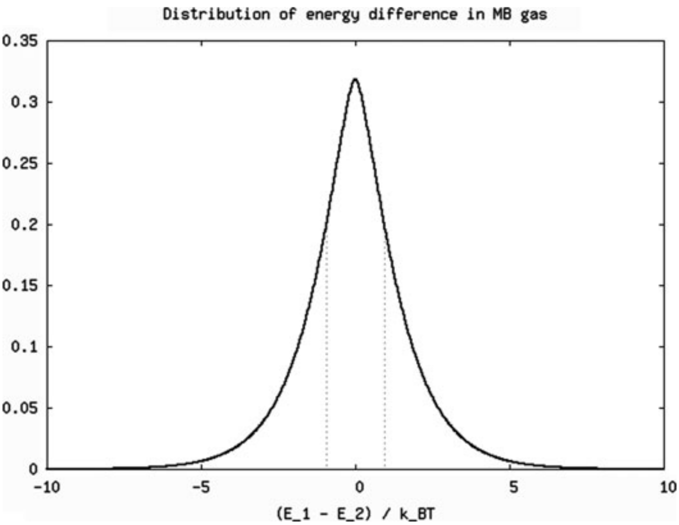

Fig. 1 The distribution of scaled energy difference, $x = (E_1 - E_2)/k_B T$, in a Maxwell–Boltzmann ideal gas. The 50% integrated probability is between the *vertical dotted lines*

3.6 The Jensen inequality,

$$\prod_{i=1}^{\mathscr{N}} a_i^{p_i} \leq \sum_{i=1}^{\mathscr{N}} p_i a_i$$

for all $a_i > 0$, $p_i \in [0,1]$ and $\sum_i p_i = 1$ is the generalization of the traditional inequality between the geometric and arithmetic means. Prove this inequality. What does it say for the information entropy?

First, we consider the $\mathscr{N} = 2$ case. Let it be $p_1 = p$, $p_2 = 1 - p$ and

$$\frac{a_1}{a_2} = (1+x)^{1/p}.$$

We assume $a_1 > a_2$, otherwise exchange the roles of indices and p with $1 - p$. Due to this assumption $x > 0$ and by construction $1/p \geq 1$. The Jensen inequality for $\mathscr{N} = 2$ states that

$$a_1^{p_1} a_2^{p_2} = (1+x)a_2 \leq p_1 a_1 + p_2 a_2 = p(1+x)^{1/p} a_2 + (1-p) a_2.$$

Dividing this by $a_2 > 0$ and denoting $w = 1/p$ one obtains

$$1 + x \leq p(1+x)^w + 1 - p$$

which after rearrangement and division by $p > 0$ becomes

$$1 + wx \leq (1+x)^w.$$

This is true for any $w > 1$ and $x > 0$, so the original inequality is also true.

Now we generalize the $\mathscr{N} = 2$ result for arbitrary $\mathscr{N}$ by induction. Denoting the generalized geometrical mean of $\mathscr{N}$ a_i values by $G_{\mathscr{N}}$ and the arithmetic mean by $A_{\mathscr{N}}$ we prove the Jensen inequality for $\mathscr{N}$ factors while assuming that it is true for $\mathscr{N} - 1$ and for 2. The geometrical mean can be written as

$$G_{\mathscr{N}} = a_{\mathscr{N}}^{p_{\mathscr{N}}} \times G_{\mathscr{N}-1}^{1-p_{\mathscr{N}}},$$

since

$$\sum_{i=1}^{\mathscr{N}-1} p_i = 1 - p_{\mathscr{N}},$$

and in order to satisfy the normalization in the expression for

$$G_{\mathscr{N}-1} = \prod_{i=1}^{\mathscr{N}-1} a_i^{\tilde{p}_i}$$

the scaled powers,

$$\tilde{p}_i = \frac{p_i}{1 - p_{\mathscr{N}}},$$

have to be used. The above expression for $G_{\mathscr{N}}$ itself is a generalized geometrical mean for two factors, so according to our previous proof

$$G_{\mathscr{N}} \leq p_{\mathscr{N}} a_{\mathscr{N}} + (1 - p_{\mathscr{N}}) G_{\mathscr{N}-1}.$$

In this the term, $G_{\mathscr{N}-1}$ can be further estimated from above by the inductive assumption that the Jensen inequality is valid for $\mathscr{N} - 1$. This leads to

$$G_{\mathscr{N}} \leq p_{\mathscr{N}} a_{\mathscr{N}} + (1 - p_{\mathscr{N}}) \sum_{i=1}^{\mathscr{N}-1} \tilde{p}_i a_i,$$

what in turn – using the scaling factor in the definition of the $\tilde{p}_i$ – simply equals to the generalized arithmetic mean of $\mathscr{N}$ factors. Therefore,

$$G_{\mathscr{N}} \leq \sum_{i=1}^{\mathscr{N}} p_i a_i = A_{\mathscr{N}},$$

completing the proof.

Taking the logarithm of the Jensen inequality one achieves

$$\sum_{i=1}^{\mathcal{N}} p_i \ln a_i \leq \ln \sum_{i=1}^{\mathcal{N}} p_i a_i.$$

Applying this formula for the special case $a_i = 1/p_i$ the following estimate arises

$$\varsigma = \sum_{i=1}^{\mathcal{N}} p_i \ln \frac{1}{p_i} \leq \ln \sum_{i=1}^{\mathcal{N}} p_i \frac{1}{p_i} = \ln \mathcal{N}.$$

Finally, this means $\varsigma \leq \ln \mathcal{N}$, the equality being fulfilled when all a_i terms are the same. In case of the BG entropy formula it means equipartition.

3.7 Second Law and Life

By thermal equilibration between two subsystems the colder heats up and the warmer body cools down until a common temperature is achieved. How is it possible then, that on the Earth, while steadily gaining energy from the hotter Sun, entropy is virtually reduced by the spontaneous evolution of highly improbable – as highly correlated – structures, shortly named Life.

The key to the solution is to observe that three bodies are involved: the Sun, the Earth and Space. Denoting the reciprocal temperatures by $\beta_i = 1/T_i$ and the energies by E_i, the Second Law demands that

$$\mathrm{d}S = \beta_1 \mathrm{d}E_1 + \beta_2 \mathrm{d}E_2 + \beta_3 \mathrm{d}E_3 \geq 0,$$

while the system is energetically closed for these three bodies:

$$\mathrm{d}E_1 + \mathrm{d}E_2 + \mathrm{d}E_3 = 0.$$

Owing to the fact that the Sun cools, the outer Space takes energy, while the Earth cools or has constant energy, $\mathrm{d}E_1 < 0$, $\mathrm{d}E_2 \leq 0$ and $\mathrm{d}E_3 > 0$. Introducing the parameter λ with values between zero and one, we have $\mathrm{d}E_1 = -\lambda \mathrm{d}E_3$ and $\mathrm{d}E_2 = -(1-\lambda)\mathrm{d}E_3$. The total entropy balance,

$$\mathrm{d}S = (-\lambda\beta_1 - (1-\lambda)\beta_2 + \beta_3)\, \mathrm{d}E_3 \geq 0,$$

translates to the following inequality:

$$(\beta_3 - \beta_2) + \lambda\,(\beta_2 - \beta_1) \geq 0.$$

This is fulfilled for arbitrary positive λ if $\beta_3 > \beta_2 > \beta_1$, or expressed by the absolute temperatures if $T_3 < T_2 < T_1$. In this setting $\mathrm{d}S_2 = \beta_2 \mathrm{d}E_2 \leq 0$, so the reduction of Earth's entropy is perfectly in accordance with the Second Law of Thermodynamics.

Problems of Chap. 4

4.1 Prove the rule for the Pascal triangle,

$$\binom{k}{n} = \binom{k-1}{n} + \binom{k-1}{n-1}.$$

It can be proved by using the definition of the binomial coefficient. One has

$$\binom{k}{n} = \frac{k!}{n!(k-n)!} = \frac{k}{n(k-n)} \frac{(k-1)!}{(n-1)!(k-1-n)!}$$

The first factor can be split as a sum

$$\frac{k}{n(k-n)} = \frac{1}{n} + \frac{1}{k-n}$$

whence

$$\binom{k}{n} = \left(\frac{1}{n} + \frac{1}{k-n}\right) \frac{(k-1)!}{(n-1)!(k-1-n)!} = \binom{k-1}{n} + \binom{k-1}{n-1}.$$

The Pascal triangle rule also can be proved by considering the identity

$$(a+b)^k = (a+b)^{k-1}(a+b)$$

and the expansion

$$(a+b)^k = \sum_{n=0}^{k} w_{n,k} a^n b^{k-n},$$

denoting the binomial coefficient "k over n" by $w_{n,k}$. Considering now

$$\sum_{n=0}^{k-1} w_{n,k-1} a^n b^{k-1-n}(a+b) = \sum_{n=0}^{k-1} w_{n,k-1} a^{n+1} b^{k-1-n} + \sum_{n=0}^{k-1} w_{n,k-1} a^n b^{k-n}$$

one redefines the running index from n to $n-1$ in the first sum and obtains

$$\sum_{n=0}^{k-1} w_{n,k-1} a^n b^{k-1-n}(a+b) = \sum_{n=1}^{k} w_{n-1,k-1} a^n b^{k-n} + \sum_{n=0}^{k-1} w_{n,k-1} a^n b^{k-n}.$$

Comparing finally the general coefficients of the terms $a^n b^{k-n}$ one concludes that

$$w_{n,k} = w_{n,k-1} + w_{n-1,k-1}.$$

4.2 What is the Pascal triangle-like recursion rule for the probabilities in the Bernoulli distribution?

The formula for the Bernoulli distribution is given as

$$P_{n,k} = \binom{k}{n} f^n (1-f)^{k-n}.$$

Using the Pascal triangle rule it splits to a sum

$$\binom{k}{n} f^n (1-f)^{k-n} = \left(\binom{k-1}{n} + \binom{k-1}{n-1} \right) f^n (1-f)^{k-n}$$

therefore

$$P_{n,k} = (1-f)P_{n,k-1} + fP_{n-1,k-1}.$$

This is a weighted Pascal rule with the average hole and particle occupation rates.

4.3 What is the Pascal triangle-like recursion rule for the probabilities in the hypergeometric distribution?

We transform the hypergeometric probability times the normalization factor:

$$\binom{K}{N} P_{n,k,N,K} = \binom{k}{n}\binom{K-k}{N-n} = \left(\binom{k-1}{n} + \binom{k-1}{n-1} \right) \binom{K-k}{N-n}.$$

The two summands can be reinterpreted as

$$\binom{k-1}{n}\binom{K-k}{N-n} + \binom{k-1}{n-1}\binom{K-k}{N-n} =$$
$$\binom{K-1}{N} P_{n,k-1,N,K-1} + \binom{K-1}{N-1} P_{n-1,k-1,N-1,K-1}.$$

Finally, one observes that

$$\frac{\binom{K-1}{N}}{\binom{K}{N}} = \frac{\frac{(K-1)!}{N!(K-1-N)!}}{\frac{K!}{N!(K-N)!}} = \frac{K-N}{K} = 1-\bar{f}$$

and

$$\frac{\binom{K-1}{N-1}}{\binom{K}{N}} = \frac{\frac{(K-1)!}{(N-1)!(K-N)!}}{\frac{K!}{N!(K-N)!}} = \frac{N}{K} = \bar{f}.$$

Using this information we conclude that

$$P_{n,k,N,K} = (1-\bar{f})P_{n,k-1,N,K-1} + \bar{f}P_{n-1,k-1,N-1,K-1}.$$

4.4 What is the recursion rule for the probabilities in the bosonic Bernoulli distribution?

The bosonic Bernoulli probability is

$$B_{n,k} = \binom{k+n}{n} f^n (1+f)^{-n-k-1}.$$

Using Pascal's rule it is split as

$$B_{n,k} = \binom{k-1+n}{n} f^n (1+f)^{-n-k-1} + \binom{k+n-1}{n-1} f^n (1+f)^{-n-k-1}.$$

The terms are reinterpreted:

$$B_{n,k} = \frac{1}{1+f} B_{n,k-1} + \frac{f}{1+f} B_{n-1,k}.$$

Note that the last term is also on the k-th level, not on the $(k-1)$-th. It means that an actual element is the weighted sum of the left in the same row and the right one above. Since for bosons $n > k$ is possible, the probabilities fill infinite stripes instead of a triangle.

4.5 Derive the generating function $Z(\gamma)$ for the bosonic occupation probability, given in (4.50).

The generating function is defined by

$$Z(\gamma) = \langle e^{\gamma n} \rangle = \sum_{n=0}^{\infty} B_{n,k} e^{n\gamma},$$

with the distribution

$$B_{n,k} = \binom{k+n}{n} f^n (1+f)^{-n-k-1}.$$

This gives rise to the infinite sum representation

$$Z(\gamma) = (1+f)^{-k-1} \sum_{n=0}^{\infty} \frac{(k+n)!}{k!} \frac{x^n}{n!}$$

with $x = f e^{\gamma}/(1+f)$. We compare this sum with the Taylor expansion

$$(1-x)^{-k-1} = 1 + (k+1)x + \frac{1}{2}(k+1)(k+2)x^2 + \cdots$$

and realize that this is the same infinite sum. Therefore,

$$Z(\gamma) = (1+f)^{-k-1}\left(1-\frac{f}{1+f}\mathrm{e}^{\gamma}\right)^{-k-1} = (1+f-f\mathrm{e}^{\gamma})^{-k-1}.$$

4.6 Determine the double generating function,

$$Z_P(\gamma,\alpha,f) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} P_n(k;f)\mathrm{e}^{\gamma n}\,\mathrm{e}^{\alpha k}$$

for the negative binomial distribution $P_n(k;f)$. It plays the role of the partition function in the pressure ensemble.

We use the integral representation

$$P_n(k;f) = \int_0^{\infty} \frac{(xf)^n}{n!}\mathrm{e}^{-xf}\,\frac{x^k}{k!}\mathrm{e}^{-x}\,\mathrm{d}x$$

to obtain

$$Z_P(\gamma,\alpha,f) = \int_0^{\infty} \sum_{n=0}^{\infty} \frac{(xf\mathrm{e}^{\gamma})^n}{n!}\mathrm{e}^{-xf}\,\sum_{k=0}^{\infty}\frac{(x\mathrm{e}^{\alpha})^k}{k!}\mathrm{e}^{-x}\,\mathrm{d}x.$$

Carrying out the summations, both lead to a respective exponential:

$$Z_P(\gamma,\alpha,f) = \int_0^{\infty} \mathrm{e}^{xf(\mathrm{e}^{\gamma}-1)}\,\mathrm{e}^{x(\mathrm{e}^{\alpha}-1))}\,\mathrm{d}x.$$

It remains a simple integral of the exponential function. The final result is given by

$$Z_P(\gamma,\alpha,f) = \frac{1}{(1-\mathrm{e}^{\alpha})+f(1-\mathrm{e}^{\gamma})}.$$

4.7 Find a transformation formula between the fermionic and bosonic type Bernoulli distributions.

The transformation formula relies on the following property of the negative binomial:

$$\binom{k+n}{n} = (-1)^n\binom{-k-1}{n}.$$

Applying this to the bosonic Bernoulli distribution,

$$B_n(k,f) = \binom{k+n}{n} f^n(1+f)^{-k-n-1},$$

one gets

$$B_n(k,f) = \binom{-k-1}{n}(-f)^n(1-(-f))^{(-k-1)-n} = F_n(-k-1,-f).$$

Is is easy to see that the inverse transformation is given by

$$F_n(k,f) = B_n(-k-1,-f).$$

So the transformation $k \to -k-1$, $f \to -f$ interchanges the bosonic and fermionic Bernoulli distributions.

It is another question that what can be the physical meaning of a negative number of slots for a positive number of quanta, and a negative average occupancy rate. Inspecting the formulas (4.36) for fermions and (4.48) for bosons distributed in respective subsystems of slots, the above negative binomial formula leads to the transformation property:

$$B_n(k;N,K) = F_n(-k-1;N,-K-2), \qquad F_n(k;N,K) = B_n(-k-1;N,-K-2).$$

The transformations $k \to -k-1$ and $K \to -K-2$ transform these two distributions into each other. We note that the expressions $k(k+1)$ and $K(K+2)$ are *invariants* of this statistically "supersymmetric" transformation. The negative number of slots can be interpreted as an "over-occupation," as a waiting queue for slots.

4.8 Estimate the magnitude of energy fluctuations near to thermal equilibrium for the non-extensively modified black body radiation, described by the equation of state

$$S(E,V) = \frac{4}{3}\sigma^{1/4} V \left(\frac{L(E)}{V}\right)^{3/4}.$$

Assuming $L(E) = \frac{1}{a}\ln(1+aE)$ with a small parameter $a > 0$, how do these fluctuations behave with a?

Following the general recipe presented in Chap. 4 we obtain the first and second derivatives of the entropy function.

$$\frac{\partial S}{\partial V} = \frac{1}{3}\sigma^{1/4}\left(\frac{L(E)}{V}\right)^{3/4} = \frac{p}{T}$$

defines the pressure in equilibrium, while from

$$\frac{\partial S}{\partial E} = \sigma^{1/4}\left(\frac{L(E)}{V}\right)^{-1/4} L'(E) = \frac{L'(E)}{T}$$

we obtain the temperature, $T = (L(E)/\sigma V)^{1/4}$. It follows

$$L(E) = \sigma V T^4,$$

and

$$p = \frac{1}{3}\sigma T^4 = \frac{1}{3}\frac{L(E)}{V}.$$

In this approach, the role of energy density is played by $\varepsilon = L(E)/V$.

We use the above relations to simplify expressions occurring in the second derivatives. With respect to the volume we obtain

$$\frac{\partial^2 S}{\partial V^2} = -\frac{3}{4V}\frac{p}{T} = -\frac{L(E)}{4V^2 T}.$$

The mixed second derivative becomes

$$\frac{\partial^2 S}{\partial V \partial E} = \frac{1}{4V}\frac{\partial S}{\partial E} = \frac{L'(E)}{4VT},$$

and the second derivative with respect to the energy

$$\frac{\partial^2 S}{\partial E^2} = \frac{1}{T}\left(L''(E) - \frac{L'(E)L'(E)}{4L(E)}\right).$$

Utilizing all these second derivatives the coefficient "metric" tensor in the expansion of the entropy around its maximum becomes

$$g = \frac{1}{T}\begin{pmatrix} -L'' + L'L'/4L, & -L'/4V \\ -L'/4V, & L/4V^2 \end{pmatrix}$$

Its determinant is

$$\det g = -\frac{LL''}{4V^2 T^2}$$

and the inverse matrix is given by

$$g^{-1} = T\begin{pmatrix} -1/L'', & -VL'/LL'' \\ -VL'/LL'', & 4V^2/L - V^2 L'L'/L^2 L'' \end{pmatrix}$$

This gives rise to the following expectation values for squared fluctuations:

$$\left\langle \Delta E^2 \right\rangle = -\frac{T}{L''(E)},$$

$$\left\langle \Delta E \Delta V \right\rangle = -V\frac{T}{L''(E)}\frac{L'(E)}{L(E)},$$

$$\left\langle \Delta V^2 \right\rangle = 4\frac{TV^2}{L(E)} - V^2 T\frac{L'(E)L'(E)}{L(E)L''(E)}.$$

We note that energy density $\varepsilon = L(E)/V$ has the fluctuation

$$\Delta\varepsilon = \Delta\frac{L(E)}{V} = \frac{1}{V}\left(L'(E)\Delta E - \varepsilon\Delta V\right).$$

With the help of this relation we obtain

$$\left\langle \Delta\varepsilon^2 \right\rangle = \frac{4T}{V}\varepsilon.$$

The characteristic fluctuations in the energy density scale like the inverse square root of the total volume V at a given temperature.

For the special energy function given,

$$L(E) = \frac{1}{a}\ln(1+aE) = V\varepsilon$$

and we have

$$L'(E) = \frac{1}{1+aE}$$

and

$$L''(E) = -\frac{a}{(1+aE)^2}.$$

This gives rise to the following expectation values

$$\left\langle \Delta E^2 \right\rangle = \frac{T}{a}\mathrm{e}^{2a\varepsilon V},$$

$$\left\langle \Delta E \Delta V \right\rangle = \frac{T}{a\varepsilon}\mathrm{e}^{a\varepsilon V},$$

$$\left\langle \Delta V^2 \right\rangle = \frac{T}{a\varepsilon^2}\left(1+4a\varepsilon V\right).$$

Expressed in this form one realizes, that the squared energy fluctuations at a given temperature, and consequently at a given energy density ε, grow more rapidly than linear with the volume. This exponential growth is due to the use of the particular formula for $L(E)$. Finally, we note that for too large fluctuations the Gaussian approximation utilized here is no more sufficient, so the above results are only approximate.

Problems of Chap. 5

5.1 Derive the general result (5.17) by executing the integration (5.16) and substituting into the formula (5.15). What can one tell about the limits $D \to 0$ and $C \to 0$?

The first key step is to observe that the linear expression $K_1(p)$ contains the derivative of the second order form $K_2'(p) = 2(Cp - B)$:

$$K_1(p) = F - Gp = -\frac{G}{2C}K_2'(p) + F - \frac{B}{C}G.$$

we use the abbreviation $\alpha = BG/C - F$. The argument of the exponential in (5.15) becomes

$$L(p) = \int_0^p \mathrm{d}q \, \frac{K_1(q)}{K_2(q)} = -\alpha \int_0^p \frac{\mathrm{d}q}{K_2(q)} - \frac{G}{2C} \int_0^p \mathrm{d}q \, \frac{K_2'(q)}{K_2(q)}.$$

It is straightforward to evaluate the second integral, the first we denote by $I(p)$. By doing so we have

$$L(p) = -\alpha I(p) - \frac{G}{2C} \ln \frac{K_2(p)}{K_2(0)},$$

and using (5.15) the stationary detailed balance distribution

$$f(p) = f(0) \left(1 - \frac{2B}{D} p + \frac{C}{D} p^2\right)^{-1-G/2C} \mathrm{e}^{-\alpha I(p)}.$$

We are left with task to evaluate

$$I(p) = \int_0^p \frac{\mathrm{d}q}{K_2(q)}.$$

In order to achieve this, one considers the zeros of $K_2(p)$. From

$$K_2(p) = D - 2Bp + Cp^2 = c(p - p_+)(p - p_-)$$

follows

$$p_\pm = \frac{B \pm i\vartheta}{C}$$

with $\vartheta = \sqrt{DC - B^2}$. The integral $I(p)$ with these notations becomes

$$I(p) = \int_0^p \frac{C dq}{(Cq - p_+)(Cq - p_-)} = \frac{1}{2i\vartheta} \ln \frac{1 - p/p_+}{1 - p/p_-}.$$

This expression contains the logarithm of a ratio between a complex number and its complex conjugate. This way it is an angle whose tangent is the ratio of the imaginary to the real part. As an intermediate step we obtain

$$\frac{p}{p_\pm} = \frac{Cp}{B \pm i\vartheta} = \frac{Cp}{B^2 + \vartheta^2}(B \mp i\vartheta) = \frac{p}{D}(B \mp i\vartheta).$$

Using this we obtain

$$I(p) = \frac{1}{2i\vartheta} \ln \frac{D - Bp + i\vartheta p}{D - Bp - i\vartheta p} = \frac{2i\varphi}{2i\vartheta},$$

with

$$\tan\varphi = \frac{\vartheta p}{D - Bp}.$$

Our final result is hence

$$I(p) = \frac{1}{\vartheta}\,\mathrm{atn}\frac{\vartheta p}{D - Bp},$$

and the stationary distribution

$$f(p) = f(0)\left(1 - \frac{2B}{D}p + \frac{C}{D}p^2\right)^{-1-G/2C} \mathrm{e}^{-\frac{\alpha}{\vartheta}\mathrm{atn}\frac{\vartheta p}{D-Bp}}.$$

With no cross correlation $B = 0$, and the above result simplifies to

$$f(p) = f(0)\left(1 + \frac{C}{D}p^2\right)^{-1-G/2C} \mathrm{e}^{\frac{F}{D}\frac{\sqrt{D}}{\sqrt{C}}\mathrm{atn}(p\frac{\sqrt{C}}{\sqrt{D}})}.$$

With vanishing mean value of the additive noise (no driving force) one has $F = 0$ and the stationary distribution is a pure cut power-law:

$$f(p) = f(0)\left(1 + \frac{C}{D}p^2\right)^{-1-G/2C}.$$

The $C \to 0$ limit of this expression, upon using Euler's formula, is the classical Gaussian

$$f(p) = f(0)\mathrm{e}^{-\frac{G}{2D}\frac{p^2}{2m}}.$$

The $\vartheta = 0$ is the degenerate case. The two roots of the second order expression $K_2(p)$ coincide and the inverse tangent function can be approximated by its argument. The stationary distribution is given by

$$f(p) = f(0)\left(1 - \frac{p}{p_m}\right)^{-2-G/C} \mathrm{e}^{\frac{F-Gp_m}{Dp_m}\frac{p}{p_m-p}},$$

with $p_m = \sqrt{D/C}$. Only the values $p \le p_m$ and $F \le Gp_m$ are meaningful in this case. At $p = p_m$ the probability density $f(p)$ becomes zero.

5.2 Consider the expectation value of the Taylor expansion with a Gauss-distributed deviation. What could be the next term continuing the construction recipe for Fisher's entropy?

We use a normal Gauss-distributed variable, z:

$$P(z) = \frac{1}{\kappa\sqrt{2\pi}}\mathrm{e}^{-z^2/2\kappa^2}.$$

The odd powers of z have vanishing expectation value, $\langle z^{2k+1}\rangle = 0$, while the even powers amount to $\langle z^{2k}\rangle = (2k-1)!!\kappa^{2k}$. Here, the double exclamation mark denotes the product of odd numbers only. The expectation value of the exponential function, represented by its infinite Taylor series, contains only contributions with even power:

$$\left\langle e^{\beta z} \right\rangle = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \langle z^n \rangle = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!} \beta^{2k} \kappa^{2k}.$$

Considering now that the factorial product of numbers up to $2k$ contains all the odd numbers up to $(2k-1)$ and k times the factor of 2 besides the numbers from 1 to k, one realizes that $(2k)! = 2^k (2k-1)!!k!$. Substituting this relation into the above one arrives at the short and elegant result,

$$\left\langle e^{\beta z} \right\rangle = e^{\beta^2 \kappa^2 / 2}.$$

As a last step we apply this result to the Taylor expansion formula and obtain

$$\langle f(a+z) \rangle = \left\langle e^{z \frac{\partial}{\partial a}} f(a) \right\rangle = e^{\frac{\kappa^2}{2} \frac{\partial^2}{\partial a^2}} f(a).$$

The leading terms of this Gauss-averaged Taylor expansion are

$$\langle f(a+z) \rangle \approx f(a) + \frac{\kappa^2}{2} \frac{\partial^2}{\partial a^2} f + \frac{\kappa^4}{8} \frac{\partial^4}{\partial a^4} f.$$

Applying this result to the construction of Fisher entropy from the BGS formula, the fourth order term,

$$S_4 = \frac{\kappa^4}{8} \int f \, \nabla^4 (-\ln f) \, d\Gamma,$$

becomes quite involved when expanded in terms of ∇f. Therefore, it has no high value for a practical use.

5.3 What is the asymptotic rule to Einstein's velocity addition rule:

$$u \oplus v = \frac{u+v}{1+uv/c^2}?$$

The asymptotic rule to Einstein's velocity addition formula is itself. One way to prove this is to check associativity:

$$h(h(u,v),w) = \frac{h(u,v)+w}{1+wh(u,v)/c^2} = \frac{\frac{u+v}{1+uv/c^2} + w}{1 + w \frac{u+v}{c^2+uv}}.$$

After multiplying numerator and denominator by $(c^2 + uv)$ this fraction becomes

$$h(h(u,v),w) = \frac{c^2(u+v+w) + uvw}{c^2 + uv + wu + wv}.$$

In this form, u, v and w are arranged symmetrically, so this expression could have been derived by starting with any permutation of them. This means that

$$h(h(u,v),w) = u \oplus v \oplus w = h(u,h(v,w)).$$

Another way is to utilize the scaling equation for asymptotic rules: We have $h_2'(u,0) = 1 - u^2/c^2$ and therefore the formal logarithm is given by

$$L(u) = \int_0^u \frac{\mathrm{d}z}{1 - z^2/c^2} = c\,\mathrm{Arth}\frac{u}{c}.$$

This quantity is the rapidity, additive by Lorentz-transformations. The asymptotic rule is given by

$$\varphi(u,v) = L^{-1}\left(L(u) + L(v)\right) = \frac{u+v}{1+uv/c^2},$$

which is the original Einstein-rule.

5.4 Is the following rule associative?

$$x \oplus y = x + y + \frac{a}{\frac{1}{x} + \frac{1}{y}}$$

The non-additive part can easily be casted into the form

$$\frac{a}{\frac{1}{x} + \frac{1}{y}} = a\frac{xy}{x+y}.$$

Since for the rule $h(x,y) = x + y + axy/(x+y)$ the derivative $h_2'(x,0) = 1 + a$ is constant, the asymptotic rule is the simple addition, $\varphi(x,y) = x + y$. It differs from the original rule, so that one is not associative.

5.5 Obtain the formal particle–hole correspondence for the deformed Fermi and Bose distributions:

$$f_a(x) = \frac{1}{e_a(x)+1}, \qquad \text{and} \qquad g_a(x) = \frac{1}{e_a(x)-1}.$$

What replaces the known results, $f_0(-x) = 1 - f_0(x)$ and $-g_0(-x) = 1 + g_0(x)$ for general values of the parameter a?

Since $1/e_a(x) = e_{-a}(-x)$, one obtains

$$1 - f_a(x) = \frac{e_a(x)}{e_a(x)+1} = \frac{1}{e_{-a}(-x)+1} = f_{-a}(-x)$$

for the extension of the Fermi distribution and

$$1+g_a(x)=\frac{e_a(x)}{e_a(x)-1}=\frac{1}{1-e_{-a}(-x)}=-g_{-a}(-x)$$

for that of the Bose distribution. The known results for $a=0$ generalize so that besides the $x\longrightarrow -x$ replacement also $a\longrightarrow -a$ is required.

5.6 What is the canonical energy distribution with an additive energy and the following entropy composition rule?

$$S_{12}=S_1\sqrt{1+a^2S_2^2}+S_2\sqrt{1+a^2S_1^2}.$$

The composition rule $h(x,y)=x\sqrt{1+a^2y^2}+y\sqrt{1+a^2x^2}$ is of Kaniadakis type, the corresponding formal logarithm is given by $L_a(x)=\frac{1}{a}\mathrm{Arsh}(ax)$. The entropy is therefore given as

$$S=\sum_i w_i\frac{1}{a}\sinh\left(a\ln\frac{1}{w_i}\right)=\frac{1}{2a}\sum_i\left(w_i^{1-a}-w_i^{1+a}\right).$$

The canonical maximization problem,

$$S-\alpha\sum_i w_i-\beta\sum_i w_iE_i=\max.$$

leads to the following equation for the probability w_i of having energy E_i:

$$\frac{1}{2a}\left[(1-a)w_i^{-a}-(1+a)w_i^{a}\right]=\alpha+\beta E_i.$$

This is a second order algebraic equation for w_i^a, which should be a quantity less than one for $a>0$. Denoting $\alpha+\beta E_i$ by X, this solution is given by

$$w_i=\left(\frac{\sqrt{1-a^2+a^2X^2}-aX}{1+a}\right)^{1/a}.$$

For large energies it is power-law:

$$\lim_{X\to\infty}w_i=\left(\frac{1-a}{2aX}\right)^{1/a}=KX^{-1/a}.$$

5.7 Obtain the canonical energy distribution for the following class of pairwise energy composition rule:

$$E_1\oplus E_2=E_1+E_2+\frac{a}{2}(E_1+E_2)^2.$$

The energy composition rule

$$h(x,y) = x + y + \frac{a}{2}(x+y)^2$$

leads to $h_2'(x,0) = 1 + ax$. This way the asymptotic rule is the Tsallis rule, $\varphi(x,y) = x + y + axy$, and the canonical entropy maximization principle becomes

$$\sum_i w_i \ln \frac{1}{w_i} - \alpha \sum_i w_i - \beta \sum_i w_i \frac{1}{a} \ln(1 + aE_i) = \max.$$

Its derivative with respect to w_i leads to

$$-\ln w_i - 1 - \alpha - \frac{\beta}{a} \ln(1 + aE_i) = 0,$$

resulting in a power-law tailed energy distribution

$$w_i = K\,(1 + aE_i)^{-\beta/a}$$

with an appropriate constant factor K.

5.8 Verify that the Rényi entropy is additive for factorizing probabilities.

The Rényi entropy,

$$S = \frac{1}{1-q} \ln \sum_i w_i^q,$$

applied to factorizing (uncorrelated) joint probabilities, $p_{ij} = w_i v_j$ gives

$$S_{12} = \frac{1}{1-q} \ln \sum_i \sum_j p_{ij}^q = \frac{1}{1-q} \ln \left(\sum_i w_i^q \right) \left(\sum_j v_j^q \right),$$

which is additive due to the same property of the natural logarithm, ln.

5.9 Construct a composition rule which does not have a thermodynamical limit.

It is sufficient to consider a composition rule with divergent $h_2'(x,0)$, e.g. $h(x,y) = x + y + a\ln(xy)$ for this purpose. In such a case the formal logarithm degenerates to zero and no asymptotic rule can be obtained by the standard procedure. However, using a small y value instead of zero, the calculation still can be done, and it leads to the addition as asymptotic rule in this case. Another possibility would be to consider a rule which cannot be differentiated with respect to y, e.g. $h(x,y) = \text{sign}(x-y)$.

5.10 Compare the Rényi and Tsallis entropy formulas for binary events, $p_1 = p$, $p_2 = 1-p$. Discuss the location and value of maximum and the convexity (Fig. 2).

The Tsallis entropy with the parameter $a = 1-q$ is given as

$$S_{\mathrm{T}} = \frac{1}{a}\left(p^{1-a} + (1-p)^{1-a} - 1\right).$$

Here, we use $k_B = 1$ units. The first derivative with respect to p,

$$\frac{\partial S_T}{\partial p} = \frac{1-a}{a}\left(p^{-a} - (1-p)^{-a}\right),$$

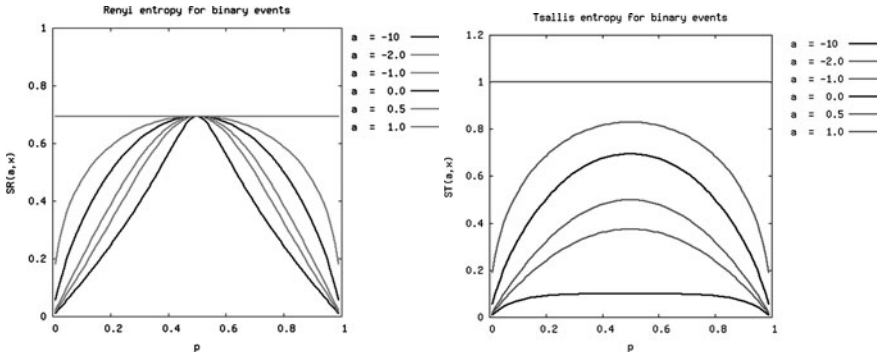

Fig. 2 The Rényi (*left*) and Tsallis (*right*) entropy formula applied to binary events. The Rényi entropy is additive and its maximal value is always 1 bit ($k_B \ln 2$). The Tsallis entropy is convex for all sensible values of the entanglement-deformity parameter, $a < 1$, while the Rényi entropy shows this property only for $a \in (-1, 1)$. (The *curves* from the top to the bottom belong to $a = 1$, 0.5, 0, -1, -2 and -10)

vanishes at $p = 1 - p = 1/2$. The extremal value of the Tsallis entropy depends on the parameter a:

$$S_T^{\text{extr}} = \frac{1}{a}\left(2^{a-1} + 2^{a-1} - 1\right) = \frac{2^a - 1}{a}.$$

Its second derivative,

$$\frac{\partial^2 S_T}{\partial p^2} = (a-1)\left(p^{-a-1} + (1-p)^{-a-1}\right),$$

is negative for every $p \in (0,1)$ provided $a < 1$ ($q > 0$).

The Rényi entropy can be obtained as the formal logarithm of the Tsallis one,

$$S_R = \frac{1}{a}\ln(1 + aS_T).$$

The first derivative,

$$\frac{\partial S_R}{\partial p} = \frac{1}{1 + aS_T}\frac{\partial S_T}{\partial p}$$

vanishes at the same probability, $p = 1/2$, where S_T did. Its extremal value is given as

$$S_R^{\text{ext}} = \ln 2$$

independently of the parameter a. The second derivative has two contributions

$$\frac{\partial^2 S_R}{\partial p^2} = -\frac{a}{(1+aS_T)^2}\left(\frac{\partial S_T}{\partial p}\right)^2 + \frac{1}{1+aS_T}\frac{\partial^2 S_T}{\partial p^2}.$$

After some algebra it can be casted into the following form

$$\frac{\partial^2 S_R}{\partial p^2} = \frac{(a-1)p^{-a}(1-p)^{-a}}{(1+aS_T)^2}$$
$$\times \quad \left[\frac{1}{a}\left(\left(\frac{p}{1-p}\right)^a + \left(\frac{1-p}{p}\right)^a - 2\right) + \left(\frac{p}{1-p} + \frac{1-p}{p} + 2\right)\right].$$

For further simplification it is useful to introduce a new variable, by $p = re^t$ and $1-p = r\mathrm{e}^{-t}$. Due to the normalization $2r\cosh(t) = 1$. On the other hand

$$\left(\frac{p}{1-p}\right)^n + \left(\frac{1-p}{p}\right)^n \pm 2 = \left(\mathrm{e}^{nt} \pm \mathrm{e}^{-nt}\right)^2.$$

This leads to the final result

$$\frac{\partial^2 S_R}{\partial p^2} = \frac{4(a-1)\cosh^2(t)}{\cosh^2((a-1)t)}\left[\frac{\sinh^2(at)}{a} + \cosh^2(t)\right]$$

with

$$t = \frac{1}{2}\ln\frac{p}{1-p}.$$

For $a > 1$ this expression is positive, the entropy would have a minimum, not a maximum. This property is shared by the Tsallis entropy. For having a maximum $a < 1$ ($q > 0$) is required. On the other hand for $a < -1$ ($q > 2$) there are probabilities for which the Rényi entropy may not be convex, $S_R'' > 0$ occurs. This depends on the sign of the expression in the square brackets above. Setting $a = -n$ the inflection points, $t = \pm t_i$, satisfy

$$\cosh(t_i) = \frac{1}{\sqrt{n}}\sinh(nt_i).$$

This is a transcendent equation with no analytic solution, but numerically it is easy to handle. For large n (large positive $q = n+1$) the inflection points shrink towards the equipartition point $t = 0$, as $t \sim \pm\ln n/2(n-1)$.

5.11 Calculate the pressure for a massless ideal Boltzmann gas with canonical Tsallis–Pareto energy distribution with no chemical potential.

The pressure,

$$p = \frac{1}{\beta}\frac{\partial}{\partial V}\ln Z,$$

for an ideal massless Bose gas is given by

$$p = -\frac{\gamma}{2\pi^2} \int_0^\infty E^2 \mathrm{d}E \, \ln\left(1 - e_{-a}(-\beta E)\right)$$

with γ being a degeneracy factor. After considering $E^2 = (E^3/3)'$ and partial integration one obtains

$$p = \frac{\gamma}{6\pi^2} \int_0^\infty E^3 \mathrm{d}E \, \frac{1}{1+a\beta E} \frac{1}{e_a(\beta E) - 1}.$$

Now, denoting βE by x and $1/\beta$ by T, we expand the generalized Bose factor and use the fact that $e_a(x) = (1+ax)^{1/a}$. In this case

$$n_a(x) = \frac{1}{(1+ax)^{1/a} - 1} = \sum_{n=1}^{\infty} (1+ax)^{-n/a}$$

and we get

$$p = \frac{\gamma}{6\pi^2} T^4 \sum_{n=1}^{\infty} \int_0^\infty x^3 \mathrm{d}x \, (1+ax)^{-(1+n/a)}.$$

The integrals can be evaluated by the substitution $t = 1 + ax$ leading to

$$I_\nu = \int_0^\infty \mathrm{d}x x^3 (1+ax)^{-\nu} = a^{-4} \int_1^\infty \mathrm{d}t \, (t-1)^3 t^{-\nu}.$$

It is easy to show that such an integral results in

$$I_\nu = \frac{1}{a^4} \left(\frac{1}{\nu-4} - \frac{3}{\nu-3} + \frac{3}{\nu-2} - \frac{1}{\nu-1} \right).$$

Applying this result for $\nu = 1 + n/a$ finally we arrive at the pressure formula

$$p = \frac{\gamma}{\pi^2} T^4 \sum_{n=1}^{\infty} \frac{1}{n(n-a)(n-2a)(n-3a)}.$$

For $a = 0$ the well-known Riemann zeta coefficient $\pi^4/90$ emerges, for a general value of the deformity parameter a the infinite sum can be expressed by incomplete Euler-gamma functions. For positive values of a (more high-energy particles than the pure exponential formula would yield) the pressure is enhanced.

Problems of Chap. 6

6.1 Express temperature transformation in rapidity variables.

By the definitions $w_i = \tanh \alpha_i$ and $v_i = \tanh \beta_i$ the equilibrium conditions read

$$\alpha_1 + \beta_1 = \alpha_2 + \beta_2$$

and

$$T_1 \cosh \alpha_1 = T_2 \cosh \alpha_2.$$

The relative rapidity of the moving bodies is $\zeta = \beta_2 - \beta_1 = \alpha_1 - \alpha_2$ so the temperature shown by the thermometer becomes

$$T_1 = T_2 \frac{\cosh \alpha_2}{\cosh(\alpha_2 + \zeta)}.$$

6.2 Can it be that $T_1/T_2 \leq |w_1/w_2|$? What does it mean for the measured value T_1?

Following the results of the previous problem (6.1) the required inequality is given as

$$\frac{T_1}{T_2} = \frac{\cosh \alpha_2}{\cosh \alpha_1} \leq \left| \frac{\tanh \alpha_1}{\tanh \alpha_2} \right| = \frac{|\sinh \alpha_1|}{\cosh \alpha_1} \frac{\cosh \alpha_2}{|\sinh \alpha_2|}.$$

This is fulfilled if $|\alpha_1| \geq |\alpha_2|$ and it has the consequence that $T_1 \leq T_2$.

6.3 At what special values of the relative velocity can it be $T_1 = 2T_2$?

Using the transformation formula we have

$$\frac{T_1}{T_2} = \frac{\sqrt{1-v^2}}{1+w_2 v} = 2.$$

Its square leads to the following second order equation for the relative velocity, v:

$$(1+4w_2^2)v^2 + 8w_2 v + 3 = 0.$$

The solutions are

$$v = -\frac{1}{1+4w_2^2}\left(-4w_2 \pm \sqrt{4w_2^2 - 3}\right).$$

Real solutions are possible as long as $w_2^2 \geq 3/4$. The extreme values of v are achieved by $w_2 = \pm 1$. The possible relative velocities leading to the above temperature ratio are $3/5 \leq v \leq 1$ and $-1 \leq v \leq -3/5$. Relative velocities with a magnitude smaller than $3/5 = 0.6$ cannot lead to such a temperature ratio in equilibrium.

Problems of Chap. 7

7.1 Prove that R^2 is flat in polar coordinates. The arc length squared is given as $ds^2 = dr^2 + r^2 d\vartheta^2$. Note that not all Christoffel symbol elements are zero, but the Riemann tensor components.

From the metric formula, $ds^2 = dr^2 + r^2 d\vartheta^2$ the following Lorentzian one-forms can be recognized: $\omega^1 = dr$ and $\omega^2 = r d\vartheta$. Their Cartan derivatives are $d\omega^1 = 0$ and $d\omega^2 = dr \wedge d\vartheta = \frac{1}{r}\omega^1 \wedge \omega^2$. This means that some of the composition coefficients are nonzero: $c_{12}{}^2 = -1/r$ and $c_{21}{}^2 = 1/r$. Since $\eta_{ij} = \delta_{ij}$ is the diagonal unit matrix, we also have $c_{122} = -1/r$ and $c_{212} = 1/r$. The antisymmetric two-index collection becomes

$$\Omega_{12} = \frac{1}{2}\left(c_{12a} + c_{1a2} - c_{2a1}\right)\omega^a = -\frac{1}{r}\omega^2 = -d\vartheta.$$

This way $\Omega^1{}_2 = -d\vartheta$ and the curvature two-from vanishes:

$$R^1{}_2 = d\Omega^1{}_2 = 0.$$

7.2 Prove that S^2, the surface of the unit sphere, has a constant curvature. The metric tensor is given by $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$.

From $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$ the basis one-forms are: $\omega^1 = d\theta$ and $\omega^2 = \sin\theta d\phi$. Their derivatives, $d\omega^1 = 0$ and

$$d\omega^2 = \cos\theta d\theta \wedge d\phi = \text{ctg}\theta\ \omega^1 \wedge \omega^2$$

reveal a nonzero entry to the Christoffel symbols. We have $c_{122} = -\text{ctg}\theta$ and $c_{212} = \text{ctg}\theta$, all the other coefficients are zero. The antisymmetric one-form matrix is given as

$$\Omega_{12} = \frac{1}{2}\left(c_{12a} + c_{1a2} - c_{2a1}\right)\omega^a = c_{122}\omega^2 = -\text{ctg}\theta \sin\theta d\phi.$$

It simplifies to $\Omega^1{}_2 = \Omega_{12} = -\cos\theta d\phi$. The curvature two-form becomes

$$R^1{}_2 = d\Omega^1{}_2 = \sin\theta d\theta \wedge d\phi = \omega^1 \wedge \omega^2.$$

This way some entries of the Riemann tensor are nonzero: $R^1{}_{212} = 1$, $R^1{}_{221} = -1$, $R^2{}_{112} = -1$ and $R^2{}_{121} = 1$. The Ricci tensor is diagonal, $\mathscr{R}_{ij} = \delta_{ij}$ (it is always proportional to the unity matrix for two-dimensional surfaces and hence the Einstein tensor vanishes in such cases). The scalar curvature is $\mathscr{R} = 2$ belonging to unit radii of the main circles in the general formula $\mathscr{R} = 2/r_1 r_2$.

7.4 Prove that (7.184) leads to a linear velocity profile.

Combining (7.184) with the relativistic formulas for the energy and angular momentum carried by the spinning string we have the following variational problem:

$$\sigma \frac{\delta}{\delta v} \int_{-\ell/2}^{+\ell/2} \frac{1-\omega s v(s)}{\sqrt{1-v^2(s)}} \, \mathrm{d}s = 0.$$

This is an $L(s, v(s))$ type Lagrangian problem, so the Euler-Lagrange equations simply end up with

$$\frac{\partial L}{\partial v} = -\frac{\omega s}{\sqrt{1-v^2}} + \frac{1-\omega s v}{(1-v^2)^{3/2}} v = 0.$$

This leads quickly to the result $v = \omega s$.

7.3 Obtain the entropy for an extremal Reissner–Nordstrøm black hole with cosmological constant, $\lambda = -3/a^2 L_\mathrm{P}^2$, according to the formula (7.134).

Observing that $M_\mathrm{P} L_\mathrm{P} = \hbar/c$ is a purely quantum mechanical, while $L_\mathrm{P}/M_\mathrm{P} = G/c^2$ is a purely gravitational combination of the Planck scales, the formula (7.134) can be written entirely in terms of these scales:

$$\frac{S}{k_\mathrm{B}} = 4\pi \int\int \delta\left(f(r;M)\right) \frac{\mathrm{d}r}{L_\mathrm{P}} \frac{\mathrm{d}M}{M_\mathrm{P}}.$$

From now on everything is understood in Planck scale and Boltzmann units. The radial metric factor for an extremal RN black hole with AdS term is given by

$$f(r;M) = \left(1 - \frac{M}{r}\right)^2 - \frac{r^2}{a^2}.$$

The horizons are determined by the solution of the fourth order algebraic equation $f(r;M) = 0$ leading to

$$M_0(r) = r(1 \mp r/a).$$

The derivative of f with respect to M gives

$$\frac{\partial f}{\partial M} = -\frac{2}{r}\left(1 - \frac{M}{r}\right),$$

which at the horizon becomes $\mp 2/a$. The entropy is therefore obtained as

$$S = 4\pi \int \frac{\mathrm{d}r}{\left|\frac{\partial f}{\partial M}\right|} = 2\pi r a.$$

This is *not* the (one fourth of the) horizon *area*, but its *perimeter* multiplied by the AdS scale a.

7.5 Find the coordinate transformation to embed the metric (7.219) as a five-dimensional hyperboloid surface in six dimensions.

Using $c = 1$ units the goal is to induce the 5-dimensional metric

$$ds_5^2 = \frac{r^2}{R^2} ds_4^2 + \frac{R^2}{r^2} dr^2$$

with

$$ds_4^2 = -dt^2 + dx^2 + dy^2 + dz^4$$

being flat Minkowskian. The embedding space is flat

$$ds_6^2 = -dt_1^2 - dt_2^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$

We use the variable $s_4^2 = -t^2 + x^2 + y^2 + z^2$ and the light cone variables $x_1 - t_1 = r$, $x_1 + t_1 = u$. By using the scaling factor r/R we set $t_2 = tr/R, x_2 = xr/R, x_3 = yr/R$ and $x_4 = zr/R$. This way we have two sub-hyperboles,

$$-t_2^2 + x_2^2 + x_3^2 + x_4^2 = (rs_4/R)^2$$

and

$$-t_1^2 + x_1^2 = ru.$$

The equation for the hyperboloid in the 6-dimensional flat spacetime becomes

$$s_6^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 - t_1^2 - t_2^2 \; = \; ru + \frac{r^2}{R^2} s_4^2.$$

By the same assumptions the subspace metrics are given by

$$-dt_2^2 + dx_2^2 + dx_3^2 + dx_4^2 = [d(s_4 r/R)]^2$$

and

$$-dt_1^2 + dx_1^2 = dr du.$$

Noting the scaled four-distance by $w = s_4 r/R$ we have

$$ds_6^2 = dr du + dw^2$$

which we restrict onto the hyperboloid defined by

$$s_6^2 = ru + w^2.$$

From this we express

$$u = \frac{1}{r} \left(s_6^2 - w^2 \right)$$

yielding the differential

$$\mathrm{d}u = -\frac{\mathrm{d}r}{r^2}\left(s_6^2 - w^2\right) - \frac{2w}{r}\mathrm{d}w.$$

This way the 6-dimensional metric restricted to the hyperboloid becomes

$$\mathrm{d}s_6^2 = \mathrm{d}w^2 - \left(s_6^2 - w^2\right)\frac{\mathrm{d}r^2}{r^2} - \frac{2w}{r}\mathrm{d}w\mathrm{d}r.$$

On the other hand $\mathrm{d}s_5^2$ can also be expressed in terms of w and r. One gets

$$\mathrm{d}s_5^2 = \left(rd\frac{w}{r}\right)^2 + \frac{R^2}{r^2}\mathrm{d}r^2.$$

This expression equals to

$$\mathrm{d}s_5^2 = \left(\mathrm{d}w - \frac{w}{r}\mathrm{d}r\right)^2 + \frac{R^2}{r^2}\mathrm{d}r^2 = \mathrm{d}w^2 - \frac{2w}{r}\mathrm{d}w\mathrm{d}r + \frac{w^2+R^2}{r^2}\mathrm{d}r^2.$$

Clearly, with the choice $s_6^2 = -R^2$ the two metrics are equivalent. In conclusion, the embedded hyperboloid interpretation is equivalent to using scaled 4-dimensional spacetime coordinates and taking r as a light cone coordinate in a $1+1$-dimensional additive spacetime.

7.6 Prove that the volume of the 5-dimensional spherical hypersurface is $\Omega_5 = \pi^3$.

The volume of S_n is the surface of a sphere with unit radius embedded in $n+1$ dimensions. It can be written as a product of angular integrals, each factor containing a higher power of $\sin\vartheta_i$ than the previous one:

$$\Omega_{n+1} = I_n \Omega_n$$

with

$$I_n = \int_0^\pi \sin^{n-1}\vartheta\,\mathrm{d}\vartheta.$$

The starting points are given by $I_1 = \pi$ and $I_2 = 2$, the rest can be obtained by a recursion formula. Such a formula is easily derived by noting that $(\sin\vartheta\cos\vartheta)' = 1 - 2\sin^2\vartheta$. In this case

$$I_{n-1} - 2I_{n+1} = \int_0^\pi (\sin\vartheta\cos\vartheta)'\,\sin^{n-2}\vartheta\,\mathrm{d}\vartheta.$$

After partial integration it becomes

$$I_{n-1} - 2I_{n+1} = -(n-2)\int_0^\pi \sin\vartheta\cos\vartheta\,\sin^{n-3}\vartheta\,\cos\vartheta\,\mathrm{d}\vartheta,$$

which upon using $\cos^2\vartheta = 1 - \sin^2\vartheta$ can be casted into the simple result

$$I_{n-1} - 2I_{n+1} = (n-2)\,(I_{n+1} - I_{n-1})\,.$$

The resolution of this linear equation is the simple recursion formula

$$I_{n+1} \;=\; \left(1 - \frac{1}{n}\right) I_{n-1}.$$

With the starting points we have $I_1 = \pi, I_2 = 2, I_3 = \pi/2, I_4 = 4/3, I_5 = 3\pi/8$ and therefore $\Omega_1 = 2\pi, \Omega_2 = 4\pi, \Omega_3 = 2\pi^2, \Omega_4 = 8\pi^2/3$ and $\Omega_5 = \pi^3$.

7.7 Obtain the horizon-entropy for a Reissner–Nordstrøm black hole with fixed charge to mass ratio, $\tilde{Q} = \mu\tilde{M}$ in Planck scale units, in the presence of a de Sitter-type cosmological constant term with $\lambda = -3/a^2$.

The radial function describing the horizon in Planck units is given by

$$f(r) = 1 - \frac{2M}{r} + \frac{\mu^2 M^2}{r^2} - \frac{r^2}{a^2} = 0.$$

Resolving this second order expression for M, one easily obtains

$$M_{\pm}(r) = \frac{r}{\mu^2}\left(1 \pm \sqrt{1 - \mu^2 + \frac{\mu^2 r^2}{a^2}}\right).$$

The derivative of f with respect to M is linear in M:

$$\frac{\partial f}{\partial M} = -\frac{2}{r} + 2\frac{\mu^2}{r^2}M,$$

which replacing $M_{\pm}(r)$ from the horizon condition becomes at the horizon

$$\left.\frac{\partial f}{\partial M}\right|_{M=M_{\pm}(r)} = \pm\frac{2}{r}\sqrt{1 - \mu^2 + \frac{\mu^2 r^2}{a^2}}.$$

The horizon-entropy in Planck scale units is given by the following integral

$$S = 2\pi \int \frac{r\,\mathrm{d}r}{\sqrt{1 - \mu^2 + \frac{\mu^2 r^2}{a^2}}}.$$

This integral can be evaluated analytically with the result:

$$S = 2\pi\left(\frac{a^2}{\mu^2}\sqrt{1 - \mu^2 + \frac{\mu^2 r^2}{a^2}} - K(a,\mu)\right).$$

The integration constant, $K(a,\mu)$ can be obtained from the requirement that at zero horizon area or zero horizon radius, $r = 0$ the entropy should be zero. This way we finally obtain:

$$S = 2\pi \frac{a^2}{\mu^2} \left(\sqrt{1-\mu^2+\frac{\mu^2 r^2}{a^2}} - \sqrt{1-\mu^2} \right).$$

For $\mu/a \to 0$ one gets back

$$\lim_{\mu/a\to 0} S = \frac{\pi r^2}{\sqrt{1-\mu^2}} \geq \frac{1}{4}A,$$

and for an extremal black hole with $\mu = 1$

$$S(1,a) = 2\pi a r.$$

Both particular cases have been discussed in Chap. 7.

Problems of Chap. 8

8.1 Obtain the parity of the canonical spectral function by expressing $\rho_{AB}(-\omega)$.

Using the definition with $-\omega$ one has

$$\rho_{AB}(-\omega) = \frac{1}{Z}\sum_{a,b} \langle a|A(0)|b\rangle \langle b|B(0)|a\rangle \, 2\pi\delta(E_a - E_b + \omega) \left[\mathrm{e}^{-\beta E_a} \mp \mathrm{e}^{-\beta E_b} \right].$$

In this formula we exchange the summation labels a and b and arrive at

$$\rho_{AB}(-\omega) = \frac{1}{Z}\sum_{a,b} \langle b|A(0)|a\rangle \langle a|B(0)|b\rangle \, 2\pi\delta(E_b - E_a + \omega) \left[\mathrm{e}^{-\beta E_b} \mp \mathrm{e}^{-\beta E_a} \right].$$

Now we use that the Dirac-delta is an even functional of its argument and factorize the ∓ 1 out of the square bracket term. We also interchange the order of the transition matrix elements $\langle b|A(0)|a\rangle$ and $\langle a|B(0)|a\rangle$. By doing so we get

$$\rho_{AB}(-\omega) = \mp\frac{1}{Z}\sum_{a,b} \langle a|B(0)|b\rangle \langle b|A(0)|a\rangle \, 2\pi\delta(E_a - E_b - \omega) \left[\mathrm{e}^{-\beta E_a} \mp \mathrm{e}^{-\beta E_b} \right].$$

In this form, the final result is easy to recognize:

$$\rho_{AB}(-\omega) = \mp\rho_{BA}(\omega).$$

8.2 Calculate the following sum rule for the canonical spectral function:

$$R = \int \frac{\mathrm{d}\omega}{2\pi} \rho_{AB}(\omega).$$

The ω-integration "consumes" the Dirac-delta and one arrives at

$$R = \frac{1}{Z} \sum_{a,b} \langle a|A(0)|b \rangle \langle b|B(0)|a \rangle \left[\mathrm{e}^{-\beta E_a} \mp \mathrm{e}^{-\beta E_b} \right].$$

This expression is equal to the canonical expectation value

$$R = \langle A(0)B(0) \mp B(0)A(0) \rangle.$$

In particular, for the creation and annihilation operator this integral is $R = 1$.

8.3 Obtain an operator formula for the Wigner transform of a convolution.

We use the notation $x_1 = x + \frac{\xi}{2}$, $x_2 = x - \frac{\xi}{2}$ for the coordinates. Let H be the convolution of F and G, then its Wigner transform is given by

$$\bar{H}(x,p) = \int \mathrm{d}\xi \mathrm{e}^{\mathrm{i}p\xi} H\left(x + \frac{\xi}{2}, x - \frac{\xi}{2}\right) = \int \mathrm{d}\xi \mathrm{e}^{\mathrm{i}p\xi} \int \mathrm{d}z F\left(x + \frac{\xi}{2}, z\right) G\left(z, x - \frac{\xi}{2}\right).$$

We introduce the quantity ε by using $z = x + \varepsilon = x - \frac{\xi}{2} + (\varepsilon + \frac{\xi}{2}) = x + \frac{\xi}{2} + (\varepsilon - \frac{\xi}{2})$. This way we obtain

$$\bar{H}(x,p) = \int \mathrm{d}z \mathrm{d}\xi \mathrm{e}^{\mathrm{i}p\xi} F\left(x + \frac{\xi}{2}, x - \frac{\xi}{2} + \left(\varepsilon + \frac{\xi}{2}\right)\right) G\left(x + \frac{\xi}{2} + \left(\varepsilon - \frac{\xi}{2}\right), x - \frac{\xi}{2}\right).$$

This formula can be rewritten by using the exponential form of the Taylor expansion for a shifted argument as

$$\bar{H}(x,p) = \int \mathrm{d}z \mathrm{d}\xi \mathrm{e}^{\mathrm{i}p\xi} \left[\mathrm{e}^{(\varepsilon + \frac{\xi}{2}) \frac{\partial}{\partial x_2}} F(x_1, x_2) \right] \left[\mathrm{e}^{(\varepsilon - \frac{\xi}{2}) \frac{\partial}{\partial x_1'}} G(x_1', x_2') \right]$$

taken at $x' = x$ at the end. As the next step we express the functions $F(x_1, x_2)$ and $G(x_1', x_2')$ from their respective Wigner transforms:

$$\bar{H}(x,p) = \int \left[\mathrm{e}^{(\varepsilon + \frac{\xi}{2}) \frac{\partial}{\partial x_2}} \mathrm{e}^{-\mathrm{i}q(x_1 - x_2)} \bar{F}\left(\frac{x_1 + x_2}{2}, q\right) \right]$$
$$\times \left[\mathrm{e}^{(\varepsilon - \frac{\xi}{2}) \frac{\partial}{\partial x_1'}} \mathrm{e}^{-\mathrm{i}r(x_1' - x_2')} \bar{G}\left(\frac{x_1' + x_2'}{2}, r\right) \right]$$

with the multiple integral

$$\int \cdots = \int \mathrm{d}z\mathrm{d}\xi \, \frac{\mathrm{d}q}{2\pi} \frac{\mathrm{d}r}{2\pi} \, \mathrm{e}^{\mathrm{i}p\xi} \ldots$$

Replacing x_1 and x_2 by their respective definitions it is equivalently written in the form

$$\bar{H}(x,p) = \int \left[\mathrm{e}^{(\varepsilon+\frac{\xi}{2})(iq+\frac{1}{2}\frac{\partial}{\partial x})} \mathrm{e}^{-\mathrm{i}q\xi} \bar{F}(x,q) \right] \left[\mathrm{e}^{(\varepsilon-\frac{\xi'}{2})(-ir+\frac{1}{2}\frac{\partial}{\partial x'})} \mathrm{e}^{-\mathrm{i}r\xi'} \bar{G}(x',r) \right]$$

taken at $x' = x$ and $\xi' = \xi$. Collecting now alike terms in the exponents (containing and not containing derivatives, respectively) we arrive at

$$\bar{H}(x,p) = \int \left[\mathrm{e}^{\mathrm{i}q(\varepsilon-\frac{\xi}{2})} \, \mathrm{e}^{(\varepsilon+\frac{\xi}{2})(\frac{1}{2}\frac{\partial}{\partial x})} \bar{F}(x,q) \right] \left[\mathrm{e}^{-\mathrm{i}r(\varepsilon+\frac{\xi'}{2})} \, \mathrm{e}^{(\varepsilon-\frac{\xi'}{2})(\frac{1}{2}\frac{\partial}{\partial x'})} \mathrm{e}^{-\mathrm{i}r\xi'} \bar{G}(x',r) \right]$$

It is purposeful to write $q = p + a$ and $r = p' + b$, and taking the result at $p' = p$ at the end. This way the exponential factors can be re-arranged and the integrations over z, q and r are replaced by those over ε, a and b:

$$\bar{H}(x,p) = \int \mathrm{e}^{\mathrm{i}\varepsilon\left(a-b+\frac{1}{2i}(\frac{\partial}{\partial x}+\frac{\partial}{\partial x'})\right)} \, \mathrm{e}^{\mathrm{i}\xi\left(-\frac{a+b}{2}+\frac{1}{4i}(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'})\right)} \, \bar{F}(x,p+a)\bar{G}(x',p'+b).$$

Now $\int = \int \mathrm{d}\xi \mathrm{d}\varepsilon \frac{\mathrm{d}a}{2\pi} \frac{\mathrm{d}b}{2\pi}$, and at the end $x' = x$ and $p' = p$ are taken. The shifted momentum arguments $p + a$ and $p' + b$ are also Taylor-expanded,

$$\bar{F}(x,p+a)\bar{G}(x',p'+b) = \mathrm{e}^{a\frac{\partial}{\partial p}+b\frac{\partial}{\partial p'}} \bar{F}(x,p)\bar{G}(x',p'),$$

to have

$$\bar{H}(x,p) = \int \mathrm{e}^{\mathrm{i}a\left(\varepsilon-\frac{\xi}{2}+\frac{1}{i}\frac{\partial}{\partial p}\right)} \, \mathrm{e}^{\mathrm{i}b\left(-\varepsilon-\frac{\xi}{2}+\frac{1}{i}\frac{\partial}{\partial p'}\right)} \, \mathrm{e}^{\mathrm{i}\frac{\varepsilon}{2}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial x'}\right)} \, \mathrm{e}^{\mathrm{i}\frac{\xi}{4}\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)} \, \bar{F}(x,p)\bar{G}(x',p').$$

The last important step is to recognize that the integration over a and b leads to the following equivalence

$$\varepsilon - \frac{\xi}{2} = -\frac{1}{i}\frac{\partial}{\partial p}$$

and

$$\varepsilon + \frac{\xi}{2} = \frac{1}{i}\frac{\partial}{\partial p'}.$$

Using these relations the leftover exponent is given as

$$\hat{\nabla} = \frac{1}{2}\left[\left(\varepsilon+\frac{\xi}{2}\right)\frac{\partial}{\partial x} + \left(\varepsilon-\frac{\xi}{2}\right)\frac{\partial}{\partial x'} \right] = \frac{1}{2i}\left[\frac{\partial}{\partial p'}\frac{\partial}{\partial x} - \frac{\partial}{\partial p}\frac{\partial}{\partial x'} \right].$$

Using this "triangle" operator notation the Wigner transform of the convolution is given by

$$\bar{H}(x,p) = \mathrm{e}^{\hat{\nabla}}\, \bar{F}(x,p)\, \bar{G}(x',p')\Big|_{x'=x,\, p'=p}$$

The two leading orders of its expansion, if the effect of $\hat{\nabla}$ is small compared to the identity (the so called "gradient – expansion"), are given by

$$\bar{H}(x,p) \approx \bar{F}(x,p)\, \bar{G}(x,p) + \frac{1}{2i}\left\{\bar{F}, \bar{G}\right\}$$

where

$$\left\{\bar{F}, \bar{G}\right\} = \frac{\partial \bar{F}}{\partial x}\frac{\partial \bar{G}}{\partial p} - \frac{\partial \bar{F}}{\partial p}\frac{\partial \bar{G}}{\partial x}$$

is the Poisson bracket.

8.4 Consider the $\ddot{x} + 2\Gamma\dot{x} + k^2 x = f(t)$ oscillator within a white noise environment, $\langle f(t)\rangle = 0$, and $\langle f(t)f(t')\rangle = 4\Gamma T\delta(t-t')/V$. What is the correlator in the real-time, $\langle x(t)x(t')\rangle$, and in the frequency representation, $\langle \tilde{x}(\omega)\tilde{x}(\omega')\rangle$, in the infrared $(k \to 0)$ limit?

Fourier transformation with respect to the time, t, leads to

$$\left(-\omega^2 + 2i\Gamma\omega + k^2\right)\tilde{x}(\omega) = \tilde{f}(\omega).$$

The correlator for the Fourier-transform of the white noise can easily be obtained as being

$$\left\langle \tilde{f}(\omega)\tilde{f}(\omega')\right\rangle = \frac{4\Gamma T}{V} 2\pi\delta(\omega+\omega').$$

The real-time correlator becomes

$$G(t-t',k) = \left\langle x(t)x(t')\right\rangle = \int \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}\omega'}{2\pi}\, \mathrm{e}^{\mathrm{i}(\omega t+\omega' t')}\left\langle \tilde{x}(\omega)\tilde{x}(\omega')\right\rangle.$$

Due to the Dirac-delta factor in the white-noise correlation in the frequency-representation this simplifies to

$$\left\langle x(t)x(t')\right\rangle = \int \frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{\mathrm{i}\omega(t-t')}\left\langle \tilde{x}(\omega)\tilde{x}(-\omega)\right\rangle.$$

The particular x-correlator in the frequency-representation, which we need here, is given by

$$\left\langle \tilde{x}(\omega)\tilde{x}(-\omega)\right\rangle = \frac{4\Gamma T/V}{(-\omega^2 + 2i\Gamma\omega + k^2)\,(-\omega^2 - 2i\Gamma\omega + k^2)}.$$

The denominator can be re-factorized as follows: Since

$$\left(-\omega^2 + 2i\Gamma\omega + k^2\right)\left(-\omega^2 - 2i\Gamma\omega + k^2\right) = \left(\omega^2 - k^2\right)^2 + 4\Gamma^2\omega^2$$

$$= \left(\omega^2 + 2\Gamma^2 - k^2\right)^2 + 4\Gamma^2\left(k^2 - \Gamma^2\right),$$

the final factors are

$$\omega^2 + 2\Gamma^2 - k^2 + 2i\Gamma\sqrt{k^2 - \Gamma^2}$$

and its conjugate version with $-\Gamma$ instead of Γ.

For small values of k the square root is purely imaginary and we arrive at

$$\frac{1}{(-\omega^2 + 2i\Gamma\omega + k^2)\,(-\omega^2 - 2i\Gamma\omega + k^2)} \approx \frac{1}{4\Gamma^2}\left(\frac{1}{\omega^2 + k^4/4\Gamma^2} - \frac{1}{\omega^2 + 4\Gamma^2}\right).$$

This way the static infrared correlator becomes

$$G(\omega = 0, k \ll \Gamma) \approx \frac{4\Gamma T}{V}\,\frac{1}{k^4}$$

describing a linear (confining) potential between heavy and static sources. In fact the stochastic background model of confinement used to be popular in the 1980s. On the other hand the real-time correlator features two damping factors according to the above result, $\gamma_1 = 2\Gamma$ and $\gamma_2 = k^2/2\Gamma$. In the infrared limit $k \to 0$ this means a very slow forgetting:

$$G(t - t', k \ll \Gamma) = \frac{T}{Vk^2}\mathrm{e}^{-k^2|t-t'|/2\Gamma} - \frac{T}{4\Gamma^2 V}\mathrm{e}^{-2\Gamma|t-t'|}.$$

the $\gamma_2 = k^2/2\Gamma$ damping factor occurs in the discussion of chaotic quantization, too.

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