

Appendix A

Lie Algebras

Definition A.1 A Lie algebra, L , over a field \mathbb{K} , is a vector space over \mathbb{K} which possesses a mapping from $L \times L$ into L , usually denoted by $[\ , \]$, such that

(i) it is bilinear

$$[u, av + bw] = a[u, v] + b[u, w], \quad (\text{A.1})$$

$$[au + bv, w] = a[u, w] + b[v, w], \quad (\text{A.2})$$

for $u, v, w \in L, a, b \in \mathbb{K}$,

(ii) it is skew-symmetric

$$[u, v] = -[v, u], \quad (\text{A.3})$$

for $u, v \in L$ (by virtue of (A.3), the linearity of the bracket on the second argument (A.1) implies its linearity on the first argument (A.2), and vice versa),

(iii) it satisfies the Jacobi identity

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0, \quad (\text{A.4})$$

for $u, v, w \in L$. A Lie algebra is Abelian if $[u, v] = 0$ for $u, v \in L$.

Let L be a Lie algebra of finite dimension (that is, L is a vector space of finite dimension), and let $\{e_i\}_{i=1}^n$ be a basis of L . Owing to the bilinearity of the bracket, the value of $[u, v]$, for $u, v \in L$ arbitrary, is determined by the values of $[e_i, e_j]$ ($i, j = 1, \dots, n$), for if $u = u^i e_i$ and $v = v^j e_j$, we have $[u, v] = [u^i e_i, v^j e_j] = u^i v^j [e_i, e_j]$.

Since $[e_i, e_j]$ must belong to L , $[e_i, e_j] = c_{ij}^k e_k$, where c_{ij}^k ($i, j, k = 1, \dots, n$) are n^3 scalars, called the *structure constants* of L . The values of the structure constants are not independent, since the bracket must be skew-symmetric, and it satisfies the Jacobi identity, which imposes the following relations among the c_{ij}^k :

$$c_{ij}^k = -c_{ji}^k \quad \text{and} \quad (\text{A.5})$$

$$c_{ij}^m c_{km}^l + c_{jk}^m c_{im}^l + c_{ki}^m c_{jm}^l = 0. \quad (\text{A.6})$$

Exercise A.2 Let V be a vector space and let $\mathfrak{gl}(V)$ be the set of the linear maps from V to V with the usual sum and multiplication by scalars, and with the bracket given by $[A, B] \equiv AB - BA$. Show that $\mathfrak{gl}(V)$ is a Lie algebra. If V is of finite dimension and $\{e_i\}_{i=1}^n$ is a basis of V , the linear transformations ϕ_j^i defined by $\phi_j^i(e_k) \equiv \delta_k^i e_j$, form a basis of $\mathfrak{gl}(V)$. Show that $[\phi_i^j, \phi_k^l] = (\delta_i^r \delta_k^j \delta_s^l - \delta_i^l \delta_k^r \delta_s^j) \phi_r^s$.

Definition A.3 Let L be a Lie algebra. A *subalgebra*, M , of L is a subset of L which is a Lie algebra with the operations inherited from L .

Since most of the properties that define a Lie algebra are automatically satisfied by any subset of a given algebra (for instance, the bilinearity and skew-symmetry of the bracket), it suffices to employ the criterion given by the following theorem in order to show that some subset is or is not a subalgebra.

Theorem A.4 Let L be a Lie algebra and let $M \subset L$. M is a subalgebra of L if and only if for $u, v \in M$ and $a \in \mathbb{K}$, the elements $u + v$, au and $[u, v]$ belong to M .

The proof of this theorem is immediate and is left to the reader.

Definition A.5 Let L be a Lie algebra and M a subalgebra of L . M is an *ideal* of L if for $u \in M$ and $v \in L$, $[u, v] \in M$.

L itself and $\{0\}$ are ideals of L , and if L is Abelian, then any subalgebra of L is invariant.

Definition A.6 A Lie algebra, L , is *simple* if it is not Abelian and does not possess other ideals apart from L and $\{0\}$. L is *semisimple* if the only Abelian ideal contained in L is $\{0\}$.

For example, the set of globally Hamiltonian vector fields of a symplectic manifold is an ideal of the Lie algebra of the locally Hamiltonian vector fields (see Sect. 8.2).

Definition A.7 Let L_1 and L_2 be two Lie algebras over the same field \mathbb{K} . A map $f : L_1 \rightarrow L_2$ is a *Lie algebra homomorphism* if

- (i) f is a linear transformation (i.e., $f(au + bv) = af(u) + bf(v)$, for $u, v \in L_1$, $a, b \in \mathbb{K}$) and
- (ii) $f([u, v]) = [f(u), f(v)]$, for $u, v \in L_1$.

If, in addition, f is bijective we say that f is a *Lie algebra isomorphism*.

Exercise A.8 Let $f : L_1 \rightarrow L_2$ be a Lie algebra homomorphism. Show that $\text{Ker } f \equiv \{u \in L_1 \mid f(u) = 0\}$ is an ideal of L_1 .

Appendix B

Invariant Metrics

Any Lie group can be turned into a Riemannian manifold in such a way that all the left translations L_g (or the right translations R_g) are isometries. Let G be a Lie group and let $\{\omega^1, \dots, \omega^n\}$ be a basis for the left-invariant 1-forms; if (a_{ij}) is any (constant) non-singular symmetric $n \times n$ matrix, then

$$a_{ij}\omega^i \otimes \omega^j \tag{B.1}$$

is a metric tensor on G , which is a *left-invariant metric* since $L_g^*(a_{ij}\omega^i \otimes \omega^j) = a_{ij}\omega^i \otimes \omega^j$, for all $g \in G$. If (a_{ij}) is positive definite, the metric (B.1) is also positive definite. If, in place of the 1-forms ω^i we employ right-invariant 1-forms, in an analogous manner we obtain a *right-invariant metric*. A metric on G is *bi-invariant* if it is left-invariant and right-invariant simultaneously.

From the results of Sect. 7.5 it follows that the right-invariant vector fields are Killing vector fields for any left-invariant metric (see Exercise 7.51). For a bi-invariant metric, the right-invariant vector fields, and the left-invariant vector fields are Killing vector fields.

Example B.1 The 2×2 real matrices of the form $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$, with $x > 0$, form a Lie subgroup of $GL(2, \mathbb{R})$. Making use of Theorem 7.35, from the equation

$$\begin{aligned} \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} dx & dy \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} x^{-1} & -yx^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dx & dy \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} x^{-1} dx & x^{-1} dy \\ 0 & 0 \end{pmatrix} \\ &= x^{-1} dx \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + x^{-1} dy \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

it follows that

$$\omega^1 \equiv x^{-1} dx, \quad \omega^2 \equiv x^{-1} dy,$$

form a basis for the left-invariant 1-forms. Using the fact that the inversion mapping, $\iota(g) = g^{-1}$, is given by $\iota^*x = x^{-1}$, $\iota^*y = -yx^{-1}$ [see (7.3)], one finds that the basis

of the right-invariant 1-forms $\dot{\omega}^i = -\iota^*\omega^i$, is

$$\dot{\omega}^1 = x^{-1} dx, \quad \dot{\omega}^2 = -yx^{-1} dx + dy,$$

and the dual basis is given by

$$\dot{\mathbf{X}}_1 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad \dot{\mathbf{X}}_2 = \frac{\partial}{\partial y} \quad (\text{B.2})$$

(cf. Example 7.47). Thus, $\dot{\mathbf{X}}_1$ and $\dot{\mathbf{X}}_2$ are Killing vector fields for the metric $a_{ij}\omega^i \otimes \omega^j = x^{-2}[a_{11} dx \otimes dx + a_{12}(dx \otimes dy + dy \otimes dx) + a_{22} dy \otimes dy]$, no matter what the values are of the constants a_{11} , a_{12} , and a_{22} . In particular, taking $a_{ij} = \delta_{ij}$, we obtain the metric

$$x^{-2}(dx \otimes dx + dy \otimes dy), \quad (\text{B.3})$$

which is the metric of Poincaré's half-plane [see (6.19)] and possesses three linearly independent Killing vector fields (see Example 6.12).

Exercise B.2 Show that if G is connected, the metric $a_{ij}\omega^i \otimes \omega^j$ is also right-invariant if and only if

$$a_{im}c_{jk}^m + a_{jm}c_{ik}^m = 0, \quad (\text{B.4})$$

where the c_{jk}^i are the structure constants of G with respect to the basis $\{\omega^i\}$.

Exercise B.3 Find a basis for the left-invariant 1-forms and its dual basis for the group formed by the 3×3 matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R},$$

which is related to the Heisenberg group [see, e.g., Baker (2002), Sect. 7.7]. Determine the structure constants of the group in this basis. Is it possible to find a bi-invariant metric?

Since the coefficients a_{ij} in (B.1) are constant, the dual basis $\{\mathbf{X}_i\}$ to $\{\omega^i\}$ is a rigid basis with respect to the metric $a_{ij}\omega^i \otimes \omega^j$; thus, comparing $[\mathbf{X}_i, \mathbf{X}_j] = c_{ij}^k \mathbf{X}_k$ with (6.62) one finds that $c_{ij}^k = \Gamma^k_{ji} - \Gamma^k_{ij} \equiv 2\Gamma^k_{[ji]}$, where the Γ^i_{jk} are the Ricci rotation coefficients for the basis $\{\mathbf{X}_i\}$. Using the identity (6.63), we obtain

$$\Gamma_{ijk} = \frac{1}{2}(a_{im}c_{kj}^m - a_{jm}c_{ki}^m - a_{km}c_{ji}^m). \quad (\text{B.5})$$

The foregoing expression is simplified if the metric (B.1) is bi-invariant because in that case the last two terms on the right-hand side of (B.5) cancel [see (B.4)], leaving

$$\Gamma_{ijk} = \frac{1}{2}a_{im}c_{kj}^m, \quad (\text{B.6})$$

so that the connection and curvature forms in this basis are

$$\Gamma^i_j = \frac{1}{2}c^i_{kj}\omega^k \quad \text{and} \quad \mathcal{R}^i_j = \frac{1}{4}c^m_{jk}c^i_{ml}\omega^k \wedge \omega^l, \tag{B.7}$$

respectively [see (5.26), (7.34), and (A.6)]. Hence, the components of the curvature with respect to the basis $\{\mathbf{X}_i\}$ are

$$R^i_{jkl} = \frac{1}{4}(c^m_{jk}c^i_{ml} - c^m_{jl}c^i_{mk}) = -\frac{1}{4}c^i_{mj}c^m_{kl}. \tag{B.8}$$

It may be noticed that in the expressions (B.7) the matrix (a_{ij}) does not appear and, furthermore, that they make sense *independently* of choosing a metric on the group. It can be directly verified that, with respect to a basis for the left-invariant 1-forms, $\{\omega^1, \dots, \omega^n\}$, the connection 1-forms (B.7) define a connection with torsion equal to zero. Hence, in any Lie group there exists a torsion-free connection, defined in a natural way, without having to specify a Riemannian metric.

From (B.6) it follows that, if the metric (B.1) is bi-invariant, the coefficients Γ_{ijk} are totally skew-symmetric, since, in general, $\Gamma_{ijk} = -\Gamma_{jik}$, while from the relation $c^m_{kj} = -c^m_{jk}$ it follows that $\Gamma_{ijk} = -\Gamma_{ikj}$. Combining these formulas one finds that $\Gamma_{ijk} = -\Gamma_{kji}$. If the dimension of G is two, then the total skew-symmetry of the Ricci rotation coefficients implies that they are equal to zero and, since (a_{im}) must be invertible, $c^m_{kj} = 0$ and, therefore, G must be Abelian.

If the dimension of G is three, the skew-symmetry of Γ_{ijk} implies that $\Gamma_{ijk} = b \varepsilon_{ijk}$, where b is some constant. Then, from (B.6), we have

$$c^m_{kj} = 2a^{im}b \varepsilon_{ijk}, \tag{B.9}$$

where (a^{im}) is the inverse of the matrix (a_{im}) ; therefore

$$\frac{1}{4}c^i_{mj}c^m_{kl} = b^2 a^{pi} \varepsilon_{pjm} a^{qm} \varepsilon_{qlk} = b^2 a^{pi} \det(a^{rs}) (a_{lp}a_{kj} - a_{lj}a_{kp})$$

and from (B.8) we obtain

$$R_{ijkl} = b^2 \det(a^{rs}) (a_{ik}a_{jl} - a_{il}a_{jk}), \tag{B.10}$$

which means that G is a constant curvature space (see Examples B.6 and B.8). For any value of b , the structure constants (B.9) satisfy the Jacobi identity (A.6). It can be noticed that in this case, if the six vector fields \mathbf{X}_i and $\check{\mathbf{X}}_i$ ($i = 1, 2, 3$) are linearly independent, then they form a basis for the Killing vector fields of G , since the maximum dimension of the Lie algebra of the Killing vector fields of a Riemannian manifold of dimension n is $n(n + 1)/2$.

Exercise B.4 Show that for any Lie group, G , the left-invariant vector fields \mathbf{X}_i , and the right-invariant vector fields $\check{\mathbf{X}}_i$ are linearly independent if and only if the center of the Lie algebra of G is $\{0\}$; that is, if and only if zero is the only element of \mathfrak{g} whose Lie bracket with all the elements of the algebra is equal to zero.

Exercise B.5 Show that if $a_{ij}\omega^i \otimes \omega^j$ is a bi-invariant metric on G , where $\{\omega^i\}$ is a basis for the left-invariant 1-forms, then $\nabla_{\mathbf{Y}}\mathbf{Z} = \frac{1}{2}[\mathbf{Y}, \mathbf{Z}]$ and $R(\mathbf{Y}, \mathbf{Z})\mathbf{W} = -\frac{1}{4}[[\mathbf{Y}, \mathbf{Z}], \mathbf{W}]$, for $\mathbf{Y}, \mathbf{Z}, \mathbf{W} \in \mathfrak{g}$, where ∇ denotes the Riemannian connection associated with the bi-invariant metric and R is its curvature tensor. Show that the integral curves of any left-invariant vector field are geodesics.

If c^i_{jk} denote the structure constants of an arbitrary Lie algebra, then the constants

$$g_{ij} = -c^k_{im}c^m_{jk} \quad (\text{B.11})$$

form a symmetric matrix, $g_{ji} = -c^k_{jm}c^m_{ik} = -c^m_{ik}c^k_{jm} = g_{ij}$. Furthermore, making use of (B.11), and the identities $c^m_{ij}c^l_{mk} + c^m_{jk}c^l_{mi} + c^m_{ki}c^l_{mj} = 0$ and $c^i_{jk} = -c^i_{kj}$ [see (A.5) and (A.6)] one finds that

$$\begin{aligned} g_{im}c^m_{jk} + g_{jm}c^m_{ik} &= -c^s_{ir}c^r_{ms}c^m_{jk} - c^s_{jr}c^r_{ms}c^m_{ik} \\ &= c^s_{ir}(c^m_{ks}c^r_{mj} + c^m_{sj}c^r_{mk}) - c^s_{jr}c^r_{ms}c^m_{ik} \\ &= (-c^s_{ki}c^m_{rs} - c^s_{rk}c^m_{is})c^r_{mj} + c^s_{ir}c^m_{sj}c^r_{mk} - c^s_{jr}c^r_{ms}c^m_{ik} \\ &= 0. \end{aligned}$$

Hence, if G is a connected Lie group and $\{\omega^i\}$ is a basis for the left-invariant 1-forms, the tensor field $g_{ij}\omega^i \otimes \omega^j$, with the g_{ij} defined by (B.5), is bi-invariant (see Exercise B.2). However, the matrix (g_{ij}) can be singular, and therefore $g_{ij}\omega^i \otimes \omega^j$ does not need to be a Riemannian metric on G . It can be shown that the matrix (g_{ij}) , defined in (B.11), is invertible if and only if the Lie algebra is *semisimple* (that is, it does not have Abelian proper ideals) [see, e.g., Sattinger and Weaver (1986, Chap. 9)].

It may be noticed that the components of the Ricci tensor associated with the curvature tensor (B.8) are given by $R_{ij} = \frac{1}{4}g_{ij}$, with g_{ij} defined by (B.11).

Example B.6 Let us consider the group $G = \text{SU}(2)$ with the parametrization given by the Euler angles, ϕ, θ, ψ ,

$$g = (\exp \phi(g)\mathbf{X}_3)(\exp \theta(g)\mathbf{X}_1)(\exp \psi(g)\mathbf{X}_3), \quad (\text{B.12})$$

where $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ is the basis of $\mathfrak{su}(2)$ given in Exercise 7.19 [cf. (8.94)]. From (7.54) it follows that (B.12) is equivalent to

$$\begin{aligned} g &= \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \begin{pmatrix} \cos \theta/2 & i \sin \theta/2 \\ i \sin \theta/2 & \cos \theta/2 \end{pmatrix} \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix} \\ &= \begin{pmatrix} e^{i(\phi+\psi)/2} \cos \theta/2 & i e^{i(\phi-\psi)/2} \sin \theta/2 \\ i e^{i(\psi-\phi)/2} \sin \theta/2 & e^{-i(\phi+\psi)/2} \cos \theta/2 \end{pmatrix}, \end{aligned} \quad (\text{B.13})$$

where, by abuse of notation, we have simply written ϕ, θ, ψ , in place of $\phi(g), \theta(g)$, and $\psi(g)$, respectively. As in Example B.1, we can make use of Theorem 7.35 to

find the basis of the left-invariant 1-forms, dual to the basis $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$. Since the structure constants for the basis $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ are the same as those of the basis $\{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3\}$ of $\mathfrak{so}(3)$, equations (8.95)–(8.100) hold if \mathbf{S}_i is replaced by \mathbf{X}_i ; hence the set

$$\begin{aligned}\omega^1 &= \sin\theta \sin\psi \, d\phi + \cos\psi \, d\theta, \\ \omega^2 &= \sin\theta \cos\psi \, d\phi - \sin\psi \, d\theta, \\ \omega^3 &= \cos\theta \, d\phi + d\psi\end{aligned}\tag{B.14}$$

is the dual basis to $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$. Making use of the fact that $[\mathbf{X}_i, \mathbf{X}_j] = \sum_{k=1}^3 \varepsilon_{ijk} \mathbf{X}_k = \delta^{kl} \varepsilon_{ijk} \mathbf{X}_l$, we have $c_{ij}^l = \delta^{kl} \varepsilon_{ijk}$; therefore, from (B.11), $g_{ij} = -\delta^{pk} \varepsilon_{imp} \delta^{qm} \varepsilon_{jkq} = -\delta^{pk} (\delta_{jp} \delta_{ki} - \delta_{ji} \delta_{kp}) = 2\delta_{ij}$, which is an invertible matrix and $g_{ij} \omega^i \otimes \omega^j = 2\delta_{ij} \omega^i \otimes \omega^j$. From (B.14) we then have

$$\begin{aligned}g_{ij} \omega^i \otimes \omega^j &= 2[d\phi \otimes d\phi + d\theta \otimes d\theta + d\psi \otimes d\psi \\ &\quad + \cos\theta(d\phi \otimes d\psi + d\psi \otimes d\phi)].\end{aligned}\tag{B.15}$$

According to the foregoing results, we may conclude that the metric (B.15) is bi-invariant. As we shall show below, this metric is essentially the usual metric of the sphere S^3 .

The underlying manifold of the group $SU(2)$ can be identified with the sphere S^3 in the following manner. All the elements of $SU(2)$ are of the form

$$\begin{pmatrix} x + iy & z + iw \\ -z + iw & x - iy \end{pmatrix},\tag{B.16}$$

where x, y, z, w are real numbers such that $x^2 + y^2 + z^2 + w^2 = 1$. Hence, there is a one-to-one correspondence between the elements of $SU(2)$ and the points of $S^3 \equiv \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}$. From the expressions (B.13) and (B.16), separating the real and imaginary parts, one obtains a local expression for the inclusion of $SU(2)$, or S^3 , in \mathbb{R}^4 ($i : SU(2) \rightarrow \mathbb{R}^4$), namely

$$\begin{aligned}i^*x &= \cos\frac{1}{2}\theta \cos\frac{1}{2}(\phi + \psi), & i^*y &= \cos\frac{1}{2}\theta \sin\frac{1}{2}(\phi + \psi), \\ i^*z &= -\sin\frac{1}{2}\theta \sin\frac{1}{2}(\phi - \psi), & i^*w &= \sin\frac{1}{2}\theta \cos\frac{1}{2}(\phi - \psi).\end{aligned}$$

The pullback under i of the usual metric of \mathbb{R}^4 is then

$$\begin{aligned}i^*(dx \otimes dx + dy \otimes dy + dz \otimes dz + dw \otimes dw) \\ = \frac{1}{4}[d\phi \otimes d\phi + d\theta \otimes d\theta + d\psi \otimes d\psi + \cos\theta(d\phi \otimes d\psi + d\psi \otimes d\phi)],\end{aligned}$$

which, except for a factor 1/8, coincides with the metric (B.15). This means that the metric (B.15), which, as we have shown, is the metric of a constant curvature

space, is essentially the standard metric of S^3 (which is, clearly, a constant curvature space). Moreover, the left-invariant vector fields \mathbf{S}_i [given by (8.98)] and the right-invariant vector fields $\dot{\mathbf{S}}_i$ [given by (8.100)] of $SU(2)$, are Killing vector fields for the metric (B.15) and, therefore, for S^3 . Thus, the Lie algebra of the Killing vector fields of S^3 , which is $\mathfrak{so}(4)$ [the Lie algebra of $SO(4)$], possesses the basis $\{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, -\dot{\mathbf{S}}_1, -\dot{\mathbf{S}}_2, -\dot{\mathbf{S}}_3\}$, which satisfies the relations

$$\begin{aligned} [\mathbf{S}_i, \mathbf{S}_j] &= \sum_{k=1}^3 \varepsilon_{ijk} \mathbf{S}_k, \\ [(-\dot{\mathbf{S}}_i), (-\dot{\mathbf{S}}_j)] &= \sum_{k=1}^3 \varepsilon_{ijk} (-\dot{\mathbf{S}}_k), \\ [\mathbf{S}_i, (-\dot{\mathbf{S}}_j)] &= 0; \end{aligned} \tag{B.17}$$

hence, $\mathfrak{so}(4)$ is the direct sum of two copies of $\mathfrak{su}(2)$:

$$\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2). \tag{B.18}$$

Each $g \in SU(2)$ can be regarded as a point of S^3 (by expressing g in the form (B.16) and taking the corresponding x, y, z, w as the coordinates of a point of S^3), and for any $g_1 \in SU(2)$, both L_{g_1} and R_{g_1} are isometries for the metric (B.15). Hence, if $(g_1, g_2) \in SU(2) \times SU(2)$, the mapping $g \mapsto L_{g_1} R_{g_2} g = g_1 g g_2 = R_{g_2} L_{g_1} g$, from $SU(2)$ onto $SU(2)$, can be seen as an isometric map from S^3 onto S^3 . In fact, it turns out that any isometry of S^3 that does not change the orientation is obtained in this manner, with g_1 and g_2 determined up to sign; if $(g_1, g_2) \in SU(2) \times SU(2)$, then $(-g_1, -g_2)$ also belongs to $SU(2) \times SU(2)$ and $L_{g_1} R_{g_2} = L_{-g_1} R_{-g_2}$. From the preceding discussion it also follows that any rotation about the origin in \mathbb{R}^4 can be represented in the form

$$\begin{pmatrix} x' + iy' & z' + iw' \\ -z' + iw' & x' - iy' \end{pmatrix} = g_1 \begin{pmatrix} x + iy & z + iw \\ -z + iw & x - iy \end{pmatrix} g_2 \tag{B.19}$$

[cf. (7.65)] with $g_1, g_2 \in SU(2)$ determined up to sign. [This result is the counterpart of (B.18).]

Exercise B.7 Show that from (B.19) it follows directly that the transformation $(x, y, z, w) \mapsto (x', y', z', w')$ belongs to $SO(4)$.

Example B.8 The functions $\alpha, \beta, \gamma : SL(2, \mathbb{R}) \rightarrow \mathbb{R}$ defined by

$$g = \left(\exp \frac{1}{2} \alpha(g) \mathbf{X}_1 \right) \left(\exp \frac{1}{2} \beta(g) (\mathbf{X}_2 + \mathbf{X}_3) \right) \left(\exp \frac{1}{2} \gamma(g) \mathbf{X}_1 \right), \tag{B.20}$$

where $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ is the basis of $\mathfrak{sl}(2, \mathbb{R})$ given in Example 7.16, form a local coordinate system for $SL(2, \mathbb{R})$, alternative to that defined by (7.4). From (B.20)

and (7.51) we then have

$$\begin{aligned}
 g &= \begin{pmatrix} e^{\alpha/2} & 0 \\ 0 & e^{-\alpha/2} \end{pmatrix} \begin{pmatrix} \cosh \beta/2 & \sinh \beta/2 \\ \sinh \beta/2 & \cosh \beta/2 \end{pmatrix} \begin{pmatrix} e^{\gamma/2} & 0 \\ 0 & e^{-\gamma/2} \end{pmatrix} \\
 &= \begin{pmatrix} e^{(\alpha+\gamma)/2} \cosh \beta/2 & e^{(\alpha-\gamma)/2} \sinh \beta/2 \\ e^{-(\alpha-\gamma)/2} \sinh \beta/2 & e^{-(\alpha+\gamma)/2} \cosh \beta/2 \end{pmatrix}, \tag{B.21}
 \end{aligned}$$

where we have written α, β, γ instead of $\alpha(g), \beta(g), \gamma(g)$ [cf. (B.13)]. The dual basis to $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$, expressed in terms of the coordinates α, β, γ , can be obtained making use of (B.20) and Theorem 7.35, which leads to [see (7.20)]

$$\begin{aligned}
 g^{-1} dg &= \frac{1}{2} \exp\left(-\frac{1}{2}\gamma \mathbf{X}_1\right) \exp\left(-\frac{1}{2}\beta(\mathbf{X}_2 + \mathbf{X}_3)\right) \\
 &\quad \cdot \lambda_1 \exp\left(\frac{1}{2}\beta(\mathbf{X}_2 + \mathbf{X}_3)\right) \exp\left(\frac{1}{2}\gamma \mathbf{X}_1\right) d\alpha \\
 &\quad + \frac{1}{2} \exp\left(-\frac{1}{2}\gamma \mathbf{X}_1\right) (\lambda_2 + \lambda_3) \exp\left(\frac{1}{2}\gamma \mathbf{X}_1\right) d\beta + \frac{1}{2} \lambda_1 d\gamma \\
 &= \frac{1}{2} (\cosh \beta d\alpha + d\gamma) \lambda_1 + \frac{1}{2} e^{-\gamma} (\sinh \beta d\alpha + d\beta) \lambda_2 \\
 &\quad + \frac{1}{2} e^{\gamma} (-\sinh \beta d\alpha + d\beta) \lambda_3,
 \end{aligned}$$

and thus

$$\begin{aligned}
 \omega^1 &= \frac{1}{2} (\cosh \beta d\alpha + d\gamma), \\
 \omega^2 &= \frac{1}{2} e^{-\gamma} (\sinh \beta d\alpha + d\beta), \\
 \omega^3 &= \frac{1}{2} e^{\gamma} (-\sinh \beta d\alpha + d\beta). \tag{B.22}
 \end{aligned}$$

On the other hand, from (7.20) we find that $[\lambda_1, \lambda_2] = 2\lambda_2$, $[\lambda_2, \lambda_3] = \lambda_1$, $[\lambda_3, \lambda_1] = 2\lambda_3$ (i.e., the structure constants that are different from zero are given by $c_{12}^2 = 2 = c_{31}^3$, $c_{23}^1 = 1$) and from (B.11) it follows that

$$(g_{ij}) = \begin{pmatrix} -8 & 0 & 0 \\ 0 & 0 & -4 \\ 0 & -4 & 0 \end{pmatrix}; \tag{B.23}$$

therefore, using (B.22) and (B.23),

$$\begin{aligned}
 g_{ij} \omega^i \otimes \omega^j &= -4(2\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^3 + \omega^3 \otimes \omega^2) \\
 &= -2[d\alpha \otimes d\alpha + d\beta \otimes d\beta + d\gamma \otimes d\gamma \\
 &\quad + \cosh \beta(d\alpha \otimes d\gamma + d\gamma \otimes d\alpha)] \tag{B.24}
 \end{aligned}$$

is a pseudo-Riemannian bi-invariant metric on $\mathrm{SL}(2, \mathbb{R})$ and, with this metric, $\mathrm{SL}(2, \mathbb{R})$ is a constant curvature space. (Note that $g_{ij}\omega^i \otimes \omega^j = -8\omega^1 \otimes \omega^1 - 2(\omega^2 + \omega^3) \otimes (\omega^2 + \omega^3) + 2(\omega^2 - \omega^3) \otimes (\omega^2 - \omega^3)$, which explicitly shows that this metric is pseudo-Riemannian.) In a similar manner to the case of $\mathrm{SU}(2)$, considered in the foregoing example, $\mathrm{SL}(2, \mathbb{R})$ with the metric (B.24) can be identified with a submanifold of \mathbb{R}^4 , provided that in the latter we introduce a pseudo-Riemannian flat metric [cf. Conlon (2001), Sect. 10.7].

Indeed, any element of $\mathrm{SL}(2, \mathbb{R})$ is of the form

$$\begin{pmatrix} x+w & y+z \\ z-y & x-w \end{pmatrix}, \quad (\text{B.25})$$

where x, y, z, w are real numbers with $x^2 + y^2 - z^2 - w^2 = 1$. This means that the underlying manifold of $\mathrm{SL}(2, \mathbb{R})$ can be identified with the hyperboloid $N \equiv \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 - z^2 - w^2 = 1\}$. Comparing (B.21) with (B.25), one finds the following local expression for the inclusion of $\mathrm{SL}(2, \mathbb{R})$ in \mathbb{R}^4 :

$$\begin{aligned} i^*x &= \cosh \frac{1}{2}\beta \cosh \frac{1}{2}(\alpha + \gamma), & i^*y &= \sinh \frac{1}{2}\beta \sinh \frac{1}{2}(\alpha - \gamma), \\ i^*z &= \sinh \frac{1}{2}\beta \cosh \frac{1}{2}(\alpha - \gamma), & i^*w &= \cosh \frac{1}{2}\beta \sinh \frac{1}{2}(\alpha + \gamma), \end{aligned}$$

hence, the metric induced on $\mathrm{SL}(2, \mathbb{R})$, or on N , by the pseudo-Riemannian metric $dx \otimes dx + dy \otimes dy - dz \otimes dz - dw \otimes dw$ of \mathbb{R}^4 is

$$\begin{aligned} & i^*(dx \otimes dx + dy \otimes dy - dz \otimes dz - dw \otimes dw) \\ &= -\frac{1}{4} [d\alpha \otimes d\alpha + d\beta \otimes d\beta + d\gamma \otimes d\gamma + \cosh \beta (d\alpha \otimes d\gamma + d\gamma \otimes d\alpha)] \end{aligned}$$

and coincides, except for a factor 1/8, with the metric (B.24). Then, owing to the bi-invariance of (B.24), the left-invariant vector fields of $\mathrm{SL}(2, \mathbb{R})$, together with the right-invariant ones are Killing vector fields for the metric (B.24) and for the metric induced on N . On the other hand, N and the metric $dx \otimes dx + dy \otimes dy - dz \otimes dz - dw \otimes dw$ are invariant under the linear transformations of \mathbb{R}^4 into \mathbb{R}^4 represented by the real 4×4 matrices, A , with determinant equal to 1, such that

$$A^t \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} A = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad (\text{B.26})$$

which form the group $\mathrm{SO}(2, 2)$, whose dimension is six. Thus, in an analogous way to (B.18), we have

$$\mathfrak{so}(2, 2) = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}). \quad (\text{B.27})$$

Since for any $g \in \text{SL}(2, \mathbb{R})$, the transformations L_g and R_g are isometries of the metric (B.24), if $(g_1, g_2) \in \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$, the transformation $g \mapsto L_{g_1} R_{g_2} g = g_1 g g_2$, from $\text{SL}(2, \mathbb{R})$ onto $\text{SL}(2, \mathbb{R})$, is an isometry and can be identified with an isometric transformation from N onto N . That is, using (B.25), the expression

$$\begin{pmatrix} x' + w' & y' + z' \\ z' - y' & x' - w' \end{pmatrix} = g_1 \begin{pmatrix} x + w & y + z \\ z - y & x - w \end{pmatrix} g_2 \tag{B.28}$$

gives an isometric transformation from N onto N , for any pair of elements $g_1, g_2 \in \text{SL}(2, \mathbb{R})$, and it turns out that any transformation belonging to $\text{SO}(2, 2)$ can be represented in this manner with g_1 and g_2 determined up to a common sign.

Harmonic Maps The harmonic mapping equations constitute a generalization of the geodesic equations (5.7). In their general form, given two Riemannian manifolds, N and M , of dimensions n and m , respectively, a differentiable map $\phi : N \rightarrow M$ is *harmonic* if

$$\frac{1}{\sqrt{|h|}} \frac{\partial}{\partial y^\alpha} \left(\sqrt{|h|} h^{\alpha\beta} \frac{\partial(\phi^* x^k)}{\partial y^\beta} \right) + (\phi^* \Gamma_{ji}^k) h^{\alpha\beta} \frac{\partial(\phi^* x^j)}{\partial y^\alpha} \frac{\partial(\phi^* x^i)}{\partial y^\beta} = 0, \tag{B.29}$$

where $(h^{\alpha\beta})$ is the inverse of the matrix $(h_{\alpha\beta})$, formed by the components of the metric tensor of N with respect to a local coordinate system (y^1, \dots, y^n) , $h \equiv \det(h_{\alpha\beta})$, (x^1, \dots, x^m) is a coordinate system on M and the Γ_{ji}^k are the Christoffel symbols corresponding to the metric tensor of M in the coordinate system x^i [see, e.g., Hélein (2002)]. When $N = \mathbb{R}$, with $y^1 = t$ and $h_{11} = 1$, equations (B.29) reduce to the equations of the geodesics (5.7). When $M = \mathbb{R}$, with its usual metric, equations (B.29) reduce to the Laplace equation, $\nabla^2 \phi = 0$ [see (6.113)].

An interesting fact is that in the case of a harmonic map $\phi : N \rightarrow G$, where G is a Lie group that admits a bi-invariant metric, equations (B.29) amount to

$$\frac{1}{\sqrt{|h|}} \frac{\partial}{\partial y^\alpha} \left[\sqrt{|h|} h^{\alpha\beta} (\phi^* \omega^k) \left(\frac{\partial}{\partial y^\beta} \right) \right] = 0, \tag{B.30}$$

where the ω^k are left-invariant 1-forms on G . In effect, the 1-forms ω^k can be expressed locally in the form

$$\omega^k = M_i^k dx^i, \tag{B.31}$$

with each $M_i^k \in C^\infty(G)$. Then

$$\frac{\partial}{\partial x^i} = M_i^k \mathbf{X}_k, \tag{B.32}$$

where the \mathbf{X}_k are the left-invariant fields that form the dual basis to $\{\omega^k\}$. Using the properties of a connection [see (5.1)], from Exercise B.5 it follows that the Christof-

fel symbols for the bi-invariant metric of G with respect to the coordinate system x^i are given by

$$\begin{aligned}\Gamma_{jk}^i \frac{\partial}{\partial x^i} &= M_k^s \nabla_{\mathbf{X}_s} (M_j^r \mathbf{X}_r) \\ &= M_k^s \mathbf{X}_s (M_j^r) \mathbf{X}_r + M_k^s M_j^r \nabla_{\mathbf{X}_s} \mathbf{X}_r \\ &= \left(\frac{\partial}{\partial x^k} M_j^r \right) \mathbf{X}_r + \frac{1}{2} M_k^s M_j^r [\mathbf{X}_s, \mathbf{X}_r].\end{aligned}$$

Since the Christoffel symbols Γ_{jk}^i are symmetric in the indices j, k , while $M_k^s M_j^r [\mathbf{X}_s, \mathbf{X}_r]$ is antisymmetric in these indices, using (B.32), it follows that

$$\Gamma_{jk}^i = \tilde{M}_r^i \frac{\partial}{\partial x^{(k}} M_{j)}^r, \quad (\text{B.33})$$

where (\tilde{M}_j^i) is the inverse of the matrix (M_j^i) , and the parentheses denote symmetrization on the enclosed indices [e.g., $t_{(ij)} = \frac{1}{2}(t_{ij} + t_{ji})$].

Thus, from (B.31), (1.23), and (1.24) we have

$$\begin{aligned}h^{\alpha\beta} \frac{\partial}{\partial y^\alpha} \left[(\phi^* \omega^k) \left(\frac{\partial}{\partial y^\beta} \right) \right] \\ &= h^{\alpha\beta} \frac{\partial}{\partial y^\alpha} \left[(\phi^* M_i^k) \frac{\partial(\phi^* x^i)}{\partial y^\beta} \right] \\ &= h^{\alpha\beta} (\phi^* M_i^k) \frac{\partial}{\partial y^\alpha} \frac{\partial(\phi^* x^i)}{\partial y^\beta} + h^{\alpha\beta} \frac{\partial(\phi^* x^i)}{\partial y^\beta} \frac{\partial(\phi^* x^j)}{\partial y^\alpha} \phi^* \left(\frac{\partial M_i^k}{\partial x^j} \right).\end{aligned}$$

Using the fact that $(h^{\alpha\beta})$ is symmetric, from (B.33) we then have

$$\begin{aligned}h^{\alpha\beta} \frac{\partial}{\partial y^\alpha} \left[(\phi^* \omega^k) \left(\frac{\partial}{\partial y^\beta} \right) \right] \\ &= (\phi^* M_s^k) \left[h^{\alpha\beta} \frac{\partial}{\partial y^\alpha} \frac{\partial(\phi^* x^s)}{\partial y^\beta} + h^{\alpha\beta} \frac{\partial(\phi^* x^i)}{\partial y^\beta} \frac{\partial(\phi^* x^j)}{\partial y^\alpha} \phi^* \Gamma_{ij}^s \right],\end{aligned}$$

which shows the equivalence of (B.29) and (B.30) in the case where M is a Lie group with a bi-invariant metric.

As pointed out previously, when $N = \mathbb{R}$ with the usual metric, the equations for a harmonic map reduce to the geodesic equations. Hence, the equations for a geodesic, C , of a group G with a bi-invariant metric, can be expressed as

$$\frac{d}{dt} \left[(C^* \omega^k) \left(\frac{\partial}{\partial t} \right) \right] = 0$$

[see (B.30)]; therefore $(C^* \omega^k)(\partial/\partial t) = a^k$, where each a^k is a real constant. That is, $\omega^k(C'_t) = a^k$, which amounts to $C'_t = a^k \mathbf{X}_k(C(t))$. Thus, in this case, a geodesic is an integral curve of some left-invariant vector field (cf. Exercise B.5).

Taking into account that, when G is some subgroup of $\text{GL}(p, \mathbb{R})$, a basis for the left-invariant 1-forms can be found from the relation $g^{-1}dg = \lambda_a \omega^a$ [see (7.46)], where the λ_a are constant matrices that form a basis for a representation of the Lie algebra of G , it follows that equations (B.30) amount to the matrix equation

$$\frac{1}{\sqrt{|h|}} \frac{\partial}{\partial y^\alpha} \left(\sqrt{|h|} h^{\alpha\beta} g^{-1} \frac{\partial g}{\partial y^\beta} \right) = 0, \quad (\text{B.34})$$

where it is understood that g is an arbitrary element of G , parameterized in terms of the y^α through the map $\phi : N \rightarrow G$.

Each Killing vector field of a Riemannian manifold, M , gives rise to a conserved quantity, constant of motion, or first integral of the geodesic equations (Theorem 6.28). This result can be extended to the equations for the harmonic maps: with each Killing vector field of a Riemannian manifold M and each harmonic map $\phi : N \rightarrow M$ one obtains a vector field on N whose divergence is equal to zero. (Such vector fields are called *conserved currents*.)

This assertion can be proved using (B.29), (1.23), and (1.24), denoting by K^i the components of a Killing vector field with respect to the coordinate system x^i and by g_{ij} the components of the metric tensor of M ,

$$\begin{aligned} & \frac{1}{\sqrt{|h|}} \frac{\partial}{\partial y^\alpha} \left[\sqrt{|h|} h^{\alpha\beta} \phi^*(g_{ik} K^i) \frac{\partial(\phi^* x^k)}{\partial y^\beta} \right] \\ &= h^{\alpha\beta} \frac{\partial(\phi^* x^k)}{\partial y^\beta} \frac{\partial}{\partial y^\alpha} \phi^*(g_{ik} K^i) - \phi^*(g_{ik} K^i) (\phi^* \Gamma_{js}^k) h^{\alpha\beta} \frac{\partial(\phi^* x^j)}{\partial y^\alpha} \frac{\partial(\phi^* x^s)}{\partial y^\beta} \\ &= h^{\alpha\beta} \frac{\partial(\phi^* x^k)}{\partial y^\beta} \frac{\partial(\phi^* x^s)}{\partial y^\alpha} \phi^* \left[\frac{\partial(g_{ik} K^i)}{\partial x^s} - \Gamma_{ks}^i g_{ij} K^j \right] \\ &= 0, \end{aligned} \quad (\text{B.35})$$

where the last equality follows from (6.14) and (6.55), and the fact that the factor $h^{\alpha\beta} [\partial(\phi^* x^k)/\partial y^\beta][\partial(\phi^* x^s)/\partial y^\alpha]$ is symmetric in the indices k, s . The left-hand side of this equality is the divergence of the vector field

$$\mathbf{J} \equiv h^{\alpha\beta} \phi^*(g_{ik} K^i) \frac{\partial(\phi^* x^k)}{\partial y^\beta} \frac{\partial}{\partial y^\alpha}$$

[cf. (6.108)].

As pointed out at the beginning of this appendix, the left-invariant and the right-invariant vector fields are Killing vector fields for a Lie group with a bi-invariant metric; therefore, the relation (B.35) holds if the K^i are the components with respect to the coordinate system x^i of a left-invariant or right-invariant vector field, when M is a Lie group with a bi-invariant metric. In fact, the m relations (B.30), applicable in the case where M is a Lie group with a bi-invariant metric, are particular cases of (B.35).

Exercise B.9 Show that for each value of the index p , the functions $K^i = g^{ij} M_j^p$ are components of a Killing vector field with respect to the coordinate system x^i , where the M_j^i are the functions defined in (B.31). (In fact, they are components of a left-invariant vector field.) Show that the relations (B.30) follow from (B.35), using these m Killing vector fields.

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