A

Some homological algebra

A.1 The language of categories and functors

This is not an introduction to category theory but just a summary of some of the standard terminology used therein.

A **category** $\mathcal{C}$ is a class $\mathrm{Obj}(\mathcal{C})$ of **objects** together with a class $\mathrm{Mor}(\mathcal{C})$ of **morphisms**. Each morphism f has a unique source object $A \in \mathrm{Obj}(\mathcal{C})$ and a unique target object $B \in \mathrm{Obj}(\mathcal{C})$. If A is the source and B the target of f one writes $f\colon A \to B$ and says that f is a morphism from A to B. The class of morphisms from A to B is denoted by $\mathrm{Hom}(A, B)$.

For every three objects A, B and C a map

$$\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C), \quad (f, g) \mapsto g \circ f,$$

called **composition of morphisms** is given such that the following axioms hold:

(A) (Associativity) If $f\colon A \to B$, $g\colon B \to C$ and $h\colon C \to D$ then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(B) (Identity) For every object X, there exists a morphism $\mathrm{id}_X\colon X \to X$ called the **identity morphism** for X, such that for every morphism $f\colon A \to B$, we have $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$.

Examples of categories appear in all branches of mathematics. The simplest example of a category is the category of sets $\mathcal{S}$ whose objects are the sets and whose morphisms are the maps between sets. Other examples are for instance: the category $\mathcal{T}$ of topological spaces, whose morphisms are the continuous maps, or for a given ring R, the category $\mathcal{M}_R$ of R-modules, whose morphisms are the R-module homomorphisms. If R is graded we can also consider the category $\mathcal{G}_R$ of graded R-modules. The morphisms in this case are the homogeneous R-module homomorphisms of degree 0. As a special case of

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the last type of category we considered in Section 5.1 the category $\mathcal{G}$ of graded modules over the exterior algebra.

Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A **covariant functor** $F\colon \mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ to $\mathcal{B}$ is a mapping that assigns to each object $A \in \mathcal{A}$ and object $F(A) \in \mathcal{B}$ and to each morphism $f\colon A \to B$ in $\mathcal{A}$ a morphism $F(f)\colon F(A) \to F(B)$ such that the following axioms hold:

(C) For all morphisms $f\colon B \to C$ and $g\colon A \to B$ in $\mathcal{A}$ one has

$$F(f \circ g) = F(f) \circ F(g).$$

(D) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all objects X in $\mathcal{A}$.

A **contravariant functor** $F\colon \mathcal{A} \to \mathcal{B}$ is defined similarly. The only difference is that it reverses the arrows of the maps. In other words to each morphism $f\colon A \to B$ in $\mathcal{A}$ the contravariant functor F assigns a morphism $F(f)\colon F(B) \to F(A)$, and for compositions of morphisms one has $F(f \circ g) = F(g) \circ F(f)$.

A typical example of this concept is the functor from the category of topological spaces to the category of abelian groups which assigns to each topological space X its ith singular homology group $H_i(X;\mathbb{Z})$. Indeed, a continuous map $f\colon X \to Y$ induces a group homomorphism $H_i(f;\mathbb{Z})\colon H_i(X;\mathbb{Z}) \to H_i(Y;\mathbb{Z})$ satisfying the axioms (C) and (D).

Other important examples are the functors Tor and Ext: let R be a ring, $\mathcal{M}_R$ the category of R-modules and $N \in \mathrm{Obj}(\mathcal{M}_R)$. Then for each integer $i \geq 0$, the assignments $\mathrm{Tor}_i^R(N,-)\colon \mathcal{M}_R \to \mathcal{M}_R$, $M \mapsto \mathrm{Tor}_i^R(N,M)$, and $\mathrm{Ext}_R^i(N,-)\colon \mathcal{M}_R \to \mathcal{M}_R$, $M \mapsto \mathrm{Ext}_R^i(N,M)$, are covariant functors, while $\mathrm{Ext}_R^i(-,N)\colon \mathcal{M}_R \to \mathcal{M}_R$, $M \mapsto \mathrm{Ext}_R^i(M,N)$, is a contravariant functor.

Special cases of these examples are the covariant functors $- \otimes_R N$ and $\mathrm{Hom}_R(N,-)$ and the contravariant functor $\mathrm{Hom}_R(-,N)$. The first of these functors is right exact, while the other two functors are left exact. Quite generally, if we have categories $\mathcal{A}$ and $\mathcal{B}$ where we can talk about exact sequences, for example in the categories $\mathcal{M}_R$, $\mathcal{G}_R$ or $\mathcal{G}$ mentioned above, we say that a functor $F\colon \mathcal{A} \to \mathcal{B}$ is **left exact** if for any exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$, the sequence $0 \to F(A) \to F(B) \to F(C)$ is exact for covariant F and $0 \to F(C) \to F(B) \to F(A)$ is exact for contravariant F. Similarly one defines right exactness. Finally F is called **exact** if F is left and right exact.

Let $\mathcal{A}$ be any one of the module categories $\mathcal{M}_R$, $\mathcal{G}_R$ or $\mathcal{G}$, and let $M \in \mathrm{Obj}(\mathcal{A})$. Then M is called **injective** if the functor $\mathrm{Hom}(-,M)$ is exact, it is called **projective** if $\mathrm{Hom}(M,-)$ is exact and it is called **flat** if $-\otimes M$ is exact. In Section 5.1 we have seen that the exterior algebra viewed as an object in $\mathcal{G}$ is injective.

Given two covariant functors $F, G\colon \mathcal{A} \to \mathcal{B}$. A family of morphisms $\eta_A\colon F(A) \to G(A)$ in $\mathcal{B}$ with $A \in \mathcal{A}$ is called a **natural transformation** from F to G, written $\eta\colon F \to G$, if for all $A, B \in \mathrm{Obj}(\mathcal{A})$ and all morphisms $f\colon A \to B$ the following diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ {\scriptstyle F(f)}\big\downarrow & & {\scriptstyle G(f)}\big\downarrow \\ F(B) & \xrightarrow{\eta_B} & G(B). \end{array}$$

is commutative.

A natural transformation $\eta\colon F \to G$ is called a **functorial isomorphism** if there exists a natural transformation $\tau\colon G \to F$ such that $\tau_A \circ \eta_A = \mathrm{id}_{F(A)}$ and $\eta_A \circ \tau_A = \mathrm{id}_{G(A)}$ for all $A \in \mathrm{Obj}(\mathcal{A})$. It is customary to call an isomorphism $\alpha\colon F(A) \to G(A)$ functorial if there exists a functorial isomorphism $\eta\colon F \to G$ such that $\alpha = \eta_A$. An example of a functorial isomorphism is the isomorphism $M^\vee \to M^*$ given in Theorem 5.1.3.

A.2 Graded free resolutions

For this and the following sections of Appendix A we fix the following assumptions and notation. We let K be a field, $(R, \mathfrak{m})$ a Noetherian local ring with residue field K or a standard graded K-algebra with graded maximal ideal $\mathfrak{m}$. As usual we write S for the polynomial ring $K[x_1, \ldots, x_n]$. We let M be a finitely generated R-module, and will assume that M is graded if R is graded.

We let $\mathcal{M}(S)$ be the category of finitely generated graded S-modules, the morphisms being the homogeneous homomorphisms $M \to N$ of degree 0, simply called homogeneous homomorphisms. A **homogeneous homomorphism** $\varphi\colon M \to N$ of graded S-modules of degree d is an S-module homomorphism such that $\varphi(M_i) \subset N_{i+d}$ for all i. For example, if $f \in S$ is homogeneous of degree d, then the multiplication map $S(-d) \to S$ with $g \mapsto fg$ is a homogeneous homomorphism. Here, for a graded S-module W and an integer a, one denotes by $W(a)$ the graded S-module whose graded components are given by $W(a)_i = W_{a+i}$. One says that $W(a)$ arises from W by applying the **shift** a.

Now let M be a finitely generated graded S-module with homogeneous generators $m_1, \ldots, m_r$ and $\deg(m_i) = a_i$ for $i = 1, \ldots, r$. Then there exists a surjective S-module homomorphism $F_0 = \bigoplus_{i=1}^r Se_i \to M$ with $e_i \mapsto m_i$. Assigning to e_i the degree a_i for $i = 1, \ldots, r$, the map $F_0 \to M$ becomes a morphism in $\mathcal{M}(S)$ and F_0 becomes isomorphic to $\bigoplus_{i=1}^r S(-a_i)$. Thus we obtain the exact sequence

$$0 \longrightarrow U \longrightarrow \bigoplus_j S(-j)^{\beta_{0j}} \longrightarrow M \longrightarrow 0,$$

where $\beta_{0j} = |\{i\colon a_i = j\}|$, and where $U = \mathrm{Ker}(\bigoplus_j S(-j)^{\beta_{0j}} \to M)$.

The module U is a graded submodule of $F_0 = \bigoplus_j S(-j)^{\beta_{0j}}$. By Hilbert's basis theorem for modules we know that U is finitely generated, and hence we find again an epimorphism $\bigoplus_j S(-j)^{\beta_{1j}} \to U$. Composing this epimorphism with the inclusion map $U \to \bigoplus_j S(-j)^{\beta_{0j}}$ we obtain the exact sequence

$$\bigoplus_j S(-j)^{\beta_{1j}} \longrightarrow \bigoplus_j S(-j)^{\beta_{0j}} \longrightarrow M \longrightarrow 0.$$

of graded S-modules. Proceeding in this way we obtain a long exact sequence

$$\mathbb{F}\colon \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

of graded S-modules with $F_i = \bigoplus_j S(-j)^{\beta_{ij}}$. Such an exact sequence is called a **graded free S-resolution** of M.

It is clear from our construction that the resolution obtained is by no means unique. On the other hand, if we choose in each step of the resolution a minimal presentation, the resolution will be unique up to an isomorphism, as we shall see now.

A set of homogeneous generators $m_1, \ldots, m_r$ of M is called **minimal** if no proper subset of it generates M.

Lemma A.2.1. *Let $m_1, \ldots, m_r$ be a homogeneous set of generators of the graded S-module M. Let $F_0 = \bigoplus_{i=1}^r Se_i$ and let $\varepsilon\colon F_0 \to M$ be the epimorphism with $e_i \mapsto m_i$ for $i = 1, \ldots, r$. Then the following conditions are equivalent:*

(a) *$m_1, \ldots, m_r$ is a minimal system of generators of M;*
(b) *$\operatorname{Ker}(\varepsilon) \subset \mathfrak{m}F_0$, where $\mathfrak{m} = (x_1, x_2, \ldots, x_n)$.*

Proof. (a) $\Rightarrow$ (b): Suppose $\operatorname{Ker}(\varepsilon) \not\subset \mathfrak{m}F_0$. Then there exists a homogeneous element $f = \sum_{i=1}^r f_i e_i$ such that $f \notin \mathfrak{m}F_0$. This implies that at least one of the coefficients f_i is of degree 0, say $\deg f_1 = 0$. Therefore $f_1 \in K \setminus \{0\}$, and it follows that

$$m_1 = f_1^{-1} f_2 m_2 + \cdots + f_1^{-1} f_r m_r,$$

a contradiction.

(b) $\Rightarrow$ (a): Suppose m_1 can be omitted, so that $m_2, \ldots, m_r$ is a system of generators of M as well. Then we have $m_1 = \sum_{i=2}^r f_i m_i$ for suitable homogeneous elements $f_i \in S$. This yields the element $f = e_1 - \sum_{i=2}^r f_i e_i$ in $\operatorname{Ker}(\varepsilon)$ with $f \notin \mathfrak{m}F_0$, a contradiction. □

Let M be a finitely generated graded S-module. A graded free S-resolution $\mathbb{F}$ of M is called **minimal** if for all i, the image of $F_{i+1} \to F_i$ is contained in $\mathfrak{m}F_i$. Lemma A.2.1 implies at once that each finitely generated graded S-module admits a minimal free resolution.

The next result shows that the numerical data given by a graded minimal free S-resolution of M depend only on M and not on the particular chosen resolution.

Proposition A.2.2. *Let M be a finitely generated graded S-module and*

$$\mathbb{F}: \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

a minimal graded free S-resolution of M with $F_i = \bigoplus_j S(-j)^{\beta_{ij}}$ for all i. Then

$$\beta_{ij} = \dim_K \operatorname{Tor}_i(K, M)_j \quad \text{for all } i \text{ and } j.$$

Proof. As a graded K-vector space $\operatorname{Tor}_i(K, M)$ is isomorphic to $H_i(\mathbb{F}/\mathfrak{m}\mathbb{F})$. However, since the resolution $\mathbb{F}$ is minimal all maps in the complex $\mathbb{F}/\mathfrak{m}\mathbb{F}$ are zero. Therefore $H_i(\mathbb{F}/\mathfrak{m}\mathbb{F}) = \mathbb{F}/\mathfrak{m}\mathbb{F} \cong \bigoplus_j K(-j)^{\beta_{ij}}$. □

The numbers $\beta_{ij} = \dim \operatorname{Tor}_i(K, M)_j$ are called the **graded Betti numbers** of M, and $\beta_i = \sum_j \beta_{ij} (= \operatorname{rank} F_i)$ is called the ***i*th Betti number** of M.

We conclude this section by showing that not only are the graded Betti numbers determined by a minimal graded free resolution but that in fact a minimal graded free resolution of M is unique up to isomorphisms.

Proposition A.2.3. *Let M be a finitely generated graded S-module and let $\mathbb{F}$ and $\mathbb{G}$ be two minimal graded free S-resolutions of M. Then the complexes $\mathbb{F}$ and $\mathbb{G}$ are isomorphic, that is, there exist isomorphisms of graded S-modules α_i: $F_i \to G_i$ such that the diagram*

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & F_i & \longrightarrow & \cdots & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_i} & & & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ \cdots & \longrightarrow & G_i & \longrightarrow & \cdots & \longrightarrow & G_1 & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

is commutative.

Proof. The existence of the isomorphism α_i will follow by induction on i once we have shown the following: let $\varphi: U \to V$ be an isomorphism of finitely generated graded S-modules, and let $\varepsilon: F \to U$ and $\eta: G \to V$ be homogeneous surjective homomorphisms with $\operatorname{Ker}(\varepsilon) \subset \mathfrak{m}F$ and $\operatorname{Ker}(\eta) \subset \mathfrak{m}G$. Then there exists a homogeneous isomorphism $\alpha: F \to G$ such that

$$\begin{array}{ccc} F & \xrightarrow{\varepsilon} & U \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \varphi} \\ G & \xrightarrow{\eta} & V \end{array}$$

is commutative. Indeed, let $f_1, \ldots, f_r$ be a homogeneous basis of F. Then

$$\varphi(\varepsilon(f_1)), \ldots, \varphi(\varepsilon(f_r))$$

is a homogeneous system of generators of V. Since η is a homogeneous surjective homomorphism, there exist homogeneous elements $g_1, \ldots, g_r \in G$ with

$\eta(g_i) = \varphi(\varepsilon(f_i))$ for $i = 1, \dots, r$. Thus if we set $\alpha(f_i) = g_i$ for $i = 1, \dots, r$, then $\alpha: F \to G$ is a homogeneous homomorphism which makes the above diagram commutative. Modulo $\mathfrak{m}$ we obtain the commutative diagram

$$\begin{array}{ccc} F/\mathfrak{m}F & \xrightarrow{\bar{\varepsilon}} & U/\mathfrak{m}U \\ {\scriptstyle\bar{\alpha}}\downarrow & & {\scriptstyle\bar{\varphi}}\downarrow \\ G/\mathfrak{m}G & \xrightarrow{\bar{\eta}} & V/\mathfrak{m}V. \end{array}$$

Since $\operatorname{Ker}(\varepsilon) \subset \mathfrak{m}F$, it follows that $\bar{\varepsilon}: F/\mathfrak{m}F \to U/\mathfrak{m}U$ is an isomorphism. Similarly, $\bar{\eta}$ and $\bar{\varphi}$ are isomorphisms. Thus $\bar{\alpha} = \bar{\eta}^{-1} \circ \bar{\varphi} \circ \bar{\varepsilon}$ is an isomorphism. Now by a homogeneous version of the Nakayama lemma it follows that α itself is an isomorphism. □

A.3 The Koszul complex

We recall the basic properties of Koszul homology that are used in this book. Let R be any commutative ring (with unit) and $\mathbf{f} = f_1, \dots, f_m$ a sequence of elements of R. The **Koszul complex** $K(\mathbf{f}; R)$ attached to the sequence $\mathbf{f}$ is defined as follows: let F be a free R-module with basis $e_1, \dots, e_m$. We let $K_j(\mathbf{f}; R)$ be the jth exterior power of F, that is, $K_j(\mathbf{f}; R) = \bigwedge^j F$. A basis of the free R-module $K_j(\mathbf{f}; R)$ is given by the wedge products $e_F = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_j}$ where $F = \{i_1 < i_2 < \cdots < i_j\}$. In particular, it follows that $\operatorname{rank} K_j(\mathbf{f}; R) = \binom{m}{j}$.

We define the differential $\partial : K_j(\mathbf{f}; R) \to K_{j-1}(\mathbf{f}; R)$ by the formula

$$\partial(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_j}) = \sum_{k=1}^{j} (-1)^{k+1} f_{i_k} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}} \wedge e_{i_{k+1}} \wedge \cdots \wedge e_{i_j}.$$

One readily verifies that $\partial \circ \partial = 0$, so that $K_{\bullet}(\mathbf{f}; R)$ is indeed a complex.

Now let M be an R-module. We define the complexes

$$K_{\bullet}(\mathbf{f}; M) = K_{\bullet}(\mathbf{f}; R) \otimes_R M \quad \text{and} \quad K^{\bullet}(\mathbf{f}; M) = \operatorname{Hom}_R(K_{\bullet}(\mathbf{f}; R), M).$$

$H_i(\mathbf{f}; M) = H_i(K(\mathbf{f}; M))$ is the ith **Koszul homology module** of $\mathbf{f}$ with respect to M, and $H^i(\mathbf{f}; M) = H^i(\operatorname{Hom}_R(K_{\bullet}(\mathbf{f}; R), M))$ is the ith **Koszul cohomology module** of $\mathbf{f}$ with respect to M.

Let $I \subset R$ be the ideal generated by $f_1, \dots, f_m$. Then

$$H_0(\mathbf{f}; M) = M/IM \quad \text{and} \quad H_m(\mathbf{f}; M) \cong 0 :_M I = \{x \in M \colon Ix = 0\}.$$

The Koszul complex $K_{\bullet}(\mathbf{f}; R)$ is a graded R-algebra, namely the exterior algebra of F, with multiplication the wedge product. We have the following rules whose verification we leave to the reader.

(i) $a \wedge b = (-1)^{\deg a \deg b} b \wedge a$ for homogeneous elements $a, b \in K(\mathbf{f}; R)$.
(ii) $\partial(a \wedge b) = \partial(a) \wedge b + (-1)^{\deg a} a \wedge \partial(b)$ for $a, b \in K(\mathbf{f}; R)$ and a homogeneous.

We denote by $Z_{\bullet}(\mathbf{f}; R)$ the cycles of the Koszul complex and by $B_{\bullet}(\mathbf{f}; R)$ its boundaries. Rule (ii) has an interesting consequence.

Proposition A.3.1. *The R-module $Z_{\bullet}(\mathbf{f}; R)$ is a graded subalgebra of $K_{\bullet}(\mathbf{f}; R)$ and $B_{\bullet}(\mathbf{f}; R) \subset Z_{\bullet}(\mathbf{f}; R)$ is a graded two-sided ideal in $Z_{\bullet}(\mathbf{f}; R)$. In particular, $H_{\bullet}(\mathbf{f}; R) = Z_{\bullet}(\mathbf{f}; R)/B_{\bullet}(\mathbf{f}; R)$ has a natural structure as graded $H_0(\mathbf{f}; R)$-algebra. Moreover, if I is the ideal generated by the sequence $\mathbf{f}$, then*

$$IH_{\bullet}(\mathbf{f}; R) = 0.$$

Proof. Let z_1 and z_2 be two homogeneous cycles. Then $\partial(z_1 \wedge z_2) = \partial(z_1) \wedge z_2 + (-1)^{\deg z_1} z_1 \wedge \partial(z_2) = 0$, since $\partial(z_1) = \partial(z_2) = 0$. So $z_1 \wedge z_2$ is again a cycle, which shows that $Z(\mathbf{f}; R)$ is a subalgebra of $K(\mathbf{f}; R)$. Now let b be a homogeneous boundary and z a cycle. There exists $a \in K(\mathbf{f}; R)$ such that $\partial(a) = b$. It then follows that $\partial(a \wedge z) = \partial(a) \wedge z + (-1)^{\deg a} a \wedge \partial(z) = b \wedge z$, which shows that $b \wedge z \in B(\mathbf{f}; R)$. Similarly, we have $z \wedge b \in B(\mathbf{f}; R)$. This shows that $B(\mathbf{f}; R)$ is indeed a two-sided ideal in $Z(\mathbf{f}; R)$. Finally, since $H(\mathbf{f}; R)$ is a $H_0(\mathbf{f}; R)$-algebra, and since $H_0(\mathbf{f}; R) = R/I$, it follows that $IH(\mathbf{f}; R) = 0$. □

Corollary A.3.2. *If $(\mathbf{f}) = R$, then $H_{\bullet}(\mathbf{f}; R) = 0$.*

Given an R-module M, then as in the preceding proof one shows that $Z_{\bullet}(\mathbf{f}; R)Z_{\bullet}(\mathbf{f}; M) \subset Z_{\bullet}(\mathbf{f}; M)$ and that $B_{\bullet}(\mathbf{f}; R)Z_{\bullet}(\mathbf{f}; M) \subset B_{\bullet}(\mathbf{f}; M)$. This then implies that $H_{\bullet}(\mathbf{f}; M)$ is a graded $H_{\bullet}(\mathbf{f}; R)$-module.

For computing the Koszul homology there are two fundamental long exact sequences of importance.

Theorem A.3.3. *Let $\mathbf{f} = f_1, \ldots, f_m$ be a sequence of elements in R, and denote by $\mathbf{g}$ the sequence $f_1, \ldots, f_{m-1}$. Furthermore, let M be an R-module and $0 \to U \to M \to N \to 0$ a short exact sequence of R-modules. Then we obtain the following long exact sequences:*

$$\begin{aligned} 0 \to H_m(\mathbf{f}; U) \to H_m(\mathbf{f}; M) \to H_m(\mathbf{f}; N) \to H_{m-1}(\mathbf{f}; U) \to H_{m-1}(\mathbf{f}; M) \to \cdots \\ \cdots \to H_{i+1}(\mathbf{f}; N) \to H_i(\mathbf{f}; U) \to H_i(\mathbf{f}; M) \to H_i(\mathbf{f}; N) \to H_{i-1}(\mathbf{f}; U) \to \cdots \\ \cdots \to H_1(\mathbf{f}; N) \to H_0(\mathbf{f}; U) \to H_0(\mathbf{f}; M) \to H_0(\mathbf{f}; N) \to 0. \end{aligned}$$

and

$$\begin{aligned} 0 \to H_m(\mathbf{f}; M) \to H_{m-1}(\mathbf{g}; M) \to H_{m-1}(\mathbf{g}; M) \to H_{m-1}(\mathbf{f}; M) \to \cdots \\ \cdots \to H_{i+1}(\mathbf{f}; M) \to H_i(\mathbf{g}; M) \to H_i(\mathbf{g}; M) \to H_i(\mathbf{f}; M) \to \cdots \\ \cdots \to H_1(\mathbf{f}; M) \to H_0(\mathbf{g}; M) \to H_0(\mathbf{g}; M) \to H_0(\mathbf{f}; M) \to 0, \end{aligned}$$

where for all i, the map $H_i(\mathbf{g}; M) \to H_i(\mathbf{g}; M)$ is multiplication by $\pm f_m$.

Proof. The short exact sequence $0 \to U \to M \to N \to 0$ induces the short exact sequence of complexes

$$0 \to K_{\bullet}(\mathbf{f}; U) \to K_{\bullet}(\mathbf{f}; M) \to K_{\bullet}(\mathbf{f}; N) \to 0$$

whose corresponding long exact sequence is the first fundamental long exact sequence.

As for the proof of the second fundamental long exact homology sequence we consider for each i the map

$$\alpha_i : K_i(\mathbf{f}; R) \longrightarrow K_{i-1}(\mathbf{g}; R)$$

defined as follows: let $a \in K_i(\mathbf{f}; R)$; then a can be uniquely written in the form $a = a_0 + a_1 \wedge e_m$ with $a_0 \in K_i(\mathbf{g}; R)$ and $a_1 \in K_{i-1}(\mathbf{g}; R)$. We then set $\alpha(a) = a_1$. Applying rule (ii) one immediately checks that $\partial \circ \alpha = \alpha \circ \partial$, so that

$$\alpha : K_{\bullet}(\mathbf{f}; R) \longrightarrow K_{\bullet}(\mathbf{g}; R)[-1],$$

where $[-1]$ denotes the shifting of the homological degree by -1.

Notice that $K(\mathbf{g}; R)$ is a subcomplex of $K(\mathbf{f}; R)$ and indeed is the kernel of α. Hence we obtain the short exact sequence of complexes

$$0 \longrightarrow K_{\bullet}(\mathbf{g}; M) \longrightarrow K_{\bullet}(\mathbf{f}; M) \longrightarrow K_{\bullet}(\mathbf{g}; M)[-1] \longrightarrow 0$$

whose corresponding long exact sequence homology sequence is the second fundamental long exact sequence.

It remains to be shown that the map $H_i(\mathbf{g}; M) \to H_i(\mathbf{g}; M)$ is multiplication by $\pm f_m$. In fact, the map is the connecting homomorphism. Thus for $[a] \in H_i(\mathbf{g}; M)$, we have to choose a preimage $b \in K_{i+1}(\mathbf{f}; M)$ under the map α for the cycle $a \in K_i(\mathbf{f}; M)$. Then the image of $[a]$ in $H_i(\mathbf{g}; M)$ is the homology class $\partial(b)$. In our case we may choose $b = a \wedge e_m$. Then $\partial(b) = \pm f_m a$, and $[a]$ maps to $\pm f_m[a]$, as desired. □

The sequence $\mathbf{f} = f_1, \ldots, f_m$ is called **regular** on M, or an M**-sequence**, if the following two conditions hold: (i) the multiplication map

$$M/(f_1, \ldots, f_{i-1})M \xrightarrow{f_i} M/(f_1, \ldots, f_{i-1})M$$

is injective for all i, and (ii) $M/(\mathbf{f})M \neq 0$.

Regular sequences can be characterized by the Koszul complex.

Theorem A.3.4. *Let $\mathbf{f} = f_1, \ldots, f_m$ be a sequence of elements of R and M an R-module.*

(a) *If $\mathbf{f}$ is an M-sequence, then $H_i(\mathbf{f}; M) = 0$ for $i > 0$.*
(b) *Suppose in addition that M is a finitely generated R-module and that R is either* (i) *a Noetherian local ring with maximal ideal $\mathfrak{m}$, or* (ii) *a graded K-algebra with graded maximal ideal $\mathfrak{m}$, and that $(\mathbf{f}) \subset \mathfrak{m}$. In case* (ii) *we also assume that $\mathbf{f}$ is a sequence of homogeneous elements. Then we have: if $H_1(\mathbf{f}; M) = 0$, then the sequence $\mathbf{f}$ is an M-sequence,*

Proof. (a) We proceed by induction on m. Let $m = 1$. We have $H_i(f_1; M) = 0$ for $i > 1$, and the exact sequence

$$0 \longrightarrow H_1(f_1; M) \longrightarrow M \xrightarrow{f_1} M \tag{A.1}$$

Since f_1 is regular on M, the kernel of the multiplication map $f_1\colon M \to M$ is zero. Hence $H_1(f_1; M) = 0$, as well.

Now let $m > 1$, and set $\mathbf{g} = f_1, \ldots, f_{m-1}$. By induction hypothesis we have $H_i(\mathbf{g}; M) = 0$ for $i > 0$. Thus the second fundamental long exact sequence yields the exact sequence

$$0 \longrightarrow H_1(\mathbf{f}; M) \longrightarrow H_0(\mathbf{g}; M) \longrightarrow H_0(\mathbf{g}; M),$$

and for each $i > 1$ the exact sequence

$$0 = H_i(\mathbf{g}; M) \longrightarrow H_i(\mathbf{f}; M) \longrightarrow H_{i-1}(\mathbf{g}; M) = 0.$$

It follows that $H_i(\mathbf{f}; M) = 0$ for $i > 1$. Since $H_0(\mathbf{g}; M) = M/(\mathbf{g})M$, we see that $H_1(\mathbf{f}; M)$ is the kernel of the multiplication map $f_m\colon M/(\mathbf{g})M \to M/(\mathbf{g})M$. Thus $H_1(\mathbf{f}; M) = 0$ as well, since f_m is regular on $M/(\mathbf{g})M$.

(b) Again we proceed by induction on m. For $m = 1$ the assertion follows from the exact sequence (A.1). Now let $m > 1$. Since $H_1(\mathbf{f}; M) = 0$ by assumption, and since $H_0(\mathbf{g}; M) = M/(\mathbf{g})M$, we deduce from the exact sequence

$$H_1(\mathbf{g}; M) \to H_1(\mathbf{g}; M) \to H_1(\mathbf{f}; M) \to H_0(\mathbf{g}; M) \to H_0(\mathbf{g}; M)$$

that f_m is regular on $M/(\mathbf{g})M$ and that $H_1(\mathbf{g}; M)/(f_m)H_1(\mathbf{g}; M) = 0$. Since $f_m \in \mathfrak{m}$, Nakayama's lemma implies that $H_1(\mathbf{g}; M) = 0$. By our induction hypothesis we then know that $\mathbf{g}$ is an M-sequence, and since f_m is regular on $M/(\mathbf{g})M$, we conclude that $\mathbf{f}$ is an M-sequence. □

Theorem A.3.4 has the following important consequence

Corollary A.3.5. *Let K be a field, $S = K[x_1, \ldots, x_n]$ the polynomial ring in n variables and M be a finitely generated graded S-module. Moreover, let $\mathbf{f} = f_1, \ldots, f_k$ be a homogeneous S-sequence. Then for each i there exists an isomorphism of graded $S/(\mathbf{f})$-modules*

$$\operatorname{Tor}_i^S(S/(\mathbf{f}), M) \cong H_i(\mathbf{f}; M).$$

In particular, for $\mathbf{x} = x_1, \ldots, x_n$ we have $\beta_{ij}(M) = \dim_K H_i(\mathbf{x}; M)_j$ and hence $\operatorname{proj}\dim M \leq n$.

Proof. We compute $\operatorname{Tor}^S(S/(\mathbf{f}), M)$ by means of a free S-resolution of $S/(\mathbf{f})$. Since $\mathbf{f}$ is an S-sequence, Theorem A.3.4 implies that the Koszul complex $K_\bullet(\mathbf{f}; S)$ provides a minimal graded free S-resolution of $S/(\mathbf{f})$, so that

$$\operatorname{Tor}_i^S(S/(\mathbf{f}), M) \cong H_i(K_\bullet(\mathbf{f}; S) \otimes M) = H_i(\mathbf{f}; M).$$

Since this isomorphism respects the grading, the desired conclusion follows. □

A.4 Depth

The **depth** of M, denoted $\operatorname{depth} M$, is the common length of a maximal M-sequence contained in $\mathfrak{m}$ (consisting of homogeneous elements if M is graded). In homological terms the depth of M is given by

$$\operatorname{depth} M = \min\{i : \operatorname{Ext}^i_R(R/\mathfrak{m}, M) \neq 0\} = \min\{i : H^i_{\mathfrak{m}}(M) \neq 0\}.$$

Here $H^i_{\mathfrak{m}}(M)$ is the ith local cohomology module of M; see A.7.

Proposition A.4.1. $\mathbf{f} = f_1, \ldots, f_m$ *be an M-sequence contained in* $\mathfrak{m}$ *(consisting of homogeneous elements if M is graded). Then* $\operatorname{depth} M/(\mathbf{f})M = \operatorname{depth} M - m$.

Proof. We may assume $m = 1$. The general case follows by an easy induction argument. Thus let f by a regular element on M. The short exact sequence

$$0 \longrightarrow M \xrightarrow{\ f\ } M \longrightarrow M/fM \longrightarrow 0$$

gives rise to the long exact sequence

$$\begin{aligned}\cdots \to &\operatorname{Ext}^{i-1}(R/\mathfrak{m}, M) \xrightarrow{\ f\ } \operatorname{Ext}^{i-1}(R/\mathfrak{m}, M) \to \operatorname{Ext}^{i-1}(R/\mathfrak{m}, M/fM)\\ \to \quad &\operatorname{Ext}^{i}(R/\mathfrak{m}, M) \longrightarrow \cdots\end{aligned}$$

Since f is in the annihilator of each $\operatorname{Ext}^i(R/\mathfrak{m}, M)$, this long exact sequence splits into the short exact sequences

$$0 \longrightarrow \operatorname{Ext}^{i-1}(R/\mathfrak{m}, M) \longrightarrow \operatorname{Ext}^{i-1}(R/\mathfrak{m}, M/fM) \longrightarrow \operatorname{Ext}^{i}(R/\mathfrak{m}, M) \longrightarrow 0.$$

Let $t = \operatorname{depth} M$. Then $\operatorname{Ext}^i(R/\mathfrak{m}, M) = 0$ for $i < t$, and the short exact sequences imply that $\operatorname{Ext}^i(R/\mathfrak{m}, M/fM) = 0$ for $i < t-1$, while for $i = t$ they yield the isomorphism

$$\operatorname{Ext}^{t-1}(R/\mathfrak{m}, M/fM) \cong \operatorname{Ext}^{t}(R/\mathfrak{m}, M). \tag{A.2}$$

This shows that $\operatorname{depth} M/fM = t - 1$, as desired. □

The depth of a module can also be characterized by Koszul homology.

Proposition A.4.2. *Let* $\mathbf{x} = x_1, \ldots, x_n$ *be a minimal system of generators of* $\mathfrak{m}$. *Then*

$$\operatorname{depth} M = n - \max\{i\colon H_i(\mathbf{x}; M) \neq 0\}.$$

Proof. We proceed by induction on the depth of M. If $\operatorname{depth} M = 0$, then $\operatorname{Hom}_R(R/\mathfrak{m}, M) \neq 0$. Hence there exists $x \in M$ such that $\mathfrak{m}x = 0$, and consequently $H_n(\mathbf{x}; M) \neq 0$.

Now let $\operatorname{depth} M = t > 0$, and let $f \in \mathfrak{m}$ be a regular element on M. Since $fH_i(\mathbf{x}; M) = 0$ for all i, the long exact sequence of Koszul homology attached with the short exact sequence $0 \to M \to M \to M/fM \to 0$ splits into the short exact sequences

$$0 \longrightarrow H_i(\mathbf{x}; M) \longrightarrow H_i(\mathbf{x}; M/fM) \longrightarrow H_{i-1}(\mathbf{x}; M) \longrightarrow 0.$$

Since by Proposition A.4.1 we have $\operatorname{depth} M/fM = t-1$, our induction hypothesis implies that $H_i(\mathbf{x}; M/fM) = 0$ for $i > n-t+1$ and that $H_i(\mathbf{x}; M/fM) \neq 0$ for $i = n-t+1$. Thus the short exact sequences of Koszul homology imply first that $H_i(\mathbf{x}; M) = 0$ for $i > n-t$, and then by choosing $i = n-t+1$ that $H_{n-t}(\mathbf{x}; M) \cong H_{n-t+1}(\mathbf{x}; M/fM) \neq 0$. □

Combining Proposition A.4.2 with Corollary A.3.5 we obtain

Corollary A.4.3 (Auslander–Buchsbaum). *Let M be a finitely generated graded $S = K[x_1, \ldots, x_n]$-module. Then*

$$\operatorname{proj\,dim} M + \operatorname{depth} M = n.$$

This is a special version of the Auslander–Buchsbaum theorem which is used several times in this book. More generally the Auslander–Buchsbaum theorem says that $\operatorname{proj\,dim} M + \operatorname{depth} M = \operatorname{depth} R$, if $\operatorname{proj\,dim} M < \infty$.

A.5 Cohen–Macaulay modules

Let M be an R-module. Since every M-sequence which is contained in $\mathfrak{m}$ is part of a system of parameters of M, it follows that $\operatorname{depth} M \leq \dim M$. The module M is said to be **Cohen–Macaulay** if $\operatorname{depth} M = \dim M$. The ring R is called a **Cohen–Macaulay ring** if R is a Cohen–Macaulay module viewed as a module over itself.

One important property of Cohen–Macaulay rings is that they are **unmixed**. In other words, $\dim R = \dim R/P$ for all $P \in \operatorname{Ass}(R)$. More generally, we have

$$\dim M = \dim R/P \quad \text{for all} \quad P \in \operatorname{Ass}(M), \tag{A.3}$$

if M is Cohen–Macaulay. This follows from the fact that $\operatorname{depth} M \leq \dim R/P$ for all $P \in \operatorname{Ass}(M)$. In particular, we see that a Cohen–Macaulay module has no embedded prime ideals, that is, all associated prime ideals of the module are minimal in its support.

An unmixed ring, however, need not be Cohen–Macaulay. For example, the ring

$$R = K[x_1, x_2, x_3, x_4]/(x_1, x_2) \cap (x_3, x_4)$$

is unmixed but not Cohen–Macaulay, since $\operatorname{depth} R = 1$, while $\dim R = 2$.

The Cohen–Macaulay property is preserved under two important module operations.

Proposition A.5.1. *Let M be a Cohen–Macaulay module, $\mathbf{f}$ an M-sequence with $(\mathbf{f}) \subset \mathfrak{m}$, and P a prime ideal in the support of M. Then $M/(\mathbf{f})M$ and M_P are again Cohen–Macaulay.*

Proof. Let $\mathbf{f} = f_1, \ldots, f_m$. Since $\mathbf{f}$ is part of a system of parameters of M, it follows that $\dim M/(\mathbf{f})M = \dim M - m$. Hence Proposition A.4.1 implies that $M/(\mathbf{f})M$ is Cohen–Macaulay.

In order to prove that M_P is Cohen–Macaulay, we use induction on $\operatorname{depth} M_P$. If $\operatorname{depth} M_P = 0$, then $P \in \operatorname{Ass}(M)$, and hence, according to (A.3), P is a minimal prime ideal of M. Thus $\dim M_P = 0$, so M_P is Cohen–Macaulay. If $\operatorname{depth} M_P > 0$, then (A.3) implies that P is not contained in any associated prime ideal of M. Thus there exists $f \in P$ which is regular on M, and one may apply the induction hypothesis to $(M/fM)_P = M_P/fM_P$ to see that $\dim M_P - 1 = \dim M_P/fM_P = \operatorname{depth} M_P/fM_P = \operatorname{depth} M_P - 1$, from which the desired conclusion follows. □

A.6 Gorenstein rings

Let M be an R-module. The **socle** of M, denoted $\operatorname{Soc}(M)$, is the submodule of M consisting of all elements $x \in M$ with $\mathfrak{m}x = 0$. Observe that $\operatorname{Soc}(M)$ has a natural structure as an $R/\mathfrak{m}$-module, and hence is a finite-dimensional K-vector space.

Proposition A.6.1. *Let M be a Cohen–Macaulay R-module of dimension d, and $\mathbf{f} = f_1, \ldots, f_d$ an M-sequence. Then*

$$\operatorname{Ext}^d_R(R/\mathfrak{m}, M) \cong \operatorname{Hom}_R(R/\mathfrak{m}, M/(\mathbf{f})M) \cong \operatorname{Soc}(M/(\mathbf{f})M).$$

In particular, $\operatorname{Ext}^d_R(R/\mathfrak{m}, M)$ *is a finite-dimensional K-vector space.*

Proof. We proceed by induction on d. If $d = 0$, we need only to observe that $\operatorname{Hom}_R(R/\mathfrak{m}, M) \cong \operatorname{Soc}(M)$. Now assume that $d > 0$. Then the isomorphism (A.2) yields $r_R(M) = r_R(M/f_1M)$. Applying our induction hypothesis to M/f_1M, the desired result follows. □

Let M be a d-dimensional Cohen–Macaulay R-module. We set

$$r_R(M) = \dim_K \operatorname{Ext}^d_R(R/\mathfrak{m}, M).$$

The number $r_R(M)$ is called the **Cohen–Macaulay type** of M. A Cohen–Macaulay ring R is called a **Gorenstein ring** if the Cohen–Macaulay type of R is one.

H. Bass [Bas62] introduced Gorenstein rings as rings which have finite injective dimension, and showed that these are exactly the Cohen–Macaulay rings whose Cohen–Macaulay type is one.

In the proof of Proposition A.6.1 we have seen that if f is regular on M, then $r_R(M) = r_R(M/fM)$. Therefore induction on the length of an M-sequence yields

Proposition A.6.2. *Let M be a Cohen–Macaulay R-module and $\mathbf{f}$ an M-sequence. Then $r_R(M) = r_R(M/(\mathbf{f})M)$. In particular, if R is a Cohen–Macaulay ring and $\mathbf{f}$ is an R-sequence, then R is Gorenstein if and only if $R/(\mathbf{f})$ is Gorenstein.*

Corollary A.6.3. *Let $R = S/I$ where $I \subset S$ is a graded ideal, and let M be a graded Cohen–Macaulay R-module. Then $r_R(M) = r_S(M)$.*

Proof. Let $d = \dim M$ and $\mathbf{f}$ an M-sequence of length $n - d$. Then Proposition A.6.2 implies that

$$\begin{aligned} r_R(M) &= \dim_K \operatorname{Hom}_R(R/\mathfrak{m}, M/(\mathbf{f})M) \\ &= \dim_K \operatorname{Hom}_S(S/(x_1, \ldots, x_n), M/(\mathbf{f})M) = r_S(M). \end{aligned}$$

□

The sequence $\mathbf{x} = x_1, \ldots, x_n$ is an S-sequence and $S/(\mathbf{x}) \cong K$. Thus it follows from Proposition A.6.2 that S is a Gorenstein ring. A standard graded K-algebra R of the form $R = S/(\mathbf{f})$ with $\mathbf{f}$ is a homogeneous S-sequence, is called a **complete intersection**. As an immediate consequence of Proposition A.6.2 and Corollary A.6.3 we obtain

Corollary A.6.4. *Let R be a complete intersection. Then R is a Gorenstein ring.*

Not every Gorenstein ring needs to be a complete intersection. Indeed, the class of Gorenstein rings is much larger than that of complete intersections. A simple example of a Gorenstein ring which is not a complete intersection is the following: let $R = S/I$ where $S = K[x_1, x_2, x_3]$ and $I = (x_1^2 - x_2^2, x_1^2 - x_2^2, x_1x_2, x_1x_3, x_2x_3)$. The ideal I is not generated by an S-sequence, because any S-sequence has length at most 3, while I is minimally generated by 5 elements. So R is not a complete intersection.

Next observe that $x_1^3 = x_1(x_1^2 - x_2^2) - x_2(x_1x_2)$, and hence $x_1^3 \in I$. Similarly, we see that $x_2^3, x_3^3 \in I$. Obviously all other monomials of degree 3 belong to I, so that $(x_1, x_2, x_3)^3 \in I$. Since the generators of I generate a 5-dimensional K-subspace of S_2, we see that $H_R(t) = 1 + 3t + t^2$. The element $x_1^2 + I$ generates the 1-dimensional K-vector space R_2, and obviously belongs to the socle of R. In order to see that R is Gorenstein it suffices therefore to show that no nonzero element $f \in R_1$ belongs to the socle of R. In fact, let $ax_1 + bx_2 + cx_3$ be a nonzero linear form in S with $a, b, c, \in K$. We may assume that $a \neq 0$. Then $x_1(ax_1 + bx_2 + cx_3) \notin I$, because $x_1^2 \notin I$ and $x_1x_2, x_1x_3 \in I$.

In contrast to this example we have the following result.

Proposition A.6.5. *Let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal such that $\dim S/I = 0$. Then S/I is Gorenstein if and only if S/I is a complete intersection. If the equivalent conditions hold, then I is generated by pure powers of the variables.*

Proof. Let $I \subset S$ be graded ideal such that S/I is a zero-dimensional Gorenstein ring. We claim that I is an irreducible ideal. In fact, let $J \subset S$ be a graded ideal which properly contains I. There exists an integer k such that $\mathfrak{m}^k J \subset I$. Let k be the smallest such integer. Then $k > 0$ and $\mathfrak{m}^{k-1} J \not\subset I$. Let $x \in \mathfrak{m}^{k-1} J \setminus I$. Since $\mathrm{Soc}(S/I) \cong S/\mathfrak{m}$, and since $x + I$ is a nonzero element in $\mathrm{Soc}(S/I)$, it follows that $\mathrm{Soc}(S/I) \subset J/I$. Therefore, I can never be the intersection of two ideals properly containing I. In other words, I is irreducible, as asserted.

Now assume that I is a monomial ideal. Then I is irreducible if and only if I is generated by pure powers of the variables; see Corollary 1.3.2. Thus all assertions follow. □

The Cohen–Macaulay type of a graded S-module has the following interesting interpretation.

Proposition A.6.6. *Let M be a Cohen–Macaulay graded S-module of dimension d. Then $r_S(M) = \beta_{n-d}(M)$. In particular, if $R = S/I$ is a Cohen–Macaulay ring of dimension d, then S/I is Gorenstein if and only if $\beta_{n-d}(R) = 1$.*

Proof. We proceed by induction on $\dim M$. If $\dim M = 0$, then $\mathrm{Soc}(M) \neq 0$. Let $\mathbf{x} = x_1, \ldots, x_n$ be the sequence of the variables of S, then $H_n(\mathbf{x}; M) \cong \mathrm{Soc}(M)$. Applying Corollary A.3.5 it follows that $r_S(M) = \dim_K H_n(\mathbf{x}; M) = \beta_n(M)$.

Now assume that $\dim M > 0$. After extending K, if necessary, we may assume that K is infinite. Then we find a linear form which is regular on M. After a change of coordinates we may assume that this linear form is x_n. Then M/x_nM is a Cohen–Macaulay S/x_nS-module of dimension $d-1$. Applying the induction hypothesis and Proposition A.6.2, we see that $\beta_{n-d}^{S/x_nS}(M/x_nM) = \beta_{(n-1)-(d-1)}^{S/x_nS}(M/x_nM) = r_S(M/x_nM) = r_S(M)$. Thus it remains to be shown that $\beta_{n-d}^{S/x_nS}(M/x_nM) = \beta_{n-d}^{S}(M)$. Actually one has, $\beta_i^{S/x_nS}(M/x_nM) = \beta_i^S(M)$ for all i, because if $\mathbb{F}$ is a graded minimal free S-resolution of M, then $\mathbb{F}/x_n\mathbb{F}$ is a free S/x_nS-resolution of M/x_nM. Indeed, $H_i(\mathbb{F}/x_n\mathbb{F})$ is isomorphic to $\mathrm{Tor}_i(S/x_nS, M)$ for all i, and $\mathrm{Tor}_i(S/x_nS, M) = 0$ for $i > 0$, since x_n is regular on M. □

Let $I \subset S$ be a graded ideal such that $R = S/I$ is a d-dimensional Cohen–Macaulay ring. Then the graded R-module

$$\omega_R = \mathrm{Ext}_S^{n-d}(R, S)$$

is called the **canonical module** of R. It can be shown that ω_R is a Cohen–Macaulay module of Cohen–Macaulay type 1; see [BH98, Chapter 3].

We denote by $\mu(N)$ the minimal number of homogeneous generators of a graded S-module. As a consequence of Proposition A.6.6 we obtain

Corollary A.6.7. *Let R be a standard graded Cohen–Macaulay ring. Then $\mu(\omega_R) = r_S(R)$. In particular, R is Gorenstein if and only if ω_R is a cyclic R-module.*

Proof. Let $\mathbb{F}$ be the minimal graded free S-resolution of R. Then

$$\omega_R = \operatorname{Coker}(F_{n-d}^* \to F_{n-d}^*).$$

Here N^* denotes the S-dual for a graded S-module N. Since F_{n-d}^* is a free S-module of the same rank as F_{n-d}, and since the image of the map $F_{n-d}^* \to F_{n-d}^*$ is contained in $\mathfrak{m}F_{n-d}^*$, it follows from Proposition A.6.6 that $\mu(\omega_R) = \mu(F_{n-d}^*) = \mu(F_{n-d}) = r_S(R)$, as desired. □

The canonical module ω_R is a faithful R-module; see [BH98, Chapter 3]. Thus Corollary A.6.7 implies that R is Gorenstein if and only if $\omega_R \cong R(a)$ for some integer a.

The results stated in Proposition A.6.6 and in Corollary A.6.7 for the ring $R = S/I$ are equally valid if we replace S by a regular local ring and define ω_R in the same was as above.

A.7 Local cohomology

We maintain our assumptions on R and M from Section A.4. We set

$$\Gamma_{\mathfrak{m}}(M) = \{x \in M\colon\ \mathfrak{m}^k x = 0 \quad \text{for some} \quad k\}.$$

$\Gamma_{\mathfrak{m}}(M)$ is largest submodule of M with support $\{\mathfrak{m}\}$. It is easily checked that $\Gamma_{\mathfrak{m}}(-)$ is a left exact additive functor. The right derived functors $H_{\mathfrak{m}}^i(-)$ of $\Gamma_{\mathfrak{m}}(-)$ are called the **local cohomology functors**. Thus if $\mathbb{I}$ is an injective resolution of M it follows that

$$\begin{aligned} H_{\mathfrak{m}}^i(M) &\cong H^i(\varinjlim \operatorname{Hom}_R(R/\mathfrak{m}^k, \mathbb{I})) \cong \varinjlim H^i(\operatorname{Hom}_r(R/\mathfrak{m}^k, \mathbb{I})) \\ &\cong \varinjlim \operatorname{Ext}_R^i(R/\mathfrak{m}^k, M). \end{aligned}$$

We quote the following fundamental vanishing theorem of Grothendieck:

Theorem A.7.1 (Grothendieck). *Let $t = \operatorname{depth} M$ and $d = \dim M$. Then $H_{\mathfrak{m}}^i(M) \neq 0$ for $i = t$ and $i = d$, and $H_{\mathfrak{m}}^i(M) = 0$ for $i < t$ and $i > d$.*

Corollary A.7.2. *M is Cohen–Macaulay if and only if $H_{\mathfrak{m}}^i(M) = 0$ for $i < \dim M$.*

In the graded case all local cohomology modules $H_{\mathfrak{m}}^i(M)$ are naturally graded R-modules and one calls the number

$$\operatorname{reg}(M) = \max\{j\colon\ H_{\mathfrak{m}}^i(M)_{j-i} \neq 0 \text{ for some } i\}$$

the **regularity** of M.

It has been shown by Eisenbud and Goto ([EG84] or [BH98, Theorem 4.3.1]) that in the case that M is a (finitely generated) graded S-module one has

$$\operatorname{reg}(M) = \max\{j\colon \operatorname{Tor}_i(K,M)_{i+j} \neq 0 \text{ for some } i\}.$$

Local cohomology can be computed by means of the **modified Čech complex**. We fix a system of elements $x_1,\ldots,x_n$ in $\mathfrak{m}$ which generates an $\mathfrak{m}$-primary ideal, and define the complex

$$\mathbb{C}\colon\ 0 \to C^0 \to C^1 \to \cdots \to C^n \to 0$$

with $C^k = \bigoplus_{1\leq i_1<i_2<\cdots<i_k\leq n} R_{x_{i_1}x_{i_2}\cdots x_{i_k}}$. The differentiation $d^k\colon\ C^k \to C^{k+1}$ is defined on the component $R_{x_{i_1}x_{i_2}\cdots x_{i_k}} \to R_{x_{j_1}x_{j_2}\cdots x_{j_{k+1}}}$ to be $(-1)^{r-1}\alpha$. Here α is natural homomorphism $R_{x_{i_1}x_{i_2}\cdots x_{i_k}} \to (R_{x_{i_1}x_{i_2}\cdots x_{i_k}})_{x_{j_r}}$, if $\{i_1,i_2,\ldots,i_k\} \subset \{j_1,\ldots,\widehat{j_r},\ldots,j_{k+1}\}$, and is the zero map otherwise.

For all i one has

$$H^i_{\mathfrak{m}}(M) = H^i(\mathbb{C}\otimes_R M). \tag{A.4}$$

We use (A.4) to compute the local cohomology of a Stanley–Reisner ring. Let Δ be a simplicial on the vertex set $[n]$ and let $R = K[x_1,\ldots,x_n]/I_\Delta$. In other words, $R = K[\Delta]$ is the Stanley–Reisner ring of Δ. We let $\mathbb{C}$ be the modified Čech complex of R with respect to the sequence $x_1,\ldots,x_n$. We note that $\mathbb{C}$ is a $\mathbb{Z}^n$-graded complex. The components of $\mathbb{C}$ are of the form R_x where x is homogeneous with respect to the $\mathbb{Z}^n$-grading. Let $\mathbf{a}\in\mathbb{Z}^n$; then we set

$$(R_x)_{\mathbf{a}} = \{\frac{r}{x^m}\colon\ r \text{ is homogeneous and } \deg r - m\deg x = \mathbf{a}\},$$

and extend this $\mathbb{Z}^n$-grading naturally to $\mathbb{C}$. This $\mathbb{Z}^n$-grading is compatible with the differentials of $\mathbb{C}$ and hence all local cohomology modules $H^i_{\mathfrak{m}}(R)$ are naturally $\mathbb{Z}^n$-graded.

Theorem A.7.3 (Hochster). *Let* $\mathbb{Z}^n_- = \{\mathbf{a}\in\mathbb{Z}^n\colon\ a_i\leq 0 \text{ for } i=1,\ldots,n\}$. *Then*

$$H^i_{\mathfrak{m}}(K[\Delta])_{\mathbf{a}} = \begin{cases} \dim_K \tilde{H}_{i-|F|-1}(\operatorname{link}_\Delta F;K), & \text{if } \mathbf{a}\in\mathbb{Z}^n_-, \text{ where } F=\operatorname{supp}\mathbf{a};\\ 0, & \text{if } \mathbf{a}\notin\mathbb{Z}^n_-.\end{cases}$$

Proof. Let F be a subset of the vertex set of Δ. The **star** of F is the set $\operatorname{star}_\Delta F = \{G\in\Delta:\ F\cup G\in\Delta\}$. Notice that $\operatorname{star}_\Delta F$ is a subcomplex of Δ. Let $\mathbf{a}\in\mathbb{Z}^n$; the $\mathbf{a}$-graded component $\mathbb{C}_{\mathbf{a}}$ of the modified Čech complex $\mathbb{C}$ of $K[\Delta]$ is a complex of finite-dimensional K-vector spaces, and there exists an isomorphism of complexes

$$\alpha:\ \mathbb{C}_{\mathbf{a}} \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\tilde{\mathcal{C}}(\operatorname{link}_{\operatorname{star} H_{\mathbf{a}}} G_{\mathbf{a}};K)[-j-1],K), \quad j=|G_{\mathbf{a}}|.$$

Here $G_{\mathbf{a}} = \{i \in [n] \colon a_i < 0\}$ and $H_{\mathbf{a}} = \{i \in [n] \colon a_i > 0\}$, and

$$\tilde{\mathcal{C}}(\mathrm{link}_{\mathrm{star}_{\mathbf{a}}} G_{\mathbf{a}}; K)[-j-1]$$

denotes the augmented oriented chain complex of $\mathrm{link}_{\mathrm{star}\, H_{\mathbf{a}}} G_{\mathbf{a}}$, homologically shifted by $-j-1$. Note that

$$\mathrm{Hom}_{\mathbb{Z}}(\tilde{\mathcal{C}}(\mathrm{link}_{\mathrm{star}\, H_{\mathbf{a}}} G_{\mathbf{a}}; K)[-j-1], K) = (K\{\mathrm{link}_{\mathrm{star}\, H_{\mathbf{a}}} G_{\mathbf{a}}\}, e)[-j],$$

see Section 5.1.4.

The map α is defined as follows: let $x = x_{i_1} \cdots x_{i_k}$ with $i_1 < i_2 < \cdots < i_k$ and set $F = \{i_1, \ldots, i_k\}$. We first observe that

$$(R_x)_{\mathbf{a}} \cong \begin{cases} K, & \text{if } G_a \subset F \text{ and } F \cup H_a \in \Delta, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that $(\mathbb{C}^i)_{\mathbf{a}}$ has a K-basis consisting of basis elements b_F indexed by $F \subset [n]$ with $|F| = i$, and such that $G_a \subset F$ and $F \cup H_a \in \Delta$. Now we let α^i be the K-linear map defined by

$$\alpha^i \colon (\mathbb{C}^i)_{\mathbf{a}} \to K\{\mathrm{link}_{\mathrm{star}\, H_{\mathbf{a}}} G_{\mathbf{a}}\}_{i-j}, \quad b_F \mapsto e_{F \setminus G_{\mathbf{a}}}.$$

Passing to homology, the map of complexes α yields the following isomorphism

$$H^i_{\mathfrak{m}}(K[\Delta])_{\mathbf{a}} \cong \tilde{H}^{i-|G_{\mathbf{a}}|-1}(\mathrm{link}_{\mathrm{star}\, H_{\mathbf{a}}} G_{\mathbf{a}}; K),$$

so that $\dim_K H^i_{\mathfrak{m}}(K[\Delta])_{\mathbf{a}} = \dim_K \tilde{H}_{i-|G_{\mathbf{a}}|-1}(\mathrm{link}_{\mathrm{star}\, H_{\mathbf{a}}} G_{\mathbf{a}}; K)$.

If $H_{\mathbf{a}} \neq \emptyset$, then $\mathrm{link}_{\mathrm{star}\, H_{\mathbf{a}}} G_{\mathbf{a}}$ is acyclic, and if $H_{\mathbf{a}} = \emptyset$, then $\mathrm{star}\, H_{\mathbf{a}} = \Delta$, so that in this case $\mathrm{link}_{\mathrm{star}\, H_{\mathbf{a}}} G_{\mathbf{a}} = \mathrm{link}_{\Delta} G_{\mathbf{a}}$. Thus the theorem follows from the fact that $H_{\mathbf{a}} = \emptyset$ if and only if $\mathbf{a} \in \mathbb{Z}^n_-$. □

A.8 The Cartan complex

We give a short introduction to the Cartan complex which for the exterior algebra plays the role of the Koszul complex for the symmetric algebra.

Let K be a field, V a K-vector space with basis $e_1, \ldots, e_n$ and E the exterior algebra of V.

Let $\mathbf{v} = v_1, \ldots, v_m$ be a sequence of elements of degree 1 in E. The **Cartan complex** $C_{\bullet}(\mathbf{v}; E)$ of the sequence $\mathbf{v}$ with values in E is defined as the complex whose i-chains $C_i(\mathbf{v}; E)$ are the elements of degree i of the free divided power algebra $E\langle x_1, \ldots, x_m\rangle$. Recall that $E\langle x_1, \ldots, x_m\rangle$ is the polynomial ring over E in the set of variables

$$x_i^{(j)}, \quad i = 1, \ldots, m, \quad j = 1, 2, \ldots$$

modulo the relations

$$x_i^{(j)} x_i^{(k)} = \binom{j+k}{j} x_i^{(j+k)}.$$

We set $x_i^{(0)} = 1$, $x_i^{(1)} = x_i$ for $i = 1, \ldots, m$ and $x_i^{(a)} = 0$ for $a < 0$.

The algebra $E\langle x_1, \ldots, x_m \rangle$ is a free E-module with basis

$$\mathbf{x}^{(\mathbf{a})} = x_1^{(a_1)} x_2^{(a_2)} \cdots x_m^{(a_m)}, \quad \mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{Z}_+^m.$$

We say that $\mathbf{x}^{(\mathbf{a})}$ has degree i if $|\mathbf{a}| = i$ where $|\mathbf{a}| = a_1 + \cdots + a_m$. Thus $C_i(\mathbf{v}; E) = \bigoplus_{|a|=i} E\mathbf{x}^{(a)}$.

The E-linear differential ∂ on $C_{\bullet}(\mathbf{v}; E)$ is defined a follows: for $\mathbf{x}^{(\mathbf{a})} = x_1^{(a_1)} \cdots x_m^{(a_m)}$ we set

$$\partial(\mathbf{x}^{(\mathbf{a})}) = \sum_{i=1}^{m} v_i x_1^{(a_1)} \cdots x_i^{(a_i - 1)} \cdots x_m^{(a_m)}.$$

One readily checks that $\partial \circ \partial = 0$, so that $(C_{\bullet}(\mathbf{v}; E), \partial)$ is indeed a complex. Moreover,

$$\partial(g_1 g_2) = g_1 \partial(g_2) + \partial(g_1) g_2 \tag{A.5}$$

for any two homogeneous elements g_1 and g_2 in $C_{\bullet}(\mathbf{v}; E)$.

Let $\mathcal{G}$ be the category of graded E-modules (in the sense of Definition 5.1.1), and let $M \in \mathcal{G}$. We define the complex

$$C_{\bullet}(\mathbf{v}; M) = M \otimes_E C_{\bullet}(\mathbf{v}; E),$$

and set $H_i(\mathbf{v}; M) = H_i(C_{\bullet}(\mathbf{v}; M))$. We call $H_i(\mathbf{v}; M)$ the ith **Cartan homology module** of $\mathbf{v}$ with respect to M. Note that each $H_i(\mathbf{v}; M)$ is a naturally graded E-module.

Proposition A.8.1. *Let $J \subset E$ be the ideal generated by the sequence $\mathbf{v} = v_1, \ldots, v_m$. Then $JH_{\bullet}(\mathbf{v}; M) = 0$.*

One good reason to consider the Cartan complex is the following result:

Theorem A.8.2. *For any graded E-module M and each $i \geq 0$ there is a natural isomorphism*

$$\operatorname{Tor}_i^E(M, K) \cong H_i(e_1, \ldots, e_n; M)$$

of graded E-modules.

For the proof of the theorem it suffices to show that $C_{\bullet}(e_1, \ldots, e_n; E)$ is acyclic with $H_0(e_1, \ldots, e_n; E) = K$. This will easily be implied by the next results.

Proposition A.8.3. *Let M be a graded E-module, $\mathbf{v} = v_1, \ldots, v_m$ a sequence of elements in E_1 and $\mathbf{v}'$ the sequence $v_1, \ldots, v_{m-1}$. Then there exists an exact sequence*

$$0 \longrightarrow C_\bullet(\mathbf{v}'; M) \xrightarrow{\iota} C_\bullet(\mathbf{v}; M) \xrightarrow{\tau} C_\bullet(\mathbf{v}; M)[-1] \longrightarrow 0$$

of complexes. Here ι is the natural inclusion map, while τ is defined by the formula

$$\tau(c_0 + c_1 x_m + \cdots + c_k x_m^{(k)}) = c_1 + c_2 x_m + \cdots + c_k x_m^{(k-1)},$$

where the c_i belong to $C_{k-i}(\mathbf{v}'; M)$.

The proof is straightforward and is left to the reader.

Corollary A.8.4. *There exists a long exact homology sequence*

$$\cdots \longrightarrow H_i(\mathbf{v}'; M) \xrightarrow{\alpha_i} H_i(\mathbf{v}; M) \xrightarrow{\beta_i} H_{i-1}(\mathbf{v}; M)(-1)$$
$$\xrightarrow{\delta_{i-1}} H_{i-1}(\mathbf{v}'; M) \longrightarrow H_{i-1}(\mathbf{v}; M) \longrightarrow \cdots$$

of graded E-modules, where α_i is induced by the inclusion map ι, β_i by τ, and δ_{i-1} is the connecting homomorphism. If $z = c_0 + c_1 x_m + \cdots + c_{i-1} x_m^{(i-1)}$ is a cycle in $C_{i-1}(\mathbf{v}; M)$, then $\delta_{i-1}([z]) = [c_0 v_m]$.

We are now in a position to complete the proof of Theorem A.8.2 by showing that $C_\bullet(e_1, \ldots, e_n; E)$ is indeed acyclic: we show by induction on j that $H_i(e_1, \ldots, e_j; E) = 0$ for all $i > 0$. The assertion is clear for $j = 1$, since $C_\bullet(e_1; E)$ is the complex

$$\cdots \longrightarrow E x_1^{(2)} \xrightarrow{e_1} E x_1 \xrightarrow{e_1} E \longrightarrow 0,$$

and since the annihilator of e_1 in E is the ideal (e_1).

We now assume that the assertion is already proved for j, let $\mathbf{v} = e_1, \ldots, e_{j+1}$ and $\mathbf{v}' = e_1, \ldots, e_j$, and consider the long exact sequence

$$\cdots \longrightarrow H_i(\mathbf{v}'; E) \longrightarrow H_i(\mathbf{v}; E) \longrightarrow H_{i-1}(\mathbf{v}; E)(-1)$$
$$\longrightarrow H_{i-1}(\mathbf{v}'; E) \longrightarrow \cdots$$

We show by induction on i that $H_i(\mathbf{v}; E) = 0$ for $i > 0$. By our induction hypothesis (induction on j) we have $H_1(\mathbf{v}'; E) = 0$. Therefore we obtain the short exact sequence

$$0 \longrightarrow H_1(\mathbf{v}; E) \longrightarrow E/(\mathbf{v})(-1) \xrightarrow{\delta_0} E/(\mathbf{v}')$$

Here δ maps the residue class of 1 in $E/(\mathbf{v})$ to the residue class of e_{j+1} in $E/(\mathbf{v}')$. Since the annihilator of e_{j+1} in $E/(\mathbf{v}')$ is generated by e_{j+1}, it follows from this sequence that $H_1(\mathbf{v}; E) = 0$.

Suppose now that $i > 1$. Our induction hypothesis (induction on j) and the above exact sequence yields

$$H_i(\mathbf{v}; E) \cong H_{i-1}(\mathbf{v}; E).$$

Applying the induction hypothesis (induction on i) we see that $H_i(\mathbf{v}; E) = 0$, as desired.

Let again $M \in \mathcal{G}$. The **Cartan cohomology** with respect to the sequence $\mathbf{v} = v_1, \ldots, v_m$ is defined to be the homology of the cocomplex $C^\bullet(\mathbf{v}; M) = {}^*\mathrm{Hom}_E(C_\bullet(\mathbf{v}; E), M)$. Explicitly, we have

$$C^\bullet(\mathbf{v}; M) : 0 \xrightarrow{\partial^0} C^0(\mathbf{v}; M) \xrightarrow{\partial^1} C^1(\mathbf{v}; M) \longrightarrow \ldots,$$

where the cochains $C^\bullet(\mathbf{v}; M)$ and the differential ∂ can be described as follows: the elements of $C^i(\mathbf{v}; M)$ may be identified with all homogeneous polynomials $\sum_a m_{\mathbf{a}} \mathbf{y}^{\mathbf{a}}$ of degree i in the variables $y_1, \ldots, y_m$ with coefficients $m_{\mathbf{a}} \in M$, where as usual for $\mathbf{a} \in \mathbb{Z}_+^n$, $\mathbf{y}^{\mathbf{a}}$ denotes the monomial $y_1^{a_1} y_2^{a_2} \cdots y_n^{a_n}$. The element $m_{\mathbf{a}} \mathbf{y}^{\mathbf{a}} \in C^\bullet(\mathbf{v}; M)$ is defined by the mapping property

$$m_{\mathbf{a}} \mathbf{y}^{\mathbf{a}}(\mathbf{x}^{(\mathbf{b})}) = \begin{cases} m_{\mathbf{a}} & \text{for} \quad \mathbf{b} = \mathbf{a}, \\ 0 & \text{for} \quad \mathbf{b} \neq \mathbf{a}. \end{cases}$$

After this identification ∂ is simply multiplication by the element $y_{\mathbf{v}} = \sum_{i=1}^n v_i y_i$. In other words, we have

$$\partial^i \colon C^i(\mathbf{v}; M) \longrightarrow C^{i+1}(\mathbf{v}; M), \quad f \mapsto y_{\mathbf{v}} f.$$

In particular we see that $C^\bullet(\mathbf{v}; E)$ may be identified with the polynomial ring $E[y_1, \ldots, y_m]$ over E, and that $C^\bullet(\mathbf{v}; M)$ is a finitely generated $C^\bullet(\mathbf{v}; E)$-module. It is obvious that cocycles and coboundaries of $C^\bullet(\mathbf{v}; M)$ are $E[y_1, \ldots, y_m]$-submodules of $C^\bullet(\mathbf{v}; M)$. As $E[y_1, \ldots, y_m]$ is Noetherian, it follows that the Cartan cohomology $H^\bullet(\mathbf{v}; M)$ of M is a finitely generated $E[y_1, \ldots, y_m]$-module.

Let $J \subset E$ be the ideal generated by $\mathbf{v}$. Then $JH^\bullet(\mathbf{v}; M) = 0$, and hence $H^\bullet(\mathbf{v}; M)$ is in fact an $(E/J)[y_1, \ldots, y_m]$-module. Viewing $(E/J)[y_1, \ldots, y_m]$ a standard graded E/J-algebra, the Cartan cohomology module $H^\bullet(\mathbf{v}; M)$ is a finitely generated graded $(E/J)[y_1, \ldots, y_m]$-module whose ith graded component is $H^i(\mathbf{v}; M)$ for $i \geq 0$, Notice that each $H^i(\mathbf{v}; M)$ itself is a graded E/J-module, so that $H^\bullet(\mathbf{v}; M)$ is a bigraded $(E/J)[y_1, \ldots, y_m]$-module with each y_i of bidegree $(-1, 1)$.

As in Chapter 5 we set $M^\vee = {}^*\mathrm{Hom}_E(M, E)$. Cartan homology and cohomology are related as follows:

Proposition A.8.5. *Let $M \in \mathcal{G}$. Then*

$$H_i(\mathbf{v}; M)^\vee \cong H^i(\mathbf{v}; M^\vee) \quad \text{for all} \quad i.$$

Proof. Since E is injective as shown in Corollary 5.1.4, the functor $(-)^\vee$ commutes with homology and we obtain

$$H_i(\mathbf{v};M)^\vee \cong H^i({}^*\mathrm{Hom}_E(C_i(\mathbf{v};M),E)) \cong \\ H^i({}^*\mathrm{Hom}_E(C_i(\mathbf{v};E),M^\vee) \cong H^i(\mathbf{v};M^\vee).$$

□

By applying the functor ${}^*\mathrm{Hom}_E(-,M)$ to the short exact sequence of complexes in Proposition A.8.3 (with $M = E$) we obtain the short exact sequence of cocomplexes

$$0 \longrightarrow C^\bullet(\mathbf{v};M)[-1] \longrightarrow C^\bullet(\mathbf{v};M) \longrightarrow C^\bullet(\mathbf{v}';M) \longrightarrow 0,$$

from which we deduce

Proposition A.8.6. *Let $M \in \mathcal{G}$. Then with $\mathbf{v}$ and $\mathbf{v}'$ as in A.8.3 there exists a long exact sequence of graded E-modules*

$$\cdots \longrightarrow H^{i-1}(\mathbf{v};M) \longrightarrow H^{i-1}(\mathbf{v}';M) \longrightarrow H^{i-1}(\mathbf{v};M) \\ \xrightarrow{y_m} H^i(\mathbf{v};M)(-1) \longrightarrow H^i(\mathbf{v}';M) \longrightarrow \cdots$$

Proof. We show only that the map

$$H^{i-1}(\mathbf{v};M) \to H^i(\mathbf{v};M)(-1),$$

which is the dual of β_i, is indeed multiplication by y_m. We show this on the level of cochains. In order to simplify notation we set $C_i = C_i(v_1,\dots,v_m;E)$ for all i, and let

$$\gamma\colon {}^*\mathrm{Hom}_E(C_{i-1},M) \to {}^*\mathrm{Hom}_E(C_i,M)$$

be the map induced by $\tau\colon C_i \to C_{i-1}$, where

$$\tau(x^{(\mathbf{b})}) = \begin{cases} x_1^{(b_1)}\cdots x_m^{(b_m-1)}, & \text{if } b_m > 0,\\ 0, & \text{otherwise.}\end{cases}$$

Our assertion is that γ is multiplication by y_m.

For all $x^{(\mathbf{b})} \in C_i$ and $ny^{\mathbf{a}} \in {}^*\mathrm{Hom}_E(C_{i-1},M)$ with $n \in M$ we have $\gamma(ny^{\mathbf{a}})(x^{(\mathbf{b})}) = ny^{\mathbf{a}}(\tau(x^{(\mathbf{b})}))$. This implies that

$$\gamma(ny^{\mathbf{a}})(x^{(\mathbf{b})}) = \begin{cases} n, & \text{if } (b_1,\dots,b_m) = (a_1,\dots,a_m+1),\\ 0, & \text{otherwise.}\end{cases}$$

Hence we see that $\gamma(ny^{\mathbf{a}}) = ny^{\mathbf{a}}y_m$, as desired. □

The next proposition shows that for a generic basis $v_1,\dots,v_n$ of E_1, the y_i act as generic linear forms on $H^\bullet(v_1,\dots,v_n;M)$. We fix a basis $\mathbf{e} = e_1,\dots,e_n$ of E_1. Then we have

Proposition A.8.7. *Let* $\mathbf{v} = v_1, \ldots, v_n$ *be a* K*-basis of* E_1 *with* $v_j = \sum_{i=1}^{n} a_{ij} e_i$ *for* $j = 1, \ldots, n$*. Then there exists an isomorphism of graded* K*-vector spaces*

$$\varphi\colon H^\bullet(\mathbf{e}; M) \to H^\bullet(\mathbf{v}; M)$$

such that

$$\varphi(fc) = \alpha(f)\varphi(c)$$

for all $f \in K[y_1, \ldots, y_n]$ *and all* $c \in H^\bullet(\mathbf{v}; M)$*. Here* $\alpha : K[y_1, \ldots, y_n] \to K[y_1, \ldots, y_n]$ *is the* K*-algebra automorphism with* $\alpha(y_j) = \sum_{i=1}^{n} a_{ji} y_i$ *for* $j = 1, \ldots, n$*.*

Proof. Let $\beta\colon E[y_1, \ldots, y_n] \to E[y_1, \ldots, y_n]$ be the linear E-algebra automorphism deduced from α by the base ring extension E/K. Then $\beta(y_{\mathbf{e}}) = y_{\mathbf{v}}$, so β induces a complex isomorphism

$$C^\bullet(\mathbf{e}; M) \longrightarrow C^\bullet(\mathbf{v}; M), \quad g(y_1, \ldots, y_m) \mapsto g(\alpha(y_1), \ldots, \alpha(y_n)),$$

which induces the graded isomorphism $\varphi\colon H^\bullet(\mathbf{e}; M) \to H^\bullet(\mathbf{v}; M)$ with the desired properties. □

The proposition shows that if we identify $H^\bullet(\mathbf{v}; M)$ with $H^\bullet(\mathbf{e}; M)$ via the isomorphism φ, then multiplication by y_i has to be identified with multiplication by $\alpha^{-1}(y_i)$.

B

Geometry

B.1 Convex polytopes

We briefly summarize fundamental facts on convex polytopes. All proofs will be omitted. We refer the reader to Grünbaum [Gru03] for detailed information about convex polytopes.

A nonempty subset X in $\mathbb{R}^n$ is called **convex** if for each $\mathbf{x}$ and for each $\mathbf{y}$ belonging to X the line segment

$$\{t\mathbf{x} + (1-t)\mathbf{y} : t \in \mathbb{R},\ 0 \leq t \leq 1\}$$

joining $\mathbf{x}$ and $\mathbf{y}$ is contained in X. If $X \subset \mathbb{R}^n$ is convex, then for each finite subset $\{\alpha_1, \ldots, \alpha_s\}$ of X its **convex combination** $\sum_{i=1}^{s} a_i\alpha_i$, where each $a_i \in \mathbb{R}$ with $0 \leq a_i \leq 1$ and where $\sum_{i=1}^{s} a_i = 1$, belongs to X.

Given a nonempty subset Y in $\mathbb{R}^n$, there exists a smallest convex set X in $\mathbb{R}^n$ with $Y \subset X$. To see why this is true, write $\mathcal{A} = \{X_\lambda\}_{\lambda \in \Lambda}$ for the family of all convex sets X_λ in $\mathbb{R}^n$ with $Y \subset X_\lambda$. Clearly $\mathcal{A}$ is nonempty since $\mathbb{R}^n \in \mathcal{A}$. Since each X_λ is convex with $Y \subset X_\lambda$, the intersection $X = \bigcap_{\lambda \in \Lambda} X_\lambda$ is again a convex set which contains Y. Since $X \in \mathcal{A}$ and since $X \subset X_\lambda$ for each $\lambda \in \Lambda$, it follows that X is a smallest convex set in $\mathbb{R}^n$ with $Y \subset X$, as desired.

The notation $\mathrm{Conv}(Y)$ stands for the smallest convex set which contains Y and is called the **convex hull** of Y.

It follows that the convex hull $\mathrm{Conv}(Y)$ of a subset $Y \subset \mathbb{R}^n$ consists of all convex combinations of finite subsets of Y. In other words,

$$\mathrm{Conv}(Y) = \{\sum_{i=1}^{s} a_i\alpha_i : \alpha_i \in Y,\ a_i \in \mathbb{R},\ 0 \leq a_i \leq 1,\ \sum_{i=1}^{s} a_i = 1,\ s \geq 1\}.$$

Definition B.1.1. A **convex polytope** in $\mathbb{R}^n$ is the convex hull of a finite set in $\mathbb{R}^n$.

Recall that a **hyperplane** in $\mathbb{R}^n$ is a subset $\mathcal{H} \subset \mathbb{R}^n$ of the form

J. Herzog, T. Hibi, *Monomial Ideals*, Graduate Texts in Mathematics 260, DOI 10.1007/978-0-85729-106-6,

$$\mathcal{H} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} a_i x_i = b\},$$

where each $a_i \in \mathbb{R}$, $b \in \mathbb{R}$ and $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$. Every hyperplane $\mathcal{H} \subset \mathbb{R}^n$ determines the following two closed half-spaces in $\mathbb{R}^n$:

$$\mathcal{H}^{(+)} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} a_i x_i \geq b\};$$

$$\mathcal{H}^{(-)} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} a_i x_i \leq b\}.$$

Let $\mathcal{P} \subset \mathbb{R}^n$ be a convex polytope. A hyperplane $\mathcal{H} \subset \mathbb{R}^n$ is called a **supporting hyperplane** of $\mathcal{P}$ if the following conditions are satisfied:

- Either $\mathcal{P} \subset \mathcal{H}^{(+)}$ or $\mathcal{P} \subset \mathcal{H}^{(-)}$;
- $\emptyset \neq \mathcal{P} \cap \mathcal{H} \neq \mathcal{P}$.

Definition B.1.2. A **face** of a convex polytope $\mathcal{P} \subset \mathbb{R}^n$ is a subset of $\mathcal{P}$ of the form $\mathcal{P} \cap \mathcal{H}$, where $\mathcal{H}$ is a supporting hyperplane of $\mathcal{P}$.

Theorem B.1.3. *A convex polytope $\mathcal{P} \subset \mathbb{R}^n$ has only a finite number of faces, and each face of $\mathcal{P}$ is again a convex polytope in $\mathbb{R}^n$.*

Theorem B.1.4. (a) *If $\mathcal{F}$ is a face of a convex polytope $\mathcal{P} \subset \mathbb{R}^n$ and if $\mathcal{F}'$ is a face of $\mathcal{F}$, then $\mathcal{F}'$ is a face of $\mathcal{P}$.*

(b) *If $\mathcal{F}$ and $\mathcal{F}'$ are faces of a convex polytope $\mathcal{P} \subset \mathbb{R}^n$ and if $\mathcal{F} \cap \mathcal{F}' \neq \emptyset$, then $\mathcal{F} \cap \mathcal{F}'$ is a face of $\mathcal{P}$.*

A **vertex** of a convex polytope $\mathcal{P} \subset \mathbb{R}^n$ is a point $\alpha \in \mathcal{P}$ for which the singleton $\{\alpha\}$ is a face of $\mathcal{P}$. Let $V = \{\alpha_1, \ldots, \alpha_s\}$ denote the set of vertices of $\mathcal{P}$. Write $\mathcal{P} - \alpha_i$ for the subset $\{\mathbf{x} - \alpha_i : \mathbf{x} \in \mathcal{P}\} \subset \mathbb{R}^n$. The **dimension** $\dim \mathcal{P}$ of $\mathcal{P}$ is the dimension of the vector subspace in $\mathbb{R}^n$ spanned by $\mathcal{P} - \alpha_i$, which is independent of the particular choice of α_i. The dimension of a face $\mathcal{F}$ of $\mathcal{P}$ is the dimension of $\mathcal{F}$ as a convex polytope in $\mathbb{R}^n$. An **edge** of $\mathcal{P}$ is a face of $\mathcal{P}$ of dimension 1. A **facet** $\mathcal{P}$ is a face of $\mathcal{P}$ of dimension $\dim \mathcal{P} - 1$.

Theorem B.1.5. *Let V denote the set of vertices of a convex polytope $\mathcal{P} \subset \mathbb{R}^n$. Then*

(i) $\mathcal{P} = \mathrm{Conv}(V)$*;*
(ii) *If $\mathcal{F}$ is a face of $\mathcal{P}$, then $\mathcal{F} = \mathrm{Conv}(V \cap \mathcal{F})$. In particular the vertex set of $\mathcal{F}$ is $V \cap \mathcal{F}$.*

Theorem B.1.6. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_q$ denote the facets of a convex polytope $\mathcal{P} \subset \mathbb{R}^n$ and $\mathcal{H}_j \subset \mathbb{R}^n$ a supporting hyperplane of $\mathcal{P}$ with $\mathcal{F}_j = \mathcal{P} \cap \mathcal{H}_j$ and with $\mathcal{P} \subset \mathcal{H}_j^{(+)}$. Then*

$$\mathcal{P} = \bigcap_{j=1}^{q} \mathcal{H}_j^{(+)}.$$

Conversely,

Theorem B.1.7. *Let $\mathcal{H}_1, \ldots, \mathcal{H}_q$ be hyperplanes in $\mathbb{R}^n$ and suppose that $\mathcal{P} = \bigcap_{j=1}^{q} \mathcal{H}_j^{(+)}$ is a nonempty subset in $\mathbb{R}^n$. If $\mathcal{P}$ is bounded, then $\mathcal{P}$ is a convex polytope in $\mathbb{R}^n$. Moreover, if the decomposition $\bigcap_{j=1}^{q} \mathcal{H}_j^{(+)}$ is irredundant, then $\mathcal{P} \cap \mathcal{H}_1, \ldots, \mathcal{P} \cap \mathcal{H}_q$ are the facets of $\mathcal{P}$.*

B.2 Linear programming

Fix positive integers n and m and let $A = (a_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le m}}$ be $n \times m$ matrix with each $a_{ij} \in \mathbb{R}$. The notation $A^\top$ stands for the transpose of A. Let $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{c} \in \mathbb{R}^m$. As in Chapter 11, for vectors $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ belonging to $\mathbb{R}^n$, we write $\mathbf{u} \le \mathbf{v}$ if all component $v_i - u_i$ are nonnegative.

A **linear programming** is the problem stated as follows: Maximize the **objective function**

$$\mathbf{c}\,\mathbf{x}^\top$$

for $\mathbf{x} \in \mathbb{R}^m$ subject to the condition

$$A\mathbf{x}^\top \le \mathbf{b}^\top, \quad \mathbf{x} \ge 0. \tag{B.1}$$

Its **dual linear programming** is the problem stated as follows: minimize the objective function

$$\mathbf{b}\,\mathbf{y}^\top$$

for $\mathbf{y} \in \mathbb{R}^n$ subject to the condition

$$A^\top \mathbf{y}^\top \ge \mathbf{c}^\top, \quad \mathbf{y} \ge 0. \tag{B.2}$$

A vector $\mathbf{x} \in \mathbb{R}^m$ satisfying (B.1) is called a **feasible solution**. Similarly, a vector $\mathbf{y} \in \mathbb{R}^m$ satisfying (B.2) is called a **feasible dual solution**. A feasible solution which maximizes $\mathbf{cx}^\top$ is called an **optimal solution**, and a feasible dual solution which minimizes $\mathbf{by}^\top$ is called an **optimal dual solution**.

Theorem B.2.1 (Duality Theorem). *If $\mathbf{x}$ is a feasible solution and $\mathbf{y}$ is a feasible dual solution, then*

$$\mathbf{c}\,\mathbf{x}^\top \le \mathbf{b}\,\mathbf{y}^\top.$$

We now come to the results which characterize vertices of convex polytopes in the language of linear programming.

Theorem B.2.2. (a) *Let $\mathcal{P} \subset \mathbb{R}^n$ be a convex polytope. Then for any $c \in \mathbb{R}^n$ there is a vertex α of $\mathcal{P}$ which maximizes $\mathbf{c}\,\mathbf{x}^\top$, where $\mathbf{x}$ runs over $\mathcal{P}$.*

(b) *Let α be a vertex of a convex polytope $\mathcal{P} \subset \mathbb{R}^n$. Then there exists a vector $\mathbf{c} \in \mathbb{R}^n$ such that α is a unique member of $\mathcal{P}$ maximizing $\mathbf{c}\,\mathbf{x}^\top$, where x runs over $\mathcal{P}$.*

A standard reference on linear programming and integer programming is Schrijver [Sch98].

B.3 Vertices of polymatroids

We now come to the problem of finding the vertices of a polymatroid. Let $\mathcal{P} \subset \mathbb{R}^n_+$ be a polymatroid on the ground set $[n]$ and ρ its ground set rank function. Recall from Theorem 12.1.3 (a) that

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{x}(A) \leq \rho(A), A \subset [n]\}.$$

Thus in particular Theorem B.1.7 guarantees that $\mathcal{P}$ is a convex polytope in $\mathbb{R}^n$.

Given a permutation $\pi = (i_1, \ldots, i_n)$ of $[n]$, we set $A^1_\pi = \{i_1\}$, $A^2_\pi = \{i_1, i_2\}$, $\ldots$, $A^n_\pi = \{i_1, \ldots, i_n\}$. Let $\mathbf{v}(k, \pi) = (v_1, \ldots, v_n)$, where $k \in [n]$ and where

$$\begin{aligned} v_{i_1} &= \rho(A^1_\pi), \\ v_{i_2} &= \rho(A^2_\pi) - \rho(A^1_\pi), \\ v_{i_3} &= \rho(A^3_\pi) - \rho(A^2_\pi), \\ &\cdots \\ v_{i_k} &= \rho(A^k_\pi) - \rho(A^{k-1}_\pi), \\ v_{i_{k+1}} = v_{i_{k+2}} &= \cdots = v_{i_n} = 0. \end{aligned}$$

Lemma B.3.1. *One has* $\mathbf{v}(k, \pi) \in \mathcal{P}$.

Proof. Let $\mathbf{v} = \mathbf{v}(k, \pi)$ and $A \subset [n]$. What we must prove is $\mathbf{v}(A) \leq \rho(A)$. Since ρ is nondecreasing, it may be assumed that $A \subset \{i_1, \ldots, i_k\}$. Let $j \in [n]$ denote the biggest integer for which $i_j \in A$. By using induction on $|A|$, one has $\mathbf{v}(A \setminus \{i_j\}) \leq \rho(A \setminus \{i_j\})$. Since

$$\mathbf{v}(A) = \mathbf{v}(A \setminus \{i_j\}) + v(i_j) \leq \rho(A \setminus \{i_j\}) + \rho(A^j_\pi) - \rho(A^{j-1}_\pi)$$

and since

$$\rho(A \setminus \{i_j\}) + \rho(A^j_\pi) \leq \rho(A) + \rho(A^{j-1}_\pi),$$

one has $\mathbf{v}(A) \leq \rho(A)$, as desired. □

Lemma B.3.2. *Each point* $\mathbf{v}(k, \pi) \in \mathcal{P}$ *is a vertex of* $\mathcal{P}$.

Proof. Let $\mathcal{H}_j$ denote the hyperplane in $\mathbb{R}^n$ consisting of all points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ with

$$x_{i_1} + x_{i_2} + \cdots + x_{i_j} = \rho(A^j_\pi),$$

where $1 \leq j \leq k$. Let $\mathcal{H}'_j$ denote the hyperplane in $\mathbb{R}^n$ consisting of all points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ with

$$x_{i_j} = 0,$$

where $k + 1 \leq j \leq n$. One has $\mathcal{P} \subset \mathcal{H}^{(-)}_j$ and $\mathbf{v}(k, \pi) \in \mathcal{H}_j$ for all j. In other words, each hyperplane $\mathcal{H}_j$ is a supporting hyperplane of $\mathcal{P}$

with $\mathbf{v}(k,\pi) \in \mathcal{H}_j$. In addition, each hyperplane $\mathcal{H}'_j$ is a supporting hyperplane of $\mathcal{P}$ with $\mathbf{v}(k,\pi) \in \mathcal{H}'_j$. Hence Theorem B.1.4 (b) guarantees that $\mathcal{P} \cap (\bigcap_{j=1}^{k} \mathcal{H}_j) \cap (\bigcap_{j=k+1}^{n} \mathcal{H}'_j)$ is a face of $\mathcal{P}$. It is clear that $(\bigcap_{j=1}^{k} \mathcal{H}_j) \cap (\bigcap_{j=k+1}^{n} \mathcal{H}'_j) = \{\mathbf{v}(k,\pi)\}$. Hence $\mathbf{v}(k,\pi) \in \mathcal{P}$ is a vertex of $\mathcal{P}$, as required. □

We are now in the position to complete a proof of Theorem 12.1.4. For a vector $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$ we define a permutation $\pi = (i_1, \ldots, i_n)$ of $[n]$ such that

$$c_{i_1} \geq c_{i_2} \geq \cdots \geq c_{i_k} > 0 \geq c_{i_{k+1}} \geq \cdots \geq c_{i_n}$$

and consider the linear programming $(L_{\mathbf{c}})$ as follows:

$$\text{Maximize} \qquad \mathbf{c}\,\mathbf{x}^{\top}$$

subject to

$$\mathbf{x} \in \mathcal{P}.$$

Lemma B.3.2 guarantees that $\mathbf{v}(k,\pi)$ is a feasible solution of $(\alpha_{\mathbf{c}})$. We prepare the $2^n - 1$ variables y_A, where $\emptyset \neq A \subset [n]$, and consider the linear programming $(L^*_{\mathbf{c}})$ as follows:

$$\text{Minimize} \qquad \sum_{\emptyset \neq A \subset [n]} \rho(A) y_A$$

subject to

$$\sum_{j \in A} y_A \geq c_j, \quad j = 1, \ldots, n$$
$$y_A \geq 0, \quad \emptyset \neq A \subset [n].$$

We then introduce the point $\mathbf{y}^* = (\mathbf{y}^*_A)_{\emptyset \neq A \subset [n]}$ defined by setting

$$\begin{aligned} \mathbf{y}^*_{A_\pi^k} &= c_{i_k}; \\ \mathbf{y}^*_{A_\pi^j} &= c_{i_j} - c_{i_{j+1}}, \quad j = 1, \ldots, k-1; \\ \mathbf{y}^*_A &= 0 \quad \text{otherwise.} \end{aligned}$$

Lemma B.3.3. *The linear programming $(L^*_{\mathbf{c}})$ is the dual linear programming of $(L_{\mathbf{c}})$ with $\mathbf{y}^* = (\mathbf{y}^*_A)_{\emptyset \neq A \subset [n]}$ a feasible dual solution. Moreover, one has*

$$\mathbf{c}\,\mathbf{v}(k,\pi)^{\top} = \sum_{\emptyset \neq A \subset [n]} \rho(A) \mathbf{y}^*_A.$$

Proof. It is clear that $(L^*_{\mathbf{c}})$ is the dual linear programming of $(L_{\mathbf{c}})$ with $\mathbf{y}^* = (\mathbf{y}^*_A)_{\emptyset \neq A \subset [n]}$ a feasible dual solution. Let $\mathbf{v}(k,\pi) = (v_1, \ldots, v_n)$. Then

$$\sum_{i=1}^{n} c_i v_i = \sum_{j=2}^{k} c_{i_j}(\rho(A_\pi^j) - \rho(_\pi^{j-1})) + c_{i_1}\rho(A_\pi^1)$$
$$= \sum_{j=1}^{k-1}(c_{i_j} - c_{i_{j+1}})\rho(A_\pi^j) + c_{i_k}\rho(A_\pi^k)$$
$$= \sum_{\emptyset \neq A \subset [n]} \rho(A)\mathbf{y}_A^*,$$

as desired. □

Theorem B.2.1 now guarantees that $\mathbf{v}(k,\pi)$ is an optimal solution of $(L_{\mathbf{c}})$. Thus in particular every vertex is of the form $\mathbf{v}(k,\pi)$. This fact, together with Lemma B.3.2 completes the proof of Theorem 12.1.4.

Example B.3.4. Let $n = 3$ and $\mathcal{P}$ the polymatroid given by the linear inequalities

$$\begin{aligned}
x_1 \quad\quad\quad\quad &\leq 2;\\
x_2 \quad\quad &\leq 3;\\
x_3 &\leq 5;\\
x_1 + x_2 \quad\quad &\leq 4;\\
x_2 + x_3 &\leq 6;\\
x_1 \quad + \quad x_3 &\leq 6;\\
x_1 + x_2 + x_3 &\leq 7;\\
x_i &\geq 0.
\end{aligned}$$

Let $\mathbf{c} = (7,3,1)$. Thus $k = 3$, $\pi = (1,2,3)$ and $\mathbf{v}(3,\pi) = (2,2,3)$. The dual linear programming $(L_{\mathbf{c}}^*)$ is to minimize the objective function

$$2y_{\{1\}} + 3y_{\{2\}} + 5y_{\{3\}} + 4y_{\{1,2\}} + 6y_{\{2,3\}} + 6y_{\{1,3\}} + 7y_{\{1,2,3\}}$$

subject to

$$\begin{aligned}
y_{\{1\}} + y_{\{1,2\}} + y_{\{1,3\}} + y_{\{1,2,3\}} &\geq 7;\\
y_{\{2\}} + y_{\{1,2\}} + y_{\{2,3\}} + y_{\{1,2,3\}} &\geq 3;\\
y_{\{3\}} + y_{\{1,3\}} + y_{\{2,3\}} + y_{\{1,2,3\}} &\geq 1;\\
y_A &\geq 0.
\end{aligned}$$

One has a dual feasible solution

$$\mathbf{y}^* = (\mathbf{y}_{\{1\}}^*, \mathbf{y}_{\{2\}}^*, \mathbf{y}_{\{3\}}^*, \mathbf{y}_{\{1,2\}}^*, \mathbf{y}_{\{2,3\}}^*, \mathbf{y}_{\{1,3\}}^*, \mathbf{y}_{\{1,2,3\}}^*) = (4,0,0,2,0,0,1)$$

with

$$\mathbf{c}\,\mathbf{v}(3,\pi)^\top = \sum_{\emptyset \neq A \subset [3]} \rho(A)\mathbf{y}_A^* = 23.$$

B.4 Intersection Theorem

The intersection theorem for polymatroids due to Edmonds [Edm70] has turned out to be one of the most powerful results in combinatorial optimizations. We refer the reader to Schrijver [Sch03] and Fujishige [Fuj05] for background on Edmonds' intersection theorem.

Theorem B.4.1 (Edmonds' Intersection Theorem). *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be polymatroids on the ground set $[n]$ and ρ_i the ground set rank function of $\mathcal{P}_i$ for $i = 1, 2$. Then*

$$\max\{\mathbf{u}([n]) : \mathbf{u} \in \mathcal{P}_1 \cap \mathcal{P}_2\} = \min\{\rho_1(X) + \rho_2([n] \setminus X) : X \subset [n]\}.$$

Moreover, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are integral, then the maximum on the left-hand side is attained by an integer vector.

B.5 Polymatroidal Sums

Somewhat surprisingly, Theorem 12.1.5 is one of the direct consequences of Edmonds' intersection theorem. Let $\mathcal{P}_1, \ldots, \mathcal{P}_k$ be polymatroids on the ground set $[n]$ and ρ_i the ground set rank function of $\mathcal{P}_i$, $1 \leq i \leq k$. We introduce $\rho : 2^{[n]} \to \mathbb{R}_+$ by setting $\rho = \sum_{i=1}^{k} \rho_i$. It follows immediately that ρ is a nondecreasing and submodular function with $\rho(\emptyset) = 0$. We write $\mathcal{P}$ for the polymatroid on the ground set $[n]$ with ρ its ground set rank function.

Lemma B.5.1. *One has $\mathcal{P}_1 \vee \cdots \vee \mathcal{P}_k \subset \mathcal{P}$.*

Proof. Let $\mathbf{x} \in \mathcal{P}_1 \vee \cdots \vee \mathcal{P}_k$. Then $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_k$ with each $\mathbf{x}_i \in \mathcal{P}_k$. Hence

$$\mathbf{x}(A) = \sum_{i=1}^{k} \mathbf{x}_i(A) \leq \sum_{i=1}^{k} \rho_i(A) = \rho(A).$$

for each $A \subset [n]$. In other words, $\mathbf{x} \in \mathcal{P}$. Thus $\mathcal{P}_1 \vee \cdots \vee \mathcal{P}_k \subset \mathcal{P}$. □

Lemma B.5.2. *One has $\mathcal{P} \subset \mathcal{P}_1 \vee \cdots \vee \mathcal{P}_k$*

Proof. Let $V_i = \{1^{(i)}, \ldots, n^{(i)}\}$ be a "copy" of $[n]$ and let V stand for the disjoint union $V_1 \cup \cdots \cup V_k$. We associate each subset $X \subset V$ with

$$\overline{X} = \{a \in [n] : a^{(i)} \in X \text{ for some } 1 \leq i \leq k\} \subset [n].$$

We introduce $\mu : 2^V \to \mathbb{R}_+$ by setting $\mu(X) = \sum_{i=1}^{k} \rho_i(\overline{X \cap V_i})$, where $X \subset V$. Let $\mathbf{x} \in \mathcal{P}$. We introduce $\xi : 2^V \to \mathbb{R}_+$ by setting $\xi(X) = \mathbf{x}(\overline{X})$, where $X \subset V$. It follows that both μ and ξ are ground set rank functions of polymatroids on the ground set V. Let $\mathcal{Q}_\mu$ (resp. $\mathcal{Q}_\xi$) be the polymatroid on V with μ (resp. ξ) its ground set rank function. Now, Theorem B.4.1 guarantees that

$$\begin{aligned}\max\{\mathbf{u}(V) : \mathbf{u} \in \mathcal{Q}_\mu \cap \mathcal{Q}_\xi\} &= \min\{\mu(X) + \xi(V \setminus X) : X \subset V\} \\ &= \min\{\sum_{i=1}^{k} \rho_i(\overline{X \cap V_i}) + \mathbf{x}(\overline{V \setminus X}) : X \subset V\} \\ &= \min\{\sum_{i=1}^{k} \rho_i(X_j) + \mathbf{x}([n] \setminus \bigcap_{j=1}^{k} X_j) : X_j \subset [n]\}.\end{aligned}$$

Since each ρ_i is nondecreasing, one has $\rho_i(X_j) \geq \rho(\bigcap_{j=1}^{k} X_j)$. Hence

$$\max\{\mathbf{u}(V) : \mathbf{u} \in \mathcal{Q}_\mu \cap \mathcal{Q}_\xi\} = \min\{\sum_{i=1}^{k} \rho_i(Y) + \mathbf{x}([n] \setminus Y) : Y \subset [n]\}.$$

Since $\mathbf{x} \in \mathcal{P}$, one has $\mathbf{x}(Y) \leq \rho(Y) = \sum_{i=1}^{k} \rho_i(Y)$ for all $Y \subset V$. Thus

$$\mathbf{x}([n]) = \mathbf{x}(Y) + \mathbf{x}([n] \setminus Y) \leq \sum_{i=1}^{k} \rho_i(Y) + \mathbf{x}([n] \setminus Y).$$

Consequently,

$$\min\{\sum_{i=1}^{k} \rho_i(X_j) + \mathbf{x}([n] \setminus \bigcap_{j=1}^{k} X_j) : X_j \subset [n]\} = \sum_{i=1}^{k} \rho_i(\emptyset) + \mathbf{x}([n]) = \mathbf{x}([n]).$$

In other words,

$$\max\{\mathbf{u}(V) : \mathbf{u} \in \mathcal{Q}_\mu \cap \mathcal{Q}_\xi\} = \mathbf{x}([n]).$$

Hence there is $\mathbf{u} \in \mathcal{Q}_\mu \cap \mathcal{Q}_\xi$ with $\mathbf{u}(V) = \mathbf{x}([n])$. Thus, in particular, since $\mathbf{u} \in \mathcal{Q}_\nu$, one has $\sum_{i=1}^{k} \mathbf{u}(a^{(i)}) \leq \mathbf{x}(a)$ for all $a \in [n]$. However, since $\mathbf{u}(V) = \mathbf{x}([n])$, it follows that $\sum_{i=1}^{k} \mathbf{u}(a^{(i)}) = \mathbf{x}(a)$ for all $a \in [n]$. We define $\mathbf{x}_i \in \mathbb{R}^n$, $1 \leq i \leq k$, by setting $\mathbf{x}_i(a) = \mathbf{u}(a^{(i)})$ for all $a \in [n]$. Then $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_k$. Since $\mathbf{u} \in \mathcal{Q}_\mu$, one has $\mathbf{x}_i \in \mathcal{P}_i$, as desired. □

It follows from Lemmata B.5.1 and B.5.2 that the polymatroidal sum $\mathcal{P}_1 \vee \cdots \vee \mathcal{P}_k$ is a polymatroid on the ground set $[n]$ with $\rho = \sum_{i=1}^{k} \rho_i$ its ground set rank function. Moreover, if each ρ_i is integer valued, then $\rho = \sum_{i=1}^{k} \rho_i$ is integer valued. In other words, if each $\mathcal{P}_i$ is integral, then $\mathcal{P}_1 \vee \cdots \vee \mathcal{P}_k$ is integral. Finally, in the proof of Lemma B.5.2, if each $\mathcal{P}_i$ is integral and if $\mathbf{x} \in \mathcal{P}$ is an integer vector, then Theorem B.4.1 guarantees that $\mathbf{u} \in \mathcal{Q}_\mu \cap \mathcal{Q}_\xi$ can be chosen as an integer vector. Thus in particular each $\mathbf{x}_i \in \mathcal{P}_i$ is an integer vector. This completes the proof of Theorem 12.1.5.

B.6 Toric rings

Let $\mathcal{P} \subset \mathbb{R}^n_+$ denote an integral convex polytope of dimension d. If $\mathbf{u} = (u(1), \ldots, u(n)) \in \mathcal{P} \cap \mathbb{Z}^n$, then the notation $\mathbf{x}^{\mathbf{u}}$ stands for the monomial

$x_1^{u(1)} \cdots x_n^{u(n)}$. The **toric ring** $K[\mathcal{P}]$ is the subring of $K[x_1, \ldots, x_n, t]$ which is generated by those monomials $\mathbf{x}^{\mathbf{u}}t$ with $\mathbf{u} \in \mathcal{P} \cap \mathbb{Z}^n$. In general, we say that $\mathcal{P}$ possesses the **integer decomposition property** if, for each $\mathbf{w} \in \mathbb{Z}^n$ which belongs to $q\mathcal{P} = \{ q\mathbf{v} : \mathbf{v} \in \mathcal{P} \}$, there exists $\mathbf{u}_1, \ldots, \mathbf{u}_q$ belonging to $\mathcal{P} \cap \mathbb{Z}^n$ such that $\mathbf{w} = \mathbf{u}_1 + \cdots + \mathbf{u}_q$.

Lemma B.6.1. *If an integral convex polytope $\mathcal{P} \subset \mathbb{R}_+^n$ possesses the integer decomposition property, then its toric ring $K[\mathcal{P}]$ is normal.*

Proof. Since $\mathcal{P}$ possesses the integer decomposition property, it follows that the toric ring $K[\mathcal{P}]$ coincides with the Ehrhart ring [Hib92, pp. 97] of $\mathcal{P}$. Since the Ehrhart ring of an integral convex polytope is normal by Gordan's Lemma ([BH98, Proposition 6.1.2]), the toric ring $K[\mathcal{P}]$ is normal, as desired. □

One of the most influential results on normal toric rings, due to Hochster [Hoc72], is the following:

Theorem B.6.2 (Hochster). *A normal toric ring is Cohen–Macaulay.*

Stanley [Sta78] and Danilov [Dan78] succeeded in describing the canonical module of a normal toric ring.

Theorem B.6.3 (Stanley, Danilov). *Let $\mathcal{P} \subset \mathbb{R}_+^n$ be an integral convex polytope and suppose that its toric ring $K[\mathcal{P}]$ is normal. Then the canonical module $\Omega(K[\mathcal{P}])$ of $K[\mathcal{P}]$ coincides with the ideal of $K[\mathcal{P}]$ which is generated by those monomials $x^u t^q$ with $u \in q(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^n$.*

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