Appendix A
Results from Real and Complex Analysis

Abstract Some key results from real and complex analysis are reviewed here. These include a brief introduction to analytic functions of a complex variable with special emphasis on Rouché's theorem and the implicit function theorem. The Ascoli–Arzela lemma has special significance here as our dynamics takes place in the space of continuous functions on an interval. The fluctuation lemma and the Gronwall lemma are stated.

A.1 Analytic Functions

A good elementary reference for the material on complex functions is [17]. We also reference results in [1].

The complex plane $\mathbb{C}$ is the set of all complex numbers $z = x + iy$. The real and imaginary parts of z are defined by $\Re(z) = x$ and $\Im(z) = y$. The modulus of z is $|z| = (x^2 + y^2)^{1/2}$. We often identify $\mathbb{C}$ with the (x, y)-plane, the complex number $z = x + iy$ being identified with the point (x, y). The complex conjugate of z is its reflection in the x-axis: $\bar{z} = x - iy$. Conjugation has nice properties; the conjugate of the sum, product and quotient of two complex numbers is the sum, product or quotient, respectively, of the conjugates.

A complex-valued function of a complex variable $f : D \to \mathbb{C}$, where $D \subset \mathbb{C}$, can be represented as

$$w = f(z) = u(z) + iv(z) = u(x, y) + iv(x, y)$$

where u and v are real-valued functions defined on the domain D, now viewed as a subset of the (x, y)-plane. An example is the exponential function

$$w = e^z = e^x \cos(y) + ie^x \sin(y)$$

Here, $u = e^x \cos(y)$ and $v = e^x \sin(y)$. The reader should verify that $|e^z| = e^x = e^{\Re(z)}$ and that $\overline{e^z} = e^{\bar{z}}$. Thus, $e^z \neq 0$.

We say that f is analytic on D provided D is an open set and f is differentiable at each point of D in the sense that

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists at each $z_0 \in D$. If f is analytic on all of $\mathbb{C}$ then f is said to be an entire function. The derivative has the usual properties we learn for functions of a real variable; rules for differentiating are similar and the usual formulas hold for derivatives of polynomial functions $f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$, and the standard functions from calculus $e^z, \cos(z), \sin(z)$.

One can verify analyticity of f directly from its real and imaginary parts u and v.

Theorem A.1 *If f is analytic in D then u, v have partial derivatives u_x, u_y and v_x, v_y at all points of D that satisfy the Cauchy–Riemann equations*

$$u_x = v_y,\ u_y = -v_x \tag{A.1}$$

Moreover,

$$f'(z) = u_x + iv_x = v_y - iu_y$$

*Conversely, if u_x, u_y and v_x, v_y exist in D and are continuous and satisfy (*A.1*), then f is analytic in D.*

Exercise A.1. Verify that the complex exponential function is analytic in $\mathbb{C}$.

The Cauchy integral theorem [17] is the truly remarkable fact about analytic functions. We do not state it here but we make use of some of its consequences:

(a) An analytic function is infinitely differentiable.
(b) The Taylor series expansion of an analytic function converges and represents that function.

If $f : D \to \mathbb{C}$ is analytic, then all its derivatives:

$$f'(z), f''(z), \ldots, f^{(n)}(z), \ldots$$

exist at every point of D. Moreover, the Taylor series centered at $z_0 \in D$:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \tag{A.2}$$

converges to $f(z)$ for all z satisfying $|z - z_0| < R$ as long as $\{z : |z - z_0| < R\} \subset D$. In fact, the series converges absolutely (i.e., the series obtained by taking term-by-term modulus converges).

Conversely, if a power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

converges for some $z = z_1$, then it converges absolutely for all z satisfying $|z - z_0| < |z_1|$ to an analytic function.

One notable consequence of these facts is that an analytic function defined on a connected open domain D in $\mathbb{C}$ and vanishing at the point $z_0 \in D$ is either the identically zero function or it has a zero of finite order at z_0. Recall, we say that z_0 is a zero of order $k \geq 1$ of f if

$$f(z_0) = f'(z_0) = \cdots f^{(k-1)}(z_0) = 0, f^{(k)}(z_0) \neq 0$$

If f has a zero of order k at z_0, then from the power series we see that

$$f(z) = \sum_{n=k}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = (z - z_0)^k g(z)$$

where

$$g(z) = \sum_{n=0}^{\infty} \frac{f^{(n+k)}(z_0)}{(n+k)!}(z - z_0)^n$$

is an analytic function. Moreover, $g(z_0) = f^{(k)}(z_0) \neq 0$ so by continuity of g there is a neighborhood U of z_0 such that $g(z) \neq 0,\ z \in U$. It follows that $f(z) \neq 0$ for $z \in U \setminus \{z_0\}$.

Summarizing, an analytic function that is not identically zero in its (connected) domain has isolated zeros. This fact has important consequences.

Proposition A.2 *Let f be analytic on a connected domain D, not identically zero in D, and let K be a closed and bounded subset of D. Then f has at most finitely many zeros in K. If f is an entire function, then it has at most countably many zeros; if it has infinitely many zeros and $\{z_n\}_{n=1}^{\infty}$ is an enumeration of its distinct zeros, then $|z_n| \to \infty$ as $n \to \infty$.*

Proof. If there were infinitely many zeros in K we could find a sequence $\{z_n\} \subset K$ of distinct points such that $f(z_n) = 0$. By Bolzano–Weierstrass theorem, this sequence must have a limit point. Thus, there must be a convergent subsequence, which we rename as $\{z_n\}$, such that $z_n \to z \in K$. By continuity of f it follows that $f(z) = 0$ and thus f has a nonisolated zero. Because f is not identically zero, this gives a contradiction.

If f is entire, then it has finitely many zeros in each closed ball $\{z : |z| \leq n\}$ so it has at most countably many zeros and at most finitely many of these may lie inside any closed ball $\{z : |z| \leq R\}$ where $R > 0$. □

A.2 Implicit Function Theorem for Complex Variables

One of our main tasks in determining the stability of an equilibrium solution is to understand the characteristic roots of an analytic characteristic equation $h(z) = 0$. In practice, there are usually important parameters, such as the delay, and we would like to know how the roots vary with the parameters. Therefore, we must study the roots z of the equation

$$h(z,p) = 0 \tag{A.3}$$

where p denotes a vector of usually real parameters. The implicit function theorem is the natural tool for this. The following is an adaptation of the usual implicit function theorem (Theorem 9.28, [65]) to complex-valued functions.

Theorem A.3 *Let $h : D \times O \to \mathbb{C}$ where $D \subset \mathbb{C}$ and $O \subset \mathbb{R}^k$ are both open sets. Assume that h is analytic in $z \in D$ for each $p \in O$ and $h_z(z,p)$ is continuous in $D \times O$. Assume also that $h_p(z,p)$ exists and is continuous in $D \times O$.*

If $h(z_0,p_0) = 0$ for some $(z_0,p_0) \in D \times O$ and $h_z(z_0,p_0) \neq 0$, then there is a neighborhood U of z_0 in D and a neighborhood V of p_0 in O and a continuously differentiable function $g : V \to U$ satisfying:

(a) $g(p_0) = z_0$.

(b) $h(g(p),p) = 0,\ p \in V$.

(c) *If $(z,p) \in U \times V$ and $h(z,p) = 0$, then $z = g(p)$.*

Proof. To see that this follows from the usual implicit function theorem (Theorem 9.28, [65] or see Theorem A.5), we identify (A.3) with

$$H(x,y,p) = 0, H(x,y,p) = (u(x,y,p), v(x,y,p))$$

where $h(z,p) = u(x,y,p) + iv(x,y,p)$ and $z = x+iy$. H is continuously differentiable on its domain. Then we have $H(x_0,y_0,p_0) = 0$ and its Jacobian with respect to (x,y) satisfies

$$\frac{\partial H}{\partial(x,y)}(x_0,y_0,p_0) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}$$

where the partial derivatives are evaluated at (x_0,y_0,p_0) and where we have used the Cauchy–Riemann equations (A.1). The determinant of the Jacobian is given by $u_x^2 + v_x^2 = |h_z(z_0,p_0)|^2$ because $h_z = u_x + iv_x$ by Theorem A.1. By hypothesis, $h_z(z_0,p_0) \neq 0$ so the determinant is nonzero as required. □

A.3 Rouché' s Theorem

The following result, a special case of Rouché' s theorem (see ([1]), is useful in studying the characteristic equation.

Theorem A.4 *[Rouché' s theorem] Let γ be a simple closed curve (non-intersecting) in the complex plane and let $f(z)$ and $g(z)$ be functions analytic in the complex plane and satisfying*

$$|f(z) - g(z)| < |f(z)|, z \in \gamma$$

Then $f(z)$ and $g(z)$ have the same number of zeros, counting the order of each root, enclosed by γ.

To see why Rouché' s theorem is useful, keep in mind that in practice our linear systems always contain many parameters so usually we have $F(z) = F(p,z)$ where $p \in \mathbb{R}^k$ are parameters and we want to know how the characteristic zeros vary as p is varied. Let's suppose that F is continuous in all arguments but analytic in z for fixed p. Suppose that γ is a simple closed curve in $\mathbb{C}$ and $F(z, p_0) = 0$ has no roots on γ for parameter value p_0. Now compactness of γ and continuity of F mean that

$$\varepsilon := \min\{|F(z, p_0)| : z \in \gamma\} > 0$$

For the same reasons, there exists $\delta > 0$ such that

$$|p - p_0| < \delta, z \in \gamma \Longrightarrow |F(p,z) - F(p_0,z)| < \varepsilon$$

from which we conclude, by Theorem A.4, that the number of roots of $F(p,z) = 0$ inside γ is the same as the number of roots of $F(p_0,z) = 0$ provided $|p - p_0| < \delta$.

Exercise A.2. Use Theorem A.4 to prove that if $p(z)$ is a polynomial of degree n and $\varepsilon > 0$ is such that $p(z) = 0$ has a root z_0 of multiplicity m and no other roots in $|z - z_0| \leq \varepsilon$, then there exists $\delta > 0$ such that nth degree polynomial $q(z)$ has m zeros, counting multiplicity, in $|z - z_0| \leq \varepsilon$ provided the coefficients of q are within δ of those of p.

A.4 Ascoli–Arzela Theorem

Let $C = C([-r,0],\mathbb{R})$ be the metric space of continuous real-valued functions on the interval $[-r,0]$ with the supremum norm

$$\|\phi\| = \sup\{|\phi(x)| : -r \leq x \leq 0\}$$

We use the argument x rather than θ for ϕ for simplicity. A sequence $\{\phi_n\}_{n=1}^{\infty}$ in C converges to $\phi \in C$ relative to the supremum norm if and only if it converges uniformly on $[-r,0]$: $\forall \varepsilon > 0, \exists$ a natural number N such that

$$|\phi_n(x) - \phi(x)| < \varepsilon, x \in [-r,0], n \geq N$$

It is important to know when a given sequence $\{\phi_n\}_{n=1}^{\infty}$ in C has a convergent subsequence. The Bolzano–Weierstrass theorem for $\mathbb{R}^n$ says that every bounded sequence of vectors has a convergent subsequence. However, this property fails for continuous function spaces. For example, consider the sequence $\phi_n(x) = x^n$ in $C([0,1],\mathbb{R})$ for $n = 1,2,\cdots$. As $|\phi_n(x)| \leq 1,\ x \in [0,1],\ n \geq 1$, it is a bounded sequence but it has no subsequence that converges uniformly on $[0,1]$ to a member of $C([0,1],\mathbb{R})$. Indeed, because $\phi_n(x)$ converges pointwise to 0 if $x < 1$ and to 1 if

$x = 1$, so will any subsequence. Therefore, in the space C, we need additional conditions beside boundedness to guarantee the existence of a uniformly convergent subsequence.

A subset A of functions in C is *equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\phi(x) - \phi(y)| < \varepsilon$ whenever $\phi \in A$ and $x, y \in [-r, 0]$ satisfy $|x - y| < \delta$. Note that the same δ works for every $\phi \in A$ and every $x, y \in [-r, 0]$ with $|x - y| < \delta$. The most common method of verifying equicontinuity is to show that there exists $M > 0$ such that ϕ' exists and $|\phi'(x)| \le M$ for every $\phi \in A$ and every $x \in [-r, 0]$. Then A is equicontinuous because

$$|\phi(x) - \phi(y)| = |\phi'(\eta)||x - y| \le M|x - y|$$

holds for every $\phi \in A$ by the mean value theorem, where $\eta \in [-r, 0]$ may depend on $\phi \in A$.

We require the famous Ascoli–Arzela theorem, Theorem 7.25 [65].

Theorem A.1. *Let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence of functions in C that is equicontinuous and such that there is some $M > 0$ such that $|\phi_n(x)| \le M$ for all $n \ge 1$ and all $x \in [-r, 0]$. Then some subsequence of $\{\phi_n\}_{n=1}^{\infty}$ converges uniformly to an element of C.*

We remark that by replacing absolute value by a vector norm on $\mathbb{R}^n$, the definitions above and Theorem A.1 extend to $C([-r, 0], \mathbb{R}^n)$.

A.5 Fluctuation Lemma

Let $f : [b, \infty) \to \mathbb{R}$. Then the *limit superior* and the *limit inferior* of f as $t \to \infty$ are defined as

$$\begin{aligned} f^{\infty} &:= \limsup_{t \to \infty} f(t) = \inf_{r \ge b} \sup\{f(t); t \ge r\} \\ f_{\infty} &:= \liminf_{t \to \infty} f(t) = \sup_{r \ge b} \inf\{f(t); t \ge r\} \end{aligned} \tag{A.4}$$

It is easily shown that there is a sequence $s_k \to \infty$ such that $f(s_k) \to f_{\infty}$ and that there is a sequence $t_k \to \infty$ such that $f(t_k) \to f^{\infty}$. In fact, f_{∞} is the smallest such sequential limit and f^{∞} is the largest.

Perhaps the most useful property of f^{∞} is that for every $\varepsilon > 0$, there exists $T > b$ such that $f(t) \le f^{\infty} + \varepsilon$ for all $t \ge T$. Analogously, for every $\varepsilon > 0$, there exists $T > b$ such that $f(t) \ge f_{\infty} - \varepsilon$ for all $t \ge T$.

The following result, often called the fluctuation lemma, is remarkably useful. It is intuitive if one thinks of $f(t) = \sin t$ where $f_{\infty} = -1$ and $f^{\infty} = 1$. See [37, 75, 72] for a proof or the reader can supply it.

Lemma A.1. *Let $f : [b, \infty) \to \mathbb{R}$ be bounded and differentiable. Then there exist sequences $s_k, t_k \to \infty$ such that*

$$\left.\begin{array}{l} f(s_k) \to f_\infty, \;\; f'(s_k) \to 0 \\ f(t_k) \to f^\infty, \;\; f'(t_k) \to 0 \end{array}\right\} k \to \infty.$$

A.6 General Implicit Function Theorem

In the following appendix, we require the implicit function theorem in a Banach space setting. We follow [81]; see also [16]. Recall that a Banach space X is a complete normed linear space; we use the notation $\| \bullet \|_X$ for the norm on X. A mapping F is said to be C^m, written $F \in C^m$, if it is m-times continuously differentiable.

Theorem A.5 *Suppose that mapping* $F : U(x_0, y_0) \subset X \times Y \to Z$ *is defined on an open set* $U(x_0, y_0)$ *and* $F(x_0, y_0) = 0$*, where* X, Y*, and* Z *are Banach spaces. Assume that the partial derivative* $F_y(x, y)$ *exists for* $(x, y) \in U(x_0, y_0)$*,* F *and* F_y *are continuous at* (x_0, y_0)*, and that* $F_y(x_0, y_0) : Y \to Z$ *is bijective. Then:*

(a) There exist $r_0, r > 0$ *such that for every* $x \in X$ *with* $\|x - x_0\|_X \le r_0$*, there is exactly one* $y = y(x) \in Y$ *for which* $\|y - y_0\|_Y \le r$ *and* $F(x, y(x)) = 0$.
(b) If F *is continuous in* $U(x_0, y_0)$*, then* $y(x)$ *is continuous in a neighborhood of* x_0.
(c) If $F \in C^m$*,* $1 \le m \le \infty$ *on* $U(x_0, y_0)$*, then* $y(\bullet) \in C^m$ *on some neighborhood of* x_0.

The standard implicit function theorem from advanced calculus is the special case $X = \mathbb{R}^m, Y = Z = \mathbb{R}^n$. The proof is the same for both the finite and infinite-dimensional cases.

A.7 Gronwall's Inequality

We recall a fundamental result from ODE theory that plays more or less the same role for delay differential equations. See [40, 10] for the elementary proof.

Theorem A.6 (Gronwall Inequality) *Let* $K \ge 0$ *and let* $f, g : [a, b] \to [0, \infty)$ *be continuous functions satisfying the inequality*

$$f(t) \le K + \int_a^t f(s)g(s)ds, a \le t \le b.$$

Then

$$f(t) \le K \exp(\int_a^t g(s)ds), a \le t \le b.$$

Appendix B
Hopf Bifurcation for Delayed Negative Feedback

Abstract In Chapter 6.3, we gave a purely formal construction of the Hopf bifurcation of periodic solutions of the canonical nonlinear negative feedback equation $x'(t) = -f(x(t-r))$. Here we give a rigorous proof of the existence of these solutions using the implicit function theorem A.5.

B.1 Basic Setup and Preliminaries

In this appendix, we give a mathematically rigorous justification of the formal arguments given in Chapter 6 for the computation of the Hopf periodic solution of the scalar delay equation with negative feedback. We begin with Equation (6.13).

We seek solutions (P,R,ω) of

$$P'(z) = -\frac{R}{\omega} f(P(z-\frac{\pi}{2}\omega)) \tag{B.1}$$

where P is 2π-periodic and

$$P \approx 0, R \approx 1, \omega \approx 1$$

Our immediate goal is to reformulate this differential equation as an equation in a suitable Banach space. For this, we need some notation and preliminary work. It is better to rewrite the equation as

$$L(P)(z) = P(z-\frac{\pi}{2}) - \frac{R}{\omega} f(P(z-\frac{\pi}{2}\omega)) \tag{B.2}$$

where

$$LP(z) = P'(z) + P(z-\frac{\pi}{2})$$

P' denotes the derivative of P. We begin by studying this linear operator.

Fourier series play a big role.

$$h = \sum_{n \in Z} h_n e^{inz}$$

is the Fourier series for h, where

$$h_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(z) e^{-inz} dz, n \in \mathbb{Z}$$

It converges to h in the mean square sense. We use complex series mainly for the linear equation. When we consider the nonlinear equation, we use real Fourier series.

For $k \geq 0$, let

$$H^k = \{h \in L^2(\mathbb{T}) : \sum_{n \in Z} n^{2k} |h_n|^2 < \infty\}$$

These spaces are Hilbert spaces contained in the Hilbert space $H^0 = L^2(\mathbb{T})$ of square integrable functions on the unit circle $\mathbb{T}$. Let

$$C^k = \{f : \mathbb{R} \to \mathbb{R} : \text{f is k times continuously differentiable and } 2\pi - \text{periodic}\}$$

denote the Banach space with norm

$$\|f\|_k = \|f\|_\infty + \|f'\|_\infty + \cdots + \|f^{(k)}\|_\infty$$

where $\|\cdot\|_\infty$ denotes the supremum norm. It is well known that

$$H^{k+1} \subset C^k \subset H^k, \; k = 0, 1, 2, \ldots$$

Proposition B.1 *Let $k \geq 1$. Then $L : H^{k+1} \to H^k$ is a bounded linear operator and*

$$N(L) = span\{\cos(z), \sin(z)\}, \; R(L) = M^k := \{Q \in H^k : Q_1 = Q_{-1} = 0\}$$

where $N(L)$ denotes null space and $R(L)$ the image of L. $L^{-1} : M^k \to M^{k+1}$ is bounded.

$L : C^{k+1} \to C^k$ is a bounded linear operator and

$$N(L) = span\{\cos(z), \sin(z)\}, \; R(L) = Z^k := \{Q \in C^k : Q_1 = Q_{-1} = 0\}$$

$L^{-1} : Z^k \to Z^{k+1}$ is bounded.

Proof. Consider

$$P'(z) + P(z - \frac{\pi}{2}) = h(z) \in H^k$$

Both h and the solution P have Fourier series and we can solve for P if we can determine its Fourier coefficients in terms of those of h. The relevant series are:

$$h = \sum_{n \in Z} h_n e^{inz}, \; P = \sum_{n \in Z} P_n e^{inz}$$
$$P(\bullet - \frac{\pi}{2}) = \sum_{n \in Z} P_n e^{-in\frac{\pi}{2}} e^{inz}$$
$$\frac{dP}{dz} = \sum_{n \in Z} inP_n e^{inz}$$

Inserting these into our equation and equating coefficients of e^{inz} leads to

$$(LP)_n = (in + e^{-in\pi/2})P_n = h_n, \; n \in Z$$

Consequently, as the term in parentheses vanishes if and only if $n = \pm 1$, we find that the null space of L, $N(L)$ is spanned by e^{iz}, e^{-iz} and that there is a 2π-periodic solution P if and only if

$$h_1 = h_{-1} = 0 \tag{B.3}$$

and its Fourier coefficients are given by

$$P_n = \frac{h_n}{in + (-i)^n}, |n| > 1, P_0 = h_0 \tag{B.4}$$

and where P_1 and P_{-1} are arbitrary. Thus,

$$(L^{-1}h)_n = \frac{h_n}{in + (-i)^n}, n \neq \pm 1$$

It is a straightforward exercise to establish the assertions regarding $L : H^{k+1} \to H^k$. Clearly, $L(C^{k+1}) \subset C^k$. If $h \in C^k$ satisfies $h_1 = h_{-1} = 0$, then $LP = h$ has a solution $P \in H^{k+1}$ because $h \in H^k$. It follows that $P(z - \pi/2)$ belongs to C^k because $H^{k+1} \in C^k$ and as $P'(z) = -P(z - \pi/2) + h(z)$, we conclude that $P' \in C^k$ implying that $P \in C^{k+1}$. □

Define the projection operator $Q : C^k \to C^k$ by

$$QP = P_{-1}e^{-iz} + P_1 e^{iz}.$$

The following result is a reformulation that is more useful for solving equations.

Corollary B.2 *The equation*

$$LP = h \in C^k$$

has a solution $P \in C^{k+1}$ *if and only if*

$$Qh = 0$$

and

$$LP = (I - Q)h$$

The last equation has a unique solution satisfying $QP = 0$.

We need to know that substitution operators are smooth.

Lemma B.1. *Let $f : \mathbb{R} \to \mathbb{R}$ have continuous derivatives of all order. Then the map $F : C^k \to C^k$ given by $P \to f(P)$ is continuously differentiable and its derivative is given by*

$$DF(P)(h)(t) = f'(P(t))h(t),\ h \in C^k$$

Proof. We merely point out what needs to be shown, namely,

$$\lim_{\|h\|_k \to 0} \frac{\|F(P+h) - F(P) - DF(P)h\|_k}{\|h\|_k} = 0$$

For example, if $k = 2$, one can show that

$$g(t) = f(P(t)+h(t)) - f(P(t)) - f'(P(t))h(t)$$

satisfies

$$\|g\|_\infty + \|g'\|_\infty + \|g''\|_\infty = O((\|h\|_\infty + \|h'\|_\infty + \|h''\|_\infty)^2)$$

by simple, but tedious, calculations. □

For $\theta \in \mathbb{R}$, define the shift operator $T_\theta : C^k \to C^k$ by

$$[T_\theta P](z) = P(z - \theta)$$

Then $\{T_\theta\}_{\theta \in \mathbb{R}}$ defines a group of bounded linear operators on the spaces H^k and C^k. Note that

$$T_\theta L = L T_\theta, T_\theta Q = Q T_\theta.$$

C^k is an algebra and T_θ is an algebra homomorphism:

$$T_\theta(P+Q) = T_\theta P + T_\theta Q, \quad T_\theta(P \cdot Q) = T_\theta P \cdot T_\theta Q$$

More generally, we use that $T_\theta f(P) = f(T_\theta P)$ for $P \in C^k$.

There is a certain loss of smoothness in applying the shift map which is immediately apparent from its definition above: to differentiate with respect to θ we must differentiate P.

Lemma B.2. *The map $K : \mathbb{R} \times C^{k+1} \to C^k$ given by $(\theta, P) \to T_\theta P$ is C^1 and*

$$DK(\theta, P)(h, Q) = T_\theta(P'h + Q)$$

B.2 The Solution

In order to simplify Equation (B.2), we write

$$\frac{\pi}{2}\omega = \frac{\pi}{2} + \delta, \frac{R}{\omega} = 1 + \mu \tag{B.5}$$

where $\delta, \mu \approx 0$. Write

$$f(u) = u + u^2 G(u)$$

where $G(0) = A/2$ and G is smooth.

Then (B.2) becomes

$$LP = T_{\pi/2}P - (1+\mu)T_{(\pi/2+\delta)}\left(P + P^2 G(P)\right)$$

Note that we have dropped arguments of functions (e.g., the z variable) to emphasize that we are now seeking an abstract formulation. We continue to do this although it leads to writing cos instead of $\cos(z)$; cos belongs to C^k but $\cos(z)$ is a scalar belonging to $\mathbb{R}$.

We seek a solution $P = \varepsilon(\cos + q)$ where $q \in Z^2$. As $L\cos = 0$ and $T_{\pi/2}\cos = \sin$, this becomes

$$\begin{aligned}\varepsilon Lq = {} & \varepsilon\sin + \varepsilon T_{\pi/2}q - (1+\mu)T_{(\pi/2+\delta)}(\varepsilon\cos + \varepsilon q \\ & + (\varepsilon\cos + \varepsilon q)^2 G(\varepsilon\cos + \varepsilon q))\end{aligned}$$

According to Corollary B.2, and using that $Qq = 0$ and $Q\sin = \sin$ and $Q\cos = \cos$, this equation is equivalent to the following system:

$$0 = \varepsilon\sin - (1+\mu)T_{(\pi/2+\delta)}\left(\varepsilon\cos + \varepsilon^2 Q(\cos+q)^2 G(\varepsilon\cos + \varepsilon q)\right) \tag{B.6}$$

$$\varepsilon Lq = \varepsilon T_{\pi/2}q - (1+\mu)T_{(\pi/2+\delta)}\left(\varepsilon q + \varepsilon^2 (I-Q)(\cos+q)^2 G(\varepsilon\cos + \varepsilon q)\right)$$

B.2.1 Solve for q

We consider the equation for $q \in Z^2$ first. Dividing by ε, it becomes

$$Lq = T_{\pi/2}q - (1+\mu)T_{(\pi/2+\delta)}\{q + \varepsilon(I-Q)(\cos+q)^2 G(\varepsilon\cos + \varepsilon q)\} \tag{B.7}$$

This equation has an important symmetry: if $(\mu, \delta, \varepsilon, q)$ satisfies this equation, where $q \in Z^2$, then so does $(\mu, \delta, -\varepsilon, -T_\pi q)$. To see this, first apply T_π to both sides, use the fact that it is an algebra homomorphism, that $T_\pi\cos = -\cos$, and that it commutes with Q, then multiply both sides by -1. These steps are carried out below where $\tilde{q} = -T_\pi q$:

$$\begin{aligned}LT_\pi q = {} & T_{\pi/2}T_\pi q - (1+\mu)T_{(\pi/2+\delta)}\{T_\pi q \\ & + \varepsilon(I-Q)(-\cos + T_\pi q)^2 G(-\varepsilon\cos + \varepsilon T_\pi q)\} \\ L(-T_\pi q) = {} & T_{\pi/2}(-T_\pi q) - (1+\mu)T_{(\pi/2+\delta)}\{(-T_\pi q) \\ & + (-\varepsilon)(I-Q)(-\cos - (-T_\pi q))^2 G((-\varepsilon)\cos + (-\varepsilon)(-T_\pi q))\} \\ L\tilde{q} = {} & T_{\pi/2}\tilde{q} - (1+\mu)T_{(\pi/2+\delta)}\{\tilde{q} \\ & + (-\varepsilon)(I-Q)(\cos + \tilde{q})^2 G((-\varepsilon)\cos + (-\varepsilon)\tilde{q})\}\end{aligned}$$

By virtue of Lemma B.2 and Lemma B.1, we may view the right side of (B.7) as a C^1 map taking $(\mu,\delta,\varepsilon,q) \in \mathbb{R}^3 \times Z^2$ into Z^1. Lemma B.2 accounts for the loss of one derivative. Hence, this equation is equivalent to

$$0 = q - L^{-1}[T_{\pi/2}q - (1+\mu)T_{(\pi/2+\delta)}\{q + \varepsilon(I-Q)(\cos + q)^2 G(\varepsilon\cos + \varepsilon q)\}] \quad \text{(B.8)}$$

We view (B.8) as

$$F(\mu,\delta,\varepsilon,q) = 0$$

where $F : \mathbb{R}^3 \times Z^2 \to Z^2$ is a C^1 map satisfying

$$F(0,0,0,0) = 0, F_q(0,0,0,0) = I$$

where F_q denotes the Frechet derivative with respect to q and I is the identity. This derivative, and other partial derivatives computed hereafter, are best computed by "freezing" the other variables at their designated values first. For example, $F_q(0,0,0,0)$ is computed by first computing $F(0,0,0,q)$, then its derivative with respect to q.

In fact, F satisfies

$$F(\mu,\delta,0,0) = 0 \quad \text{(B.9)}$$

By the implicit function theorem A.5, there exists a C^1 function $q : \mathbb{R}^3 \to Z^2$, defined near $(0,0,0)$ such that $q = q(\mu,\delta,\varepsilon)$ satisfies

$$F(\mu,\delta,\varepsilon,q(\mu,\delta,\varepsilon)) = 0, \text{ and } \quad q(0,0,0) = 0$$

By the symmetry mentioned above and the uniqueness of solutions guaranteed by the implicit function theorem, we must have that

$$T_\pi q(\mu,\delta,\varepsilon) = -q(\mu,\delta,-\varepsilon) \quad \text{(B.10)}$$

and by (B.9)

$$q(\mu,\delta,0) = 0$$

It follows that

$$q_\mu(0,0,0) = q_\delta(0,0,0) = 0$$

A straightforward calculation gives:

$$\begin{aligned} F_\varepsilon(0,0,0,0) &= G(0)L^{-1}(I-Q)\sin^2 \\ &= G(0)[\frac{1}{2} + \frac{1}{10}\cos(2z) - \frac{1}{5}\sin(2z)] \end{aligned}$$

Implicit differentiation of $F = 0$ yields:

$$0 = F_\varepsilon(0,0,0,0) + F_q(0,0,0,0)q_\varepsilon = F_\varepsilon(0,0,0,0) + q_\varepsilon$$

which implies that

$$q_\varepsilon = -G(0)[\frac{1}{2} + \frac{1}{10}\cos(2z) - \frac{1}{5}\sin(2z)] \tag{B.11}$$

B.2.2 Solve for μ and δ

Having solved the second of equations (B.6) for $q = q(\mu, \delta, \varepsilon)$, we now insert this into the first equation and try to solve it for (μ, δ) in terms of ε. As the right-hand side of the second equation belongs to $R(Q) = \text{span}\{\cos, \sin\}$, it actually represents a system of two equations for the sin- and cos-components of the right-hand side. First, we divide out a factor of ε from the equation to get

$$0 = \sin - (1+\mu)T_{(\pi/2+\delta)}\left(\cos + \varepsilon Q(\cos + q)^2 G(\varepsilon\cos + \varepsilon q)\right)$$

where $q = q(\mu, \delta, \varepsilon)$. Now we break this up into components. As $T_{(\pi/2+\delta)}\cos = \cos\delta\sin - \sin\delta\cos$, we obtain the two equations for $(\mu, \delta, \varepsilon)$:

$$\begin{aligned} 0 &= 1 - (1+\mu)\cos\delta - \varepsilon(1+\mu)[T_{(\pi/2+\delta)}(\cos + q)^2 G(\varepsilon\cos + \varepsilon q)]_{sin} \\ 0 &= (1+\mu)\sin\delta - \varepsilon(1+\mu)[T_{(\pi/2+\delta)}(\cos + q)^2 G(\varepsilon\cos + \varepsilon q)]_{cos} \end{aligned}$$

where

$$[h]_{sin} = \frac{1}{\pi}\int_{-\pi}^{\pi} h(z)\sin(z)dz$$

denotes the sin-component and $[h]_{cos}$ is similar, with cos replacing sin. Note that operator Q would be redundant if left in place, so it has been removed. Finally, we divide out the factor $(1+\mu)$ so we have

$$\begin{aligned} 0 &= -\frac{\mu}{1+\mu} + 1 - \cos\delta - \varepsilon h(\mu, \delta, \varepsilon) \\ 0 &= \sin\delta - \varepsilon k(\mu, \delta, \varepsilon) \end{aligned} \tag{B.12}$$

where

$$h(\mu, \delta, \varepsilon) = [T_{(\pi/2+\delta)}(\cos + q)^2 G(\varepsilon\cos + \varepsilon q)]_{sin} \tag{B.13}$$

$$k(\mu, \delta, \varepsilon) = [T_{(\pi/2+\delta)}(\cos + q)^2 G(\varepsilon\cos + \varepsilon q)]_{cos} \tag{B.14}$$

Using (B.10) and that $[T_\pi P]_{cos} = -[P]_{cos}$ for $P \in C^k$, we see that k is odd:

$$\begin{aligned} k(\mu, \delta, -\varepsilon) &= [T_{(\pi/2+\delta)}(\cos - T_\pi q)^2 G(-\varepsilon\cos + \varepsilon T_\pi q)]_{cos} \\ &= [T_{(\pi/2+\delta)} T_\pi(-\cos - q)^2 T_\pi G(\varepsilon\cos + \varepsilon q)]_{cos} \\ &= [T_\pi(T_{(\pi/2+\delta)}(\cos + q)^2 G(\varepsilon\cos + \varepsilon q))]_{cos} \\ &= -k(\mu, \delta, \varepsilon) \end{aligned}$$

A similar calculation shows that h is odd. It follows that $k(\mu, \delta, 0) = 0 = h(\mu, \delta, 0)$.

We view system (B.12) as $G(\mu,\delta,\varepsilon)=0$ for $G:\mathbb{R}^3\to\mathbb{R}^2$ satisfying $G(0,0,0)=0$ and, as h,k are odd, $G(\mu,\delta,-\varepsilon)=G(\mu,\delta,\varepsilon)$. An easy computation shows that

$$\frac{\partial G}{\partial(\mu,\delta)}(0,0,0)=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$$

The implicit function theorem implies that the equation $G=0$ is solved by a C^1 function $(\mu,\delta)=(\mu(\varepsilon),\delta(\varepsilon))$ satisfying

$$\mu(-\varepsilon)=\mu(\varepsilon),\delta(-\varepsilon)=\delta(\varepsilon),\mu(0)=0=\delta(0)$$

Divide the second of equations (B.12) by ε^2 to get

$$\frac{\sin\delta(\varepsilon)}{\delta(\varepsilon)}\frac{\delta(\varepsilon)}{\varepsilon^2}=\frac{k(\mu(\varepsilon),\delta(\varepsilon),\varepsilon)}{\varepsilon}$$

Letting $\varepsilon\to 0$ and using that $k_\mu(0,0,0)=k_\delta(0,0,0)=0$ we find that

$$\lim_{\varepsilon\to 0}\frac{\delta(\varepsilon)}{\varepsilon^2}=k_\varepsilon(0,0,0)$$

Similarly, dividing the first of equations (B.12) by ε^2, we get

$$-\frac{\mu}{\varepsilon^2}\frac{1}{1+\mu}+\frac{1-\cos(\delta)}{\delta}\frac{\delta}{\varepsilon^2}=\frac{h(\mu(\varepsilon),\delta(\varepsilon),\varepsilon)}{\varepsilon}$$

Taking the limit as before results in

$$\lim_{\varepsilon\to 0}\frac{\mu(\varepsilon)}{\varepsilon^2}=-h_\varepsilon(0,0,0)$$

The derivative $k_\varepsilon(0,0,0)$ can be computed from

$$k(0,0,\varepsilon)=[T_{\pi/2}(\cos+q(0,0,\varepsilon))^2G(\varepsilon(\cos+q))]_{cos}$$

to be

$$\begin{aligned}k_\varepsilon(0,0,0)&=2G(0)[T_{\pi/2}(\cos q_\varepsilon(0,0,0))]_{cos}+G'(0)[\sin^3]_{cos}\\&=-2G(0)^2[\sin(z)(\frac{1}{2}+\frac{1}{10}\cos 2(z-\frac{\pi}{2})-\frac{1}{5}\sin 2(z-\frac{\pi}{2}))]_{cos}\\&=-2G(0)^2[\frac{1}{2}\sin z-\frac{1}{10}\cos 2z\sin z+\frac{1}{5}\sin 2z\sin z]_{cos}\\&=-\frac{G(0)^2}{5}\end{aligned}$$

Similarly,

$$
\begin{aligned}
h_\varepsilon(0,0,0) &= -2G(0)^2[\frac{1}{2}\sin z - \frac{1}{10}\cos 2z \sin z + \frac{1}{5}\sin 2z \sin z]_{sin} \\
&\quad + G'(0)[(3/4)\sin(z) - (1/4)\sin(3z)]_{sin} \\
&= -2G(0)^2[\frac{11}{20}] + \frac{3G'(0)}{4} \\
&= -\frac{11G(0)^2}{10} + \frac{3G'(0)}{4}
\end{aligned}
$$

where we have used trigonometric identities such as

$$
\cos(2z)\sin(z) = \frac{1}{2}[\sin(3z) - \sin(z)].
$$

We summarize our computations as follows.

$$
\delta = \delta(\varepsilon) = -\frac{G(0)^2}{5}\varepsilon^2 + o(\varepsilon^2) \tag{B.15}
$$

$$
\mu = \mu(\varepsilon) = \left(\frac{11G(0)^2}{10} - \frac{3G'(0)}{4}\right)\varepsilon^2 + o(\varepsilon^2) \tag{B.16}
$$

Returning to the original parameters R, ω from (B.5), we have

$$
\begin{aligned}
\omega &= 1 + \frac{2}{\pi}\delta \\
R &= 1 + \frac{2}{\pi}\delta + \mu + \frac{2}{\pi}\delta\mu
\end{aligned}
$$

which leads to

$$
\omega = 1 - \frac{2G(0)^2}{5\pi}\varepsilon^2 + o(\varepsilon^2) \tag{B.17}
$$

$$
R = 1 + \left(G(0)^2\frac{11\pi - 4}{10\pi} - \frac{3G'(0)}{4}\right)\varepsilon^2 + o(\varepsilon^2) \tag{B.18}
$$

Recall that

$$
P = \varepsilon\cos(z) + \varepsilon q(\mu, \delta, \varepsilon)(z)
$$

As q_μ and q_δ vanish at $(0,0,0)$ we have

$$
q = q(\mu(\varepsilon), \delta(\varepsilon), \varepsilon) = \varepsilon q_\varepsilon(0,0,0) + o(\varepsilon)
$$

and therefore, from (B.11), we have

$$
P(z) = \varepsilon\cos(z) - \varepsilon^2 G(0)[\frac{1}{2} + \frac{1}{10}\cos(2z) - \frac{1}{5}\sin(2z)] + O(\varepsilon^3) \tag{B.19}
$$

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