Appendix A
Lie Algebras

Definition A.1 A *Lie algebra*, L, over a field $\mathbb{K}$, is a vector space over $\mathbb{K}$ which possesses a mapping from $L \times L$ into L, usually denoted by $[\ ,\ ]$, such that

(i) it is bilinear

$$[u, av + bw] = a[u, v] + b[u, w], \tag{A.1}$$

$$[au + bv, w] = a[u, w] + b[v, w], \tag{A.2}$$

for $u, v, w \in L$, $a, b \in \mathbb{K}$,

(ii) it is skew-symmetric

$$[u, v] = -[v, u], \tag{A.3}$$

for $u, v \in L$ (by virtue of (A.3), the linearity of the bracket on the second argument (A.1) implies its linearity on the first argument (A.2), and vice versa),

(iii) it satisfies the Jacobi identity

$$\big[u, [v, w]\big] + \big[v, [w, u]\big] + \big[w, [u, v]\big] = 0, \tag{A.4}$$

for $u, v, w \in L$. A Lie algebra is Abelian if $[u, v] = 0$ for $u, v \in L$.

Let L be a Lie algebra of finite dimension (that is, L is a vector space of finite dimension), and let $\{e_i\}_{i=1}^n$ be a basis of L. Owing to the bilinearity of the bracket, the value of $[u, v]$, for $u, v \in L$ arbitrary, is determined by the values of $[e_i, e_j]$ ($i, j = 1, \ldots, n$), for if $u = u^i e_i$ and $v = v^j e_j$, we have $[u, v] = [u^i e_i, v^j e_j] = u^i v^j [e_i, e_j]$.

Since $[e_i, e_j]$ must belong to L, $[e_i, e_j] = c_{ij}^k e_k$, where c_{ij}^k ($i, j, k = 1, \ldots, n$) are n^3 scalars, called the *structure constants* of L. The values of the structure constants are not independent, since the bracket must be skew-symmetric, and it satisfies the Jacobi identity, which imposes the following relations among the c_{ij}^k:

$$c_{ij}^k = -c_{ji}^k \quad \text{and} \tag{A.5}$$

$$c_{ij}^m c_{km}^l + c_{jk}^m c_{im}^l + c_{ki}^m c_{jm}^l = 0. \tag{A.6}$$

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DOI 10.1007/978-0-8176-8271-2,

Exercise A.2 Let V be a vector space and let $\mathfrak{gl}(V)$ be the set of the linear maps from V to V with the usual sum and multiplication by scalars, and with the bracket given by $[A, B] \equiv AB - BA$. Show that $\mathfrak{gl}(V)$ is a Lie algebra. If V is of finite dimension and $\{e_i\}_{i=1}^n$ is a basis of V, the linear transformations ϕ^i_j defined by $\phi^i_j(e_k) \equiv \delta^i_k e_j$, form a basis of $\mathfrak{gl}(V)$. Show that $[\phi^j_i, \phi^l_k] = (\delta^r_i \delta^j_k \delta^l_s - \delta^l_i \delta^r_k \delta^j_s)\phi^s_r$.

Definition A.3 Let L be a Lie algebra. A *subalgebra*, M, of L is a subset of L which is a Lie algebra with the operations inherited from L.

Since most of the properties that define a Lie algebra are automatically satisfied by any subset of a given algebra (for instance, the bilinearity and skew-symmetry of the bracket), it suffices to employ the criterion given by the following theorem in order to show that some subset is or is not a subalgebra.

Theorem A.4 *Let L be a Lie algebra and let $M \subset L$. M is a subalgebra of L if and only if for $u, v \in M$ and $a \in \mathbb{K}$, the elements $u + v$, au and $[u, v]$ belong to M.*

The proof of this theorem is immediate and is left to the reader.

Definition A.5 Let L be a Lie algebra and M a subalgebra of L. M is an *ideal* of L if for $u \in M$ and $v \in L$, $[u, v] \in M$.

L itself and $\{0\}$ are ideals of L, and if L is Abelian, then any subalgebra of L is invariant.

Definition A.6 A Lie algebra, L, is *simple* if it is not Abelian and does not possess other ideals apart from L and $\{0\}$. L is *semisimple* if the only Abelian ideal contained in L is $\{0\}$.

For example, the set of globally Hamiltonian vector fields of a symplectic manifold is an ideal of the Lie algebra of the locally Hamiltonian vector fields (see Sect. 8.2).

Definition A.7 Let L_1 and L_2 be two Lie algebras over the same field $\mathbb{K}$. A map $f : L_1 \to L_2$ is a *Lie algebra homomorphism* if

(i) f is a linear transformation (i.e., $f(au + bv) = af(u) + bf(v)$, for $u, v \in L_1$, $a, b \in \mathbb{K}$) and
(ii) $f([u, v]) = [f(u), f(v)]$, for $u, v \in L_1$.

If, in addition, f is bijective we say that f is a *Lie algebra isomorphism*.

Exercise A.8 Let $f : L_1 \to L_2$ be a Lie algebra homomorphism. Show that $\operatorname{Ker} f \equiv \{u \in L_1 \mid f(u) = 0\}$ is an ideal of L_1.

Appendix B
Invariant Metrics

Any Lie group can be turned into a Riemannian manifold in such a way that all the left translations L_g (or the right translations R_g) are isometries. Let G be a Lie group and let $\{\omega^1, \ldots, \omega^n\}$ be a basis for the left-invariant 1-forms; if (a_{ij}) is any (constant) non-singular symmetric $n \times n$ matrix, then

$$a_{ij}\omega^i \otimes \omega^j \tag{B.1}$$

is a metric tensor on G, which is a *left-invariant metric* since $L_g{}^*(a_{ij}\omega^i \otimes \omega^j) = a_{ij}\omega^i \otimes \omega^j$, for all $g \in G$. If (a_{ij}) is positive definite, the metric (B.1) is also positive definite. If, in place of the 1-forms ω^i we employ right-invariant 1-forms, in an analogous manner we obtain a *right-invariant metric*. A metric on G is *bi-invariant* if it is left-invariant and right-invariant simultaneously.

From the results of Sect. 7.5 it follows that the right-invariant vector fields are Killing vector fields for any left-invariant metric (see Exercise 7.51). For a bi-invariant metric, the right-invariant vector fields, and the left-invariant vector fields are Killing vector fields.

Example B.1 The 2×2 real matrices of the form $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$, with $x > 0$, form a Lie subgroup of $\mathrm{GL}(2, \mathbb{R})$. Making use of Theorem 7.35, from the equation

$$\begin{aligned}\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{d}x & \mathrm{d}y \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} x^{-1} & -yx^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}x & \mathrm{d}y \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} x^{-1}\,\mathrm{d}x & x^{-1}\,\mathrm{d}y \\ 0 & 0 \end{pmatrix} \\ &= x^{-1}\mathrm{d}x \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + x^{-1}\mathrm{d}y \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\end{aligned}$$

it follows that

$$\omega^1 \equiv x^{-1}\,\mathrm{d}x, \qquad \omega^2 \equiv x^{-1}\,\mathrm{d}y,$$

form a basis for the left-invariant 1-forms. Using the fact that the inversion mapping, $\iota(g) = g^{-1}$, is given by $\iota^* x = x^{-1}$, $\iota^* y = -yx^{-1}$ [see (7.3)], one finds that the basis

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DOI 10.1007/978-0-8176-8271-2,

of the right-invariant 1-forms $\dot{\omega}^i = -\iota^* \omega^i$, is

$$\dot{\omega}^1 = x^{-1}\,\mathrm{d}x, \qquad \dot{\omega}^2 = -yx^{-1}\,\mathrm{d}x + \mathrm{d}y,$$

and the dual basis is given by

$$\dot{\mathbf{X}}_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad \dot{\mathbf{X}}_2 = \frac{\partial}{\partial y} \tag{B.2}$$

(cf. Example 7.47). Thus, $\dot{\mathbf{X}}_1$ and $\dot{\mathbf{X}}_2$ are Killing vector fields for the metric $a_{ij}\omega^i \otimes \omega^j = x^{-2}\big[a_{11}\,\mathrm{d}x \otimes \mathrm{d}x + a_{12}(\mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x) + a_{22}\,\mathrm{d}y \otimes \mathrm{d}y\big]$, no matter what the values are of the constants a_{11}, a_{12}, and a_{22}. In particular, taking $a_{ij} = \delta_{ij}$, we obtain the metric

$$x^{-2}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y), \tag{B.3}$$

which is the metric of Poincaré's half-plane [see (6.19)] and possesses three linearly independent Killing vector fields (see Example 6.12).

Exercise B.2 Show that if G is connected, the metric $a_{ij}\omega^i \otimes \omega^j$ is also right-invariant if and only if

$$a_{im}c^m_{jk} + a_{jm}c^m_{ik} = 0, \tag{B.4}$$

where the c^i_{jk} are the structure constants of G with respect to the basis $\{\omega^i\}$.

Exercise B.3 Find a basis for the left-invariant 1-forms and its dual basis for the group formed by the 3×3 matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \qquad x, y, z \in \mathbb{R},$$

which is related to the Heisenberg group [see, e.g., Baker (2002), Sect. 7.7]. Determine the structure constants of the group in this basis. Is it possible to find a bi-invariant metric?

Since the coefficients a_{ij} in (B.1) are constant, the dual basis $\{\mathbf{X}_i\}$ to $\{\omega^i\}$ is a rigid basis with respect to the metric $a_{ij}\omega^i \otimes \omega^j$; thus, comparing $[\mathbf{X}_i, \mathbf{X}_j] = c^k_{ij}\mathbf{X}_k$ with (6.62) one finds that $c^k_{ij} = \Gamma^k{}_{ji} - \Gamma^k{}_{ij} \equiv 2\Gamma^k{}_{[ji]}$, where the $\Gamma^i{}_{jk}$ are the Ricci rotation coefficients for the basis $\{\mathbf{X}_i\}$. Using the identity (6.63), we obtain

$$\Gamma_{ijk} = \frac{1}{2}\big(a_{im}c^m_{kj} - a_{jm}c^m_{ki} - a_{km}c^m_{ji}\big). \tag{B.5}$$

The foregoing expression is simplified if the metric (B.1) is bi-invariant because in that case the last two terms on the right-hand side of (B.5) cancel [see (B.4)], leaving

$$\Gamma_{ijk} = \frac{1}{2}a_{im}c^m_{kj}, \tag{B.6}$$

so that the connection and curvature forms in this basis are

$$\Gamma^i{}_j = \frac{1}{2} c^i_{kj} \omega^k \quad \text{and} \quad \mathscr{R}^i{}_j = \frac{1}{4} c^m_{jk} c^i_{ml} \omega^k \wedge \omega^l, \tag{B.7}$$

respectively [see (5.26), (7.34), and (A.6)]. Hence, the components of the curvature with respect to the basis $\{\mathbf{X}_i\}$ are

$$R^i{}_{jkl} = \frac{1}{4}\left(c^m_{jk} c^i_{ml} - c^m_{jl} c^i_{mk}\right) = -\frac{1}{4} c^i_{mj} c^m_{kl}. \tag{B.8}$$

It may be noticed that in the expressions (B.7) the matrix (a_{ij}) does not appear and, furthermore, that they make sense *independently* of choosing a metric on the group. It can be directly verified that, with respect to a basis for the left-invariant 1-forms, $\{\omega^1, \ldots, \omega^n\}$, the connection 1-forms (B.7) define a connection with torsion equal to zero. Hence, in any Lie group there exists a torsion-free connection, defined in a natural way, without having to specify a Riemannian metric.

From (B.6) it follows that, if the metric (B.1) is bi-invariant, the coefficients Γ_{ijk} are totally skew-symmetric, since, in general, $\Gamma_{ijk} = -\Gamma_{jik}$, while from the relation $c^m_{kj} = -c^m_{jk}$ it follows that $\Gamma_{ijk} = -\Gamma_{ikj}$. Combining these formulas one finds that $\Gamma_{ijk} = -\Gamma_{kji}$. If the dimension of G is two, then the total skew-symmetry of the Ricci rotation coefficients implies that they are equal to zero and, since (a_{im}) must be invertible, $c^m_{kj} = 0$ and, therefore, G must be Abelian.

If the dimension of G is three, the skew-symmetry of Γ_{ijk} implies that $\Gamma_{ijk} = b\,\varepsilon_{ijk}$, where b is some constant. Then, from (B.6), we have

$$c^m_{kj} = 2a^{im} b\,\varepsilon_{ijk}, \tag{B.9}$$

where (a^{im}) is the inverse of the matrix (a_{im}); therefore

$$\frac{1}{4} c^i_{mj} c^m_{kl} = b^2 a^{pi}\,\varepsilon_{pjm} a^{qm}\,\varepsilon_{qlk} = b^2 a^{pi} \det\left(a^{rs}\right)(a_{lp} a_{kj} - a_{lj} a_{kp})$$

and from (B.8) we obtain

$$R_{ijkl} = b^2 \det\left(a^{rs}\right)(a_{ik} a_{jl} - a_{il} a_{jk}), \tag{B.10}$$

which means that G is a constant curvature space (see Examples B.6 and B.8). For any value of b, the structure constants (B.9) satisfy the Jacobi identity (A.6). It can be noticed that in this case, if the six vector fields $\mathbf{X}_i$ and $\dot{\mathbf{X}}_i$ $(i = 1, 2, 3)$ are linearly independent, then they form a basis for the Killing vector fields of G, since the maximum dimension of the Lie algebra of the Killing vector fields of a Riemannian manifold of dimension n is $n(n+1)/2$.

Exercise B.4 Show that for any Lie group, G, the left-invariant vector fields $\mathbf{X}_i$, and the right-invariant vector fields $\dot{\mathbf{X}}_i$ are linearly independent if and only if the *center* of the Lie algebra of G is $\{0\}$; that is, if and only if zero is the only element of $\mathfrak{g}$ whose Lie bracket with all the elements of the algebra is equal to zero.

Exercise B.5 Show that if $a_{ij}\omega^i \otimes \omega^j$ is a bi-invariant metric on G, where $\{\omega^i\}$ is a basis for the left-invariant 1-forms, then $\nabla_{\mathbf{Y}}\mathbf{Z} = \frac{1}{2}[\mathbf{Y}, \mathbf{Z}]$ and $R(\mathbf{Y}, \mathbf{Z})\mathbf{W} = -\frac{1}{4}[[\mathbf{Y}, \mathbf{Z}], \mathbf{W}]$, for $\mathbf{Y}, \mathbf{Z}, \mathbf{W} \in \mathfrak{g}$, where ∇ denotes the Riemannian connection associated with the bi-invariant metric and R is its curvature tensor. Show that the integral curves of any left-invariant vector field are geodesics.

If c^i_{jk} denote the structure constants of an arbitrary Lie algebra, then the constants

$$g_{ij} = -c^k_{im}c^m_{jk} \tag{B.11}$$

form a symmetric matrix, $g_{ji} = -c^k_{jm}c^m_{ik} = -c^m_{ik}c^k_{jm} = g_{ij}$. Furthermore, making use of (B.11), and the identities $c^m_{ij}c^l_{mk} + c^m_{jk}c^l_{mi} + c^m_{ki}c^l_{mj} = 0$ and $c^i_{jk} = -c^i_{kj}$ [see (A.5) and (A.6)] one finds that

$$\begin{aligned}
g_{im}c^m_{jk} + g_{jm}c^m_{ik} &= -c^s_{ir}c^r_{ms}c^m_{jk} - c^s_{jr}c^r_{ms}c^m_{ik} \\
&= c^s_{ir}\left(c^m_{ks}c^r_{mj} + c^m_{sj}c^r_{mk}\right) - c^s_{jr}c^r_{ms}c^m_{ik} \\
&= \left(-c^s_{ki}c^m_{rs} - c^s_{rk}c^m_{is}\right)c^r_{mj} + c^s_{ir}c^m_{sj}c^r_{mk} - c^s_{jr}c^r_{ms}c^m_{ik} \\
&= 0.
\end{aligned}$$

Hence, if G is a connected Lie group and $\{\omega^i\}$ is a basis for the left-invariant 1-forms, the tensor field $g_{ij}\omega^i \otimes \omega^j$, with the g_{ij} defined by (B.5), is bi-invariant (see Exercise B.2). However, the matrix (g_{ij}) can be singular, and therefore $g_{ij}\omega^i \otimes \omega^j$ does not need to be a Riemannian metric on G. It can be shown that the matrix (g_{ij}), defined in (B.11), is invertible if and only if the Lie algebra is *semisimple* (that is, it does not have Abelian proper ideals) [see, e.g., Sattinger and Weaver (1986, Chap. 9)].

It may be noticed that the components of the Ricci tensor associated with the curvature tensor (B.8) are given by $R_{ij} = \frac{1}{4}g_{ij}$, with g_{ij} defined by (B.11).

Example B.6 Let us consider the group $G = \mathrm{SU}(2)$ with the parametrization given by the Euler angles, ϕ, θ, ψ,

$$g = \big(\exp\phi(g)\mathbf{X}_3\big)\big(\exp\theta(g)\mathbf{X}_1\big)\big(\exp\psi(g)\mathbf{X}_3\big), \tag{B.12}$$

where $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ is the basis of $\mathfrak{su}(2)$ given in Exercise 7.19 [cf. (8.94)]. From (7.54) it follows that (B.12) is equivalent to

$$\begin{aligned}
g &= \begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\phi/2} \end{pmatrix} \begin{pmatrix} \cos\theta/2 & \mathrm{i}\sin\theta/2 \\ \mathrm{i}\sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\psi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\psi/2} \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{e}^{\mathrm{i}(\phi+\psi)/2}\cos\theta/2 & \mathrm{i}\,\mathrm{e}^{\mathrm{i}(\phi-\psi)/2}\sin\theta/2 \\ \mathrm{i}\,\mathrm{e}^{\mathrm{i}(\psi-\phi)/2}\sin\theta/2 & \mathrm{e}^{-\mathrm{i}(\phi+\psi)/2}\cos\theta/2 \end{pmatrix},
\end{aligned} \tag{B.13}$$

where, by abuse of notation, we have simply written ϕ, θ, ψ, in place of $\phi(g), \theta(g)$, and $\psi(g)$, respectively. As in Example B.1, we can make use of Theorem 7.35 to find the basis of the left-invariant 1-forms, dual to the basis $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$. Since the structure constants for the basis $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ are the same as those of the basis $\{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3\}$ of $\mathfrak{so}(3)$, equations (8.95)–(8.100) hold if $\mathbf{S}_i$ is replaced by $\mathbf{X}_i$; hence the set

$$\begin{aligned}\omega^1 &= \sin\theta \sin\psi \, \mathrm{d}\phi + \cos\psi \, \mathrm{d}\theta, \\ \omega^2 &= \sin\theta \cos\psi \, \mathrm{d}\phi - \sin\psi \, \mathrm{d}\theta, \\ \omega^3 &= \cos\theta \, \mathrm{d}\phi + \mathrm{d}\psi\end{aligned} \tag{B.14}$$

is the dual basis to $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$. Making use of the fact that $[\mathbf{X}_i, \mathbf{X}_j] = \sum_{k=1}^{3} \varepsilon_{ijk}\mathbf{X}_k = \delta^{kl}\varepsilon_{ijk}\mathbf{X}_l$, we have $c_{ij}^{l} = \delta^{kl}\varepsilon_{ijk}$; therefore, from (B.11), $g_{ij} = -\delta^{pk}\varepsilon_{imp}\delta^{qm}\varepsilon_{jkq} = -\delta^{pk}(\delta_{jp}\delta_{ki} - \delta_{ji}\delta_{kp}) = 2\delta_{ij}$, which is an invertible matrix and $g_{ij}\omega^i \otimes \omega^j = 2\delta_{ij}\omega^i \otimes \omega^j$. From (B.14) we then have

$$\begin{aligned}g_{ij}\omega^i \otimes \omega^j = 2\big[&\mathrm{d}\phi \otimes \mathrm{d}\phi + \mathrm{d}\theta \otimes \mathrm{d}\theta + \mathrm{d}\psi \otimes \mathrm{d}\psi \\ &+ \cos\theta(\mathrm{d}\phi \otimes \mathrm{d}\psi + \mathrm{d}\psi \otimes \mathrm{d}\phi)\big].\end{aligned} \tag{B.15}$$

According to the foregoing results, we may conclude that the metric (B.15) is bi-invariant. As we shall show below, this metric is essentially the usual metric of the sphere S^3.

The underlying manifold of the group SU(2) can be identified with the sphere S^3 in the following manner. All the elements of SU(2) are of the form

$$\begin{pmatrix} x + \mathrm{i}y & z + \mathrm{i}w \\ -z + \mathrm{i}w & x - \mathrm{i}y \end{pmatrix}, \tag{B.16}$$

where x, y, z, w are real numbers such that $x^2 + y^2 + z^2 + w^2 = 1$. Hence, there is a one-to-one correspondence between the elements of SU(2) and the points of $\mathrm{S}^3 \equiv \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}$. From the expressions (B.13) and (B.16), separating the real and imaginary parts, one obtains a local expression for the inclusion of SU(2), or S^3, in $\mathbb{R}^4$ ($i : \mathrm{SU}(2) \to \mathbb{R}^4$), namely

$$\begin{aligned} i^*x &= \cos\tfrac{1}{2}\theta \cos\tfrac{1}{2}(\phi + \psi), \qquad & i^*y &= \cos\tfrac{1}{2}\theta \sin\tfrac{1}{2}(\phi + \psi), \\ i^*z &= -\sin\tfrac{1}{2}\theta \sin\tfrac{1}{2}(\phi - \psi), \qquad & i^*w &= \sin\tfrac{1}{2}\theta \cos\tfrac{1}{2}(\phi - \psi). \end{aligned}$$

The pullback under i of the usual metric of $\mathbb{R}^4$ is then

$$\begin{aligned}&i^*(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z + \mathrm{d}w \otimes \mathrm{d}w) \\ &\quad = \frac{1}{4}\big[\mathrm{d}\phi \otimes \mathrm{d}\phi + \mathrm{d}\theta \otimes \mathrm{d}\theta + \mathrm{d}\psi \otimes \mathrm{d}\psi + \cos\theta(\mathrm{d}\phi \otimes \mathrm{d}\psi + \mathrm{d}\psi \otimes \mathrm{d}\phi)\big],\end{aligned}$$

which, except for a factor 1/8, coincides with the metric (B.15). This means that the metric (B.15), which, as we have shown, is the metric of a constant curvature space, is essentially the standard metric of S^3 (which is, clearly, a constant curvature space). Moreover, the left-invariant vector fields $\mathbf{S}_i$ [given by (8.98)] and the right-invariant vector fields $\dot{\mathbf{S}}_i$ [given by (8.100)] of SU(2), are Killing vector fields for the metric (B.15) and, therefore, for S^3. Thus, the Lie algebra of the Killing vector fields of S^3, which is $\mathfrak{so}(4)$ [the Lie algebra of SO(4)], possesses the basis $\{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, -\dot{\mathbf{S}}_1, -\dot{\mathbf{S}}_2, -\dot{\mathbf{S}}_3\}$, which satisfies the relations

$$\begin{aligned} [\mathbf{S}_i, \mathbf{S}_j] &= \sum_{k=1}^{3} \varepsilon_{ijk} \mathbf{S}_k, \\ \big[(-\dot{\mathbf{S}}_i), (-\dot{\mathbf{S}}_j)\big] &= \sum_{k=1}^{3} \varepsilon_{ijk} (-\dot{\mathbf{S}}_k), \\ \big[\mathbf{S}_i, (-\dot{\mathbf{S}}_j)\big] &= 0; \end{aligned} \tag{B.17}$$

hence, $\mathfrak{so}(4)$ is the direct sum of two copies of $\mathfrak{su}(2)$:

$$\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2). \tag{B.18}$$

Each $g \in \mathrm{SU}(2)$ can be regarded as a point of S^3 (by expressing g in the form (B.16) and taking the corresponding x, y, z, w as the coordinates of a point of S^3), and for any $g_1 \in \mathrm{SU}(2)$, both L_{g_1} and R_{g_1} are isometries for the metric (B.15). Hence, if $(g_1, g_2) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$, the mapping $g \mapsto L_{g_1} R_{g_2} g = g_1 g g_2 = R_{g_2} L_{g_1} g$, from SU(2) onto SU(2), can be seen as an isometric map from S^3 onto S^3. In fact, it turns out that any isometry of S^3 that does not change the orientation is obtained in this manner, with g_1 and g_2 determined up to sign; if $(g_1, g_2) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$, then $(-g_1, -g_2)$ also belongs to $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and $L_{g_1} R_{g_2} = L_{-g_1} R_{-g_2}$. From the preceding discussion it also follows that any rotation about the origin in $\mathbb{R}^4$ can be represented in the form

$$\begin{pmatrix} x' + \mathrm{i}y' & z' + \mathrm{i}w' \\ -z' + \mathrm{i}w' & x' - \mathrm{i}y' \end{pmatrix} = g_1 \begin{pmatrix} x + \mathrm{i}y & z + \mathrm{i}w \\ -z + \mathrm{i}w & x - \mathrm{i}y \end{pmatrix} g_2 \tag{B.19}$$

[cf. (7.65)] with $g_1, g_2 \in \mathrm{SU}(2)$ determined up to sign. [This result is the counterpart of (B.18).]

Exercise B.7 Show that from (B.19) it follows directly that the transformation $(x, y, z, w) \mapsto (x', y', z', w')$ belongs to SO(4).

Example B.8 The functions $\alpha, \beta, \gamma : \mathrm{SL}(2, \mathbb{R}) \to \mathbb{R}$ defined by

$$g = \left(\exp \frac{1}{2}\alpha(g)\mathbf{X}_1\right)\left(\exp \frac{1}{2}\beta(g)(\mathbf{X}_2 + \mathbf{X}_3)\right)\left(\exp \frac{1}{2}\gamma(g)\mathbf{X}_1\right), \tag{B.20}$$

where $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ is the basis of $\mathfrak{sl}(2, \mathbb{R})$ given in Example 7.16, form a local coordinate system for $\mathrm{SL}(2, \mathbb{R})$, alternative to that defined by (7.4). From (B.20) and (7.51) we then have

$$g = \begin{pmatrix} e^{\alpha/2} & 0 \\ 0 & e^{-\alpha/2} \end{pmatrix} \begin{pmatrix} \cosh\beta/2 & \sinh\beta/2 \\ \sinh\beta/2 & \cosh\beta/2 \end{pmatrix} \begin{pmatrix} e^{\gamma/2} & 0 \\ 0 & e^{-\gamma/2} \end{pmatrix}$$
$$= \begin{pmatrix} e^{(\alpha+\gamma)/2}\cosh\beta/2 & e^{(\alpha-\gamma)/2}\sinh\beta/2 \\ e^{-(\alpha-\gamma)/2}\sinh\beta/2 & e^{-(\alpha+\gamma)/2}\cosh\beta/2 \end{pmatrix}, \tag{B.21}$$

where we have written α, β, γ instead of $\alpha(g)$, $\beta(g)$, $\gamma(g)$ [cf. (B.13)]. The dual basis to $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$, expressed in terms of the coordinates α, β, γ, can be obtained making use of (B.20) and Theorem 7.35, which leads to [see (7.20)]

$$\begin{aligned} g^{-1}\,dg &= \frac{1}{2}\exp\left(-\frac{1}{2}\gamma\mathbf{X}_1\right)\exp\left(-\frac{1}{2}\beta(\mathbf{X}_2+\mathbf{X}_3)\right) \\ &\quad \cdot \lambda_1 \exp\left(\frac{1}{2}\beta(\mathbf{X}_2+\mathbf{X}_3)\right)\exp\left(\frac{1}{2}\gamma\mathbf{X}_1\right)d\alpha \\ &\quad + \frac{1}{2}\exp\left(-\frac{1}{2}\gamma\mathbf{X}_1\right)(\lambda_2+\lambda_3)\exp\left(\frac{1}{2}\gamma\mathbf{X}_1\right)d\beta + \frac{1}{2}\lambda_1\,d\gamma \\ &= \frac{1}{2}(\cosh\beta\,d\alpha + d\gamma)\lambda_1 + \frac{1}{2}e^{-\gamma}(\sinh\beta\,d\alpha + d\beta)\lambda_2 \\ &\quad + \frac{1}{2}e^{\gamma}(-\sinh\beta\,d\alpha + d\beta)\lambda_3, \end{aligned}$$

and thus

$$\begin{aligned} \omega^1 &= \frac{1}{2}(\cosh\beta\,d\alpha + d\gamma), \\ \omega^2 &= \frac{1}{2}e^{-\gamma}(\sinh\beta\,d\alpha + d\beta), \\ \omega^3 &= \frac{1}{2}e^{\gamma}(-\sinh\beta\,d\alpha + d\beta). \end{aligned} \tag{B.22}$$

On the other hand, from (7.20) we find that $[\lambda_1, \lambda_2] = 2\lambda_2$, $[\lambda_2, \lambda_3] = \lambda_1$, $[\lambda_3, \lambda_1] = 2\lambda_3$ (i.e., the structure constants that are different from zero are given by $c_{12}^2 = 2 = c_{31}^3$, $c_{23}^1 = 1$) and from (B.11) it follows that

$$(g_{ij}) = \begin{pmatrix} -8 & 0 & 0 \\ 0 & 0 & -4 \\ 0 & -4 & 0 \end{pmatrix}; \tag{B.23}$$

therefore, using (B.22) and (B.23),

$$\begin{aligned} g_{ij}\omega^i \otimes \omega^j &= -4\big(2\omega^1\otimes\omega^1 + \omega^2\otimes\omega^3 + \omega^3\otimes\omega^2\big) \\ &= -2\big[d\alpha\otimes d\alpha + d\beta\otimes d\beta + d\gamma\otimes d\gamma \\ &\quad + \cosh\beta(d\alpha\otimes d\gamma + d\gamma\otimes d\alpha)\big] \end{aligned} \tag{B.24}$$

is a pseudo-Riemannian bi-invariant metric on $\mathrm{SL}(2,\mathbb{R})$ and, with this metric, $\mathrm{SL}(2,\mathbb{R})$ is a constant curvature space. (Note that $g_{ij}\omega^i \otimes \omega^j = -8\omega^1 \otimes \omega^1 - 2(\omega^2+\omega^3) \otimes (\omega^2+\omega^3) + 2(\omega^2-\omega^3) \otimes (\omega^2-\omega^3)$, which explicitly shows that this metric is pseudo-Riemannian.) In a similar manner to the case of SU(2), considered in the foregoing example, $\mathrm{SL}(2,\mathbb{R})$ with the metric (B.24) can be identified with a submanifold of $\mathbb{R}^4$, provided that in the latter we introduce a pseudo-Riemannian flat metric [cf. Conlon (2001), Sect. 10.7].

Indeed, any element of $\mathrm{SL}(2,\mathbb{R})$ is of the form

$$\begin{pmatrix} x+w & y+z \\ z-y & x-w \end{pmatrix}, \tag{B.25}$$

where x, y, z, w are real numbers with $x^2+y^2-z^2-w^2=1$. This means that the underlying manifold of $\mathrm{SL}(2,\mathbb{R})$ can be identified with the hyperboloid $N \equiv \{(x,y,z,w) \in \mathbb{R}^4 \,|\, x^2+y^2-z^2-w^2=1\}$. Comparing (B.21) with (B.25), one finds the following local expression for the inclusion of $\mathrm{SL}(2,\mathbb{R})$ in $\mathbb{R}^4$:

$$i^*x = \cosh\frac{1}{2}\beta\cosh\frac{1}{2}(\alpha+\gamma), \qquad i^*y = \sinh\frac{1}{2}\beta\sinh\frac{1}{2}(\alpha-\gamma),$$

$$i^*z = \sinh\frac{1}{2}\beta\cosh\frac{1}{2}(\alpha-\gamma), \qquad i^*w = \cosh\frac{1}{2}\beta\sinh\frac{1}{2}(\alpha+\gamma),$$

hence, the metric induced on $\mathrm{SL}(2,\mathbb{R})$, or on N, by the pseudo-Riemannian metric $\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y-\mathrm{d}z\otimes\mathrm{d}z-\mathrm{d}w\otimes\mathrm{d}w$ of $\mathbb{R}^4$ is

$$\begin{aligned} & i^*(\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y-\mathrm{d}z\otimes\mathrm{d}z-\mathrm{d}w\otimes\mathrm{d}w) \\ &\quad = -\frac{1}{4}\big[\mathrm{d}\alpha\otimes\mathrm{d}\alpha+\mathrm{d}\beta\otimes\mathrm{d}\beta+\mathrm{d}\gamma\otimes\mathrm{d}\gamma+\cosh\beta(\mathrm{d}\alpha\otimes\mathrm{d}\gamma+\mathrm{d}\gamma\otimes\mathrm{d}\alpha)\big] \end{aligned}$$

and coincides, except for a factor 1/8, with the metric (B.24). Then, owing to the bi-invariance of (B.24), the left-invariant vector fields of $\mathrm{SL}(2,\mathbb{R})$, together with the right-invariant ones are Killing vector fields for the metric (B.24) and for the metric induced on N. On the other hand, N and the metric $\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y-\mathrm{d}z\otimes\mathrm{d}z-\mathrm{d}w\otimes\mathrm{d}w$ are invariant under the linear transformations of $\mathbb{R}^4$ into $\mathbb{R}^4$ represented by the real 4×4 matrices, A, with determinant equal to 1, such that

$$A^{\mathrm{t}}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} A = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \tag{B.26}$$

which form the group SO(2, 2), whose dimension is six. Thus, in an analogous way to (B.18), we have

$$\mathfrak{so}(2,2) = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}). \tag{B.27}$$

Since for any $g \in \mathrm{SL}(2,\mathbb{R})$, the transformations L_g and R_g are isometries of the metric (B.24), if $(g_1, g_2) \in \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, the transformation $g \mapsto L_{g_1} R_{g_2} g = g_1 g g_2$, from $\mathrm{SL}(2,\mathbb{R})$ onto $\mathrm{SL}(2,\mathbb{R})$, is an isometry and can be identified with an isometric transformation from N onto N. That is, using (B.25), the expression

$$\begin{pmatrix} x' + w' & y' + z' \\ z' - y' & x' - w' \end{pmatrix} = g_1 \begin{pmatrix} x + w & y + z \\ z - y & x - w \end{pmatrix} g_2 \tag{B.28}$$

gives an isometric transformation from N onto N, for any pair of elements $g_1, g_2 \in \mathrm{SL}(2,\mathbb{R})$, and it turns out that any transformation belonging to $\mathrm{SO}(2,2)$ can be represented in this manner with g_1 and g_2 determined up to a common sign.

Harmonic Maps The harmonic mapping equations constitute a generalization of the geodesic equations (5.7). In their general form, given two Riemannian manifolds, N and M, of dimensions n and m, respectively, a differentiable map $\phi : N \to M$ is *harmonic* if

$$\frac{1}{\sqrt{|h|}} \frac{\partial}{\partial y^\alpha} \left(\sqrt{|h|}\, h^{\alpha\beta} \frac{\partial(\phi^* x^k)}{\partial y^\beta} \right) + \left(\phi^* \Gamma^k_{ji}\right) h^{\alpha\beta} \frac{\partial(\phi^* x^j)}{\partial y^\alpha} \frac{\partial(\phi^* x^i)}{\partial y^\beta} = 0, \tag{B.29}$$

where $(h^{\alpha\beta})$ is the inverse of the matrix $(h_{\alpha\beta})$, formed by the components of the metric tensor of N with respect to a local coordinate system $(y^1, \ldots, y^n)$, $h \equiv \det(h_{\alpha\beta})$, $(x^1, \ldots, x^m)$ is a coordinate system on M and the Γ^k_{ji} are the Christoffel symbols corresponding to the metric tensor of M in the coordinate system x^i [see, e.g., Hélein (2002)]. When $N = \mathbb{R}$, with $y^1 = t$ and $h_{11} = 1$, equations (B.29) reduce to the equations of the geodesics (5.7). When $M = \mathbb{R}$, with its usual metric, equations (B.29) reduce to the Laplace equation, $\nabla^2 \phi = 0$ [see (6.113)].

An interesting fact is that in the case of a harmonic map $\phi : N \to G$, where G is a Lie group that admits a bi-invariant metric, equations (B.29) amount to

$$\frac{1}{\sqrt{|h|}} \frac{\partial}{\partial y^\alpha} \left[\sqrt{|h|}\, h^{\alpha\beta} \left(\phi^* \omega^k\right) \left(\frac{\partial}{\partial y^\beta} \right) \right] = 0, \tag{B.30}$$

where the ω^k are left-invariant 1-forms on G. In effect, the 1-forms ω^k can be expressed locally in the form

$$\omega^k = M^k_i \, \mathrm{d}x^i, \tag{B.31}$$

with each $M^k_i \in C^\infty(G)$. Then

$$\frac{\partial}{\partial x^i} = M^k_i \mathbf{X}_k, \tag{B.32}$$

where the $\mathbf{X}_k$ are the left-invariant fields that form the dual basis to $\{\omega^k\}$. Using the properties of a connection [see (5.1)], from Exercise B.5 it follows that the Christoffel symbols for the bi-invariant metric of G with respect to the coordinate system x^i are given by

$$\begin{aligned}\Gamma^i_{jk}\frac{\partial}{\partial x^i} &= M^s_k \nabla_{\mathbf{X}_s}\left(M^r_j \mathbf{X}_r\right)\\ &= M^s_k \mathbf{X}_s\left(M^r_j\right)\mathbf{X}_r + M^s_k M^r_j \nabla_{\mathbf{X}_s}\mathbf{X}_r\\ &= \left(\frac{\partial}{\partial x^k}M^r_j\right)\mathbf{X}_r + \frac{1}{2}M^s_k M^r_j[\mathbf{X}_s, \mathbf{X}_r].\end{aligned}$$

Since the Christoffel symbols Γ^i_{jk} are symmetric in the indices j, k, while $M^s_k M^r_j[\mathbf{X}_s, \mathbf{X}_r]$ is antisymmetric in these indices, using (B.32), it follows that

$$\Gamma^i_{jk} = \widetilde{M}^i_r \frac{\partial}{\partial x^{(k}} M^r_{j)}, \tag{B.33}$$

where $(\widetilde{M}^i_j)$ is the inverse of the matrix (M^i_j), and the parentheses denote symmetrization on the enclosed indices [e.g., $t_{(ij)} = \frac{1}{2}(t_{ij} + t_{ji})$].

Thus, from (B.31), (1.23), and (1.24) we have

$$\begin{aligned}&h^{\alpha\beta}\frac{\partial}{\partial y^\alpha}\left[\left(\phi^*\omega^k\right)\left(\frac{\partial}{\partial y^\beta}\right)\right]\\ &\quad = h^{\alpha\beta}\frac{\partial}{\partial y^\alpha}\left[\left(\phi^* M^k_i\right)\frac{\partial(\phi^* x^i)}{\partial y^\beta}\right]\\ &\quad = h^{\alpha\beta}\left(\phi^* M^k_i\right)\frac{\partial}{\partial y^\alpha}\frac{\partial(\phi^* x^i)}{\partial y^\beta} + h^{\alpha\beta}\frac{\partial(\phi^* x^i)}{\partial y^\beta}\frac{\partial(\phi^* x^j)}{\partial y^\alpha}\,\phi^*\left(\frac{\partial M^k_i}{\partial x^j}\right).\end{aligned}$$

Using the fact that $(h^{\alpha\beta})$ is symmetric, from (B.33) we then have

$$\begin{aligned}&h^{\alpha\beta}\frac{\partial}{\partial y^\alpha}\left[\left(\phi^*\omega^k\right)\left(\frac{\partial}{\partial y^\beta}\right)\right]\\ &\quad = \left(\phi^* M^k_s\right)\left[h^{\alpha\beta}\frac{\partial}{\partial y^\alpha}\frac{\partial(\phi^* x^s)}{\partial y^\beta} + h^{\alpha\beta}\frac{\partial(\phi^* x^i)}{\partial y^\beta}\frac{\partial(\phi^* x^j)}{\partial y^\alpha}\phi^*\Gamma^s_{ij}\right],\end{aligned}$$

which shows the equivalence of (B.29) and (B.30) in the case where M is a Lie group with a bi-invariant metric.

As pointed out previously, when $N = \mathbb{R}$ with the usual metric, the equations for a harmonic map reduce to the geodesic equations. Hence, the equations for a geodesic, C, of a group G with a bi-invariant metric, can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(C^*\omega^k\right)\left(\frac{\partial}{\partial t}\right)\right] = 0$$

[see (B.30)]; therefore $(C^*\omega^k)(\partial/\partial t) = a^k$, where each a^k is a real constant. That is, $\omega^k(C'_t) = a^k$, which amounts to $C'_t = a^k \mathbf{X}_k(C(t))$. Thus, in this case, a geodesic is an integral curve of some left-invariant vector field (cf. Exercise B.5).

Taking into account that, when G is some subgroup of $\mathrm{GL}(p, \mathbb{R})$, a basis for the left-invariant 1-forms can be found from the relation $g^{-1}\mathrm{d}g = \lambda_a \omega^a$ [see (7.46)], where the λ_a are constant matrices that form a basis for a representation of the Lie algebra of G, it follows that equations (B.30) amount to the matrix equation

$$\frac{1}{\sqrt{|h|}} \frac{\partial}{\partial y^\alpha} \left(\sqrt{|h|}\, h^{\alpha\beta} g^{-1} \frac{\partial g}{\partial y^\beta} \right) = 0, \tag{B.34}$$

where it is understood that g is an arbitrary element of G, parameterized in terms of the y^α through the map $\phi : N \to G$.

Each Killing vector field of a Riemannian manifold, M, gives rise to a conserved quantity, constant of motion, or first integral of the geodesic equations (Theorem 6.28). This result can be extended to the equations for the harmonic maps: with each Killing vector field of a Riemannian manifold M and each harmonic map $\phi : N \to M$ one obtains a vector field on N whose divergence is equal to zero. (Such vector fields are called *conserved currents*.)

This assertion can be proved using (B.29), (1.23), and (1.24), denoting by K^i the components of a Killing vector field with respect to the coordinate system x^i and by g_{ij} the components of the metric tensor of M,

$$\begin{aligned}
&\frac{1}{\sqrt{|h|}} \frac{\partial}{\partial y^\alpha} \left[\sqrt{|h|}\, h^{\alpha\beta} \phi^*\big(g_{ik} K^i\big) \frac{\partial(\phi^* x^k)}{\partial y^\beta} \right] \\
&\quad = h^{\alpha\beta} \frac{\partial(\phi^* x^k)}{\partial y^\beta} \frac{\partial}{\partial y^\alpha} \phi^*\big(g_{ik} K^i\big) - \phi^*\big(g_{ik} K^i\big)\big(\phi^* \Gamma^k_{js}\big) h^{\alpha\beta} \frac{\partial(\phi^* x^j)}{\partial y^\alpha} \frac{\partial(\phi^* x^s)}{\partial y^\beta} \\
&\quad = h^{\alpha\beta} \frac{\partial(\phi^* x^k)}{\partial y^\beta} \frac{\partial(\phi^* x^s)}{\partial y^\alpha}\, \phi^* \left[\frac{\partial(g_{ik} K^i)}{\partial x^s} - \Gamma^i_{ks} g_{ij} K^j \right] \\
&\quad = 0,
\end{aligned} \tag{B.35}$$

where the last equality follows from (6.14) and (6.55), and the fact that the factor $h^{\alpha\beta}[\partial(\phi^* x^k)/\partial y^\beta][\partial(\phi^* x^s)/\partial y^\alpha]$ is symmetric in the indices k, s. The left-hand side of this equality is the divergence of the vector field

$$\mathbf{J} \equiv h^{\alpha\beta} \phi^*\big(g_{ik} K^i\big) \frac{\partial(\phi^* x^k)}{\partial y^\beta} \frac{\partial}{\partial y^\alpha}$$

[cf. (6.108)].

As pointed out at the beginning of this appendix, the left-invariant and the right-invariant vector fields are Killing vector fields for a Lie group with a bi-invariant metric; therefore, the relation (B.35) holds if the K^i are the components with respect to the coordinate system x^i of a left-invariant or right-invariant vector field, when M is a Lie group with a bi-invariant metric. In fact, the m relations (B.30), applicable in the case where M is a Lie group with a bi-invariant metric, are particular cases of (B.35).

Exercise B.9 Show that for each value of the index p, the functions $K^i = g^{ij} M^p_j$ are components of a Killing vector field with respect to the coordinate system x^i, where the M^i_j are the functions defined in (B.31). (In fact, they are components of a left-invariant vector field.) Show that the relations (B.30) follow from (B.35), using these m Killing vector fields.

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