Appendix A
Mathematical Preliminaries

A.1 Linear Algebra

A.1.1 Rank and Determinant

If A is an arbitrary $m \times n$ matrix over the field of real numbers $\mathbb{R}$, then the *row space* of A is the subspace of $\mathbb{R}^n$ generated by the rows of A, and the *column space* of A is the subspace of $\mathbb{R}^m$ generated by the columns of A. The dimensions of the row space and of the column space are called the *row rank* of A and the *column rank* of A, respectively. The following results hold linking the row space and the column space:

Theorem A.1 *The row rank and column rank of the matrix A are equal.*

Row rank and column rank (since they are equal) are usually called the *rank* of A and written rank(A). For arbitrary matrices A and B of appropriate dimension that the product AB exists has the property that

$$\operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A),\ \operatorname{rank}(B)\}. \tag{A.1}$$

For every square matrix A (i.e., a matrix with the same number of rows and columns) there can be assigned a unique scalar, called its *determinant*, and denoted det(A). This value can be assigned recursively as follows: first consider a scalar as matrix with one row and one column. The determinant of such a matrix is equal to the element itself. For a matrix of dimension 2 given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{A.2}$$

then the determinant of A is given (by definition) as

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$$\det(A) = a_{11}a_{22} - a_{12}a_{21}. \tag{A.3}$$

The determinant of a matrix A of dimension 3 may be defined by

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

The representation above demonstrates how determinants can be defined in terms of determinants of matrices of lower order. In order to develop a specific generic formula, first define the *minor* M_{rs} of any element a_{rs} of a general matrix A of dimension n is the determinant of the matrix obtained by omitting the row and column containing a_{rs}. The *cofactor* of a_{rs}, here denoted by A_{rs}, is given by

$$A_{rs} = (-1)^{r+s} M_{rs}$$

With these definitions, the determinant of a general square matrix A can then be expressed (in terms of minors and cofactors) as

$$\det(A) = a_{r1}A_{r1} + a_{r2}A_{r2} + \ldots + a_{rn}A_{rn}, \qquad r = 1, 2, \ldots, n.$$

Some useful properties of determinants are:

- If A has two identical rows (columns), then $\det(A) = 0$.
- If A is invertible then $\det(A)$ is nonzero.
- The determinant of a product of two matrices A and B is equal to the product of their determinants, $\det(AB) = \det(A)\det(B)$.

Define the *identity matrix* or order n which will be denoted as I_n as the matrix which has ones on the diagonal elements and zeros elsewhere. It is easy to verify that the determinant of such a matrix is unity.

The *transpose* of a matrix A, denoted by A^T, is defined by interchanging the rows and columns, i.e., $\{A^T\}_{ij} = A_{ji}$. A square matrix is said to be *orthogonal* if $A^T A = AA^T = I_n$.

A matrix A is a *symmetric matrix* if $A^T = A$.

A.1.2 Eigenvalues and Eigenvectors

Definition A.1. If $A \in \mathbb{R}^{n \times n}$ is a square matrix, then $\lambda \in \mathbb{C}$ is an *eigenvalue* of A, if for some nonzero vector $v \in \mathbb{C}^n$,

$$Av = \lambda v$$

In this case v is said to be the *eigenvector* corresponding to the eigenvalue λ.

The matrix $\lambda I_n - A$ is called the *characteristic matrix* of A. The determinant of the characteristic matrix is called the *characteristic polynomial* of A and

$$\det(\lambda I_n - A) = 0 \tag{A.4}$$

is called the *characteristic equation* of A. This is an nth-order polynomial in λ. Note that the values of λ which constitute roots of the characteristic equation determine the eigenvalues of A.

If A is a square matrix partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \tag{A.5}$$

where the top left and bottom right sub-blocks (A.5) are square, then the following properties hold:

Proposition A.1 *The determinant and the eigenvalues of A satisfy:*

1. $det(A) = det(A_{11})\, det(A_{22})$
2. $\lambda(A) = \lambda(A_{11}) \cup \lambda(A_{22})$

A.1.3 QR Decomposition

A frequently used decomposition is the so-called QR reduction, whereby an arbitrary matrix $X \in \mathbb{R}^{n \times m}$ can be expressed as

$$X = QR, \tag{A.6}$$

where R is an upper triangular matrix of the same dimension as X and Q is an orthogonal matrix.

A.1.4 Norms

Definition A.2. A norm is a function which assigns to every vector x in a vector space a real number $\|x\|$ such that:

1. $\|x\| \geq 0$
2. $\|x\| = 0$ if and only if $x = 0$
3. $\|kx\| = |k|\, \|x\|$ where k is a scalar and $|k|$ is the absolute value of k
4. $\|x + y\| \leq \|x\| + \|y\|$

Property 4 is usually called the *triangle inequality*.
Three common vector norms are:

1. The 1-norm
$$\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n|. \tag{A.7}$$
2. The Euclidean norm (or 2-norm)
$$\|x\|_2 = (|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2)^{1/2}. \tag{A.8}$$
3. The ∞ norm
$$\|x\|_\infty = \max\ |x|_i \quad (i = 1, \ldots, n). \tag{A.9}$$

The Euclidean norm corresponds exactly to the usual notion of distance, i.e., the length of the straight line between two points.

Corresponding to the vector norms above are three *induced matrix norms*:

1. The 1-norm
$$\|A\|_1 = \max_j (\sum_i |a_{ij}|). \tag{A.10}$$
2. The 2-norm is given by the maximum eigenvalue of $A^T A$.
3. The ∞ norm
$$\|A\|_\infty = \max_i (\sum_j |a_{ij}|). \tag{A.11}$$

For any two vectors x and y the equality
$$y^T x = \|x\|\|y\| \cos(\theta) \tag{A.12}$$
holds where θ is the angle (or more formally the direction cosine) between the two vectors and $\|\cdot\|$ denotes the Euclidean norm. From the properties of the cosine function it follows
$$y^T x \leq \|x\|\|y\| \tag{A.13}$$
for all x and y. The inequality in Eq. (A.13) is called the *Cauchy–Schwarz inequality*.

A.1.5 Quadratic Forms

A quadratic form is a function Q of n real variables $x_1, x_2, \ldots, x_n$ such that
$$Q(x_1, x_2, \ldots, x_n) = \sum_{i,j=1}^{n} q_{ij} x_i x_j \qquad q_{ij} = q_{ji}. \tag{A.14}$$

Without loss of generality, the q_{ij} can be thought of as the entries of a symmetric matrix Q and the x_i can be considered the components of the vector x. The quadratic form (A.14) can more conveniently be represented as

$$Q(x_1, x_2, \ldots, x_n) = x^T Q x, \tag{A.15}$$

where the matrix $Q \in \mathbb{R}^{ntimesn}$ is a symmetric matrix.

Proposition A.2 *Let A be a symmetric matrix, then:*

1. *The eigenvalues of A are all real.*
2. *There exists an orthogonal matrix Q such that*

$$A = Q \Lambda Q^T, \tag{A.16}$$

where Λ is a diagonal matrix formed from the eigenvalues of A and the orthogonal matrix Q is formed from the associated eigenvectors.

Quadratic forms always satisfy the Rayleigh principle, namely,

$$\lambda_{min}(Q)\|x\|^2 \le x^T Q x \le \lambda_{max}(Q)\|x\|^2. \tag{A.17}$$

In particular, if $\lambda_{min}(Q) \ge 0$ then it follows that $x^T Q x \ge 0$ for all x.

Definition A.3. The quadratic form $x^T Q x$ where Q is a real symmetric matrix is said to be positive semidefinite if

$$x^T Q x \ge 0 \qquad \text{for all } x$$

In particular if

$$x^T Q x > 0 \qquad x \ne 0$$

then the quadratic form is said to be positive definite

If $x^T Q x$ is positive definite then the matrix Q is said to be a *positive definite matrix* and this will be written $Q > 0$. It can be seen from the modal decomposition expression in Eq. (A.16) that a symmetric matrix Q will be positive definite if all its eigenvalues are positive. It also follows that a positive definite matrix Q is invertible since from Eq. (A.16)

$$\det(Q) = \det(\Lambda) = \prod_{i=1}^{n} \lambda_i > 0.$$

Given any square matrix Q let a sequence of matrices $M_1 \ldots M_n$, termed the *principal minors*, be recursively defined so that $M_1 = Q$ and M_{i+1} is obtained from deleting the first row and first column from M_i.

Theorem A.2 *A necessary and sufficient condition for the quadratic form $x^T Qx$, where Q is a square symmetric matrix, to be positive definite is that the determinant of Q be positive and the successive principal minors of the determinant of Q be positive.*

Another propertythat will be exploited is that, given a symmetric matrix P and a nonsingular matrix T of the same dimension,

$$P > 0 \quad \Leftrightarrow \quad T^T PT > 0$$

A symmetric matrix P is said to be *negative definite* if $-P$ is positive definite.

Appendix B
Describing Functions

B.1 Describing Function Fundamentals

The main use of *describing function (DF) technique* [11, 100, 171] is in studying the stability of the nonlinear systems, in particular predicting of limit cycles and their stability in nonlinear systems. A system with zero input that contains only one combined nonlinearity is depicted in Fig. B.1. The transfer function $G(s)$ is denoted as the plant. In general case the transfer function $G(s)$ can include also a transfer function of the controller and the sensor. It is assumed that there exists a *periodic solution*

$$e(t) = A \sin \omega t, \tag{B.1}$$

where ω is a frequency of *self-sustained oscillations* (*limit cycle*). The output v of the nonlinear block will be given by the Fourier series

$$v = \frac{\bar{a}_0}{2} + \sum_{n=1}^{\infty} \left[\bar{a}_n \cos(n\omega t) + \bar{b}_n \sin(n\omega t) \right], \tag{B.2}$$

where $\bar{a}_0 = 0$ for odd nonlinearities and

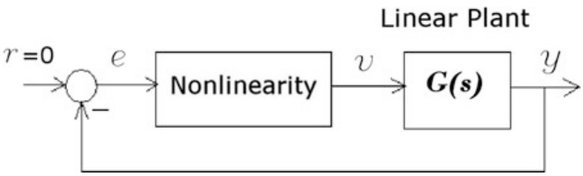

Fig. B.1 Block diagram of nonlinear system

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$$\begin{aligned} \bar{a}_n &= \frac{\omega}{\pi} \int_0^{2\pi/\omega} v(t) \cos(n\omega t)\, dt \\ \bar{b}_n &= \frac{\omega}{\pi} \int_0^{2\pi/\omega} v(t) \sin(n\omega t)\, dt. \end{aligned} \tag{B.3}$$

B.1.1 Low-Pass Filter Hypothesis and Describing Function

Assume that the transfer function of the plant, $G(s)$, has low-pass filter characteristics with respect to higher harmonics in the signal $e(t)$. It means that $|G(s)|$ is assumed to be small with respect to the higher harmonic components for $n = 2, 3, \ldots$ of the signal $v(t)$ given by Eq. (B.2) compared to the value $|G(s)|$ with respect to a fundamental component for $n = 1$. In this case the fundamental output $v(t)$ of the nonlinearity can be computed as

$$v = \bar{a}_1 \cos(\omega t) + \bar{b}_1 \sin(\omega t) = M sin(\omega t + \varphi). \tag{B.4}$$

This assumption is the foundation of the DF technique.

The DF of the nonlinearity is defined as the fundamental output (presented in a complex function or phasor format) divided by the input amplitude. This is in a Cartesian format

$$N(A, \omega) = \frac{\bar{a}_1 + j\bar{b}_1}{A} \tag{B.5}$$

or in polar coordinates

$$N(A, \omega) = \frac{M(A, \omega)}{A} e^{j\varphi(A,\omega)}, \tag{B.6}$$

where

$$\begin{aligned} M(A, \omega) &= \sqrt{\bar{a}_1^2 + \bar{b}_1^2} \\ \varphi(A, \omega) &= -\tan^{-1} \frac{\bar{b}_1}{\bar{a}_1} \end{aligned} . \tag{B.7}$$

B.1.2 Limit Cycle Analysis Using Describing Functions

In order to analyze the possibility of occurrence of limit cycles in nonlinear closed-loop system (Fig. B.1) we assume that the transfer function of the plant $G(s)$ has low-pass filter characteristics and the input $e(t)$ of the nonlinearity is sinusoidal and given by Eq. (B.1). Then the nonlinearity in Fig. B.1 is replaced by its DF $N(A, \omega)$ given by Eqs. (B.5) or (B.6) and (B.7), and the sinusoidal signals $e(t)$

and $v(t)$ are presented in a phasor format. Resulting system with $M = \sqrt{\bar{a}_1^2 + \bar{b}_1^2}$, $\varphi = -\tan^{-1}\frac{\bar{b}_1}{\bar{a}_1}$ is presented in Fig. B.2.

A steady state sinusoidal analysis is performed next. The nonlinear system will have a limit cycle if there exists a harmonic balance in this system. It means that the sinusoidal signal at the input of the nonlinearity will propagate through the cascade of the nonlinear block and the linear plant and being negated at a negative feedback will regenerate itself. Therefore, we obtain in the frequency domain:

$$\begin{array}{r} Ae^{j\omega t}N(A,\omega) = Me^{j(\omega t+\varphi)} \\ Me^{j(\omega t+\varphi)}G(j\omega) = -Ae^{j\omega t} \end{array} . \tag{B.8}$$

Equation (B.8) implies

$$1 + G(j\omega)N(A,\omega) = 0 \;\Rightarrow\; G(j\omega) = -\frac{1}{N(A,\omega)} \tag{B.9}$$

that is known as the harmonic balance equation. If Eq. (B.9) is satisfied for $A = A_c$, and $\omega = \omega_c$, then these values give the amplitude and frequency of the *predicted limit cycle*. If Eq. (B.9) does not have a real-valued positive solution, the limit cycle *is not predicted*. We have to acknowledge that due to the previous assumptions, including low-pass filter hypothesis, and due to taking into account only the fundamental harmonic when performing the harmonic balance, we can only *predict* the limit cycle. The prediction is supposed to be verified via simulations. It is worth noting the stability of the predicted limit cycle must be studied separately.

B.1.3 Stability Analysis of the Limit Cycle

The DF technique can be used to analyze the stability of the limit cycles. The Nyquist criterion of stability will be also employed. Let's assume that a solution (A_c, ω_c) of Eq. (B.9) exists. Figure B.3 illustrates the procedure of getting the solution graphically.

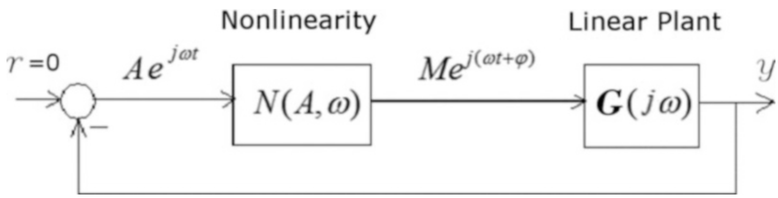

Fig. B.2 Block diagram of nonlinear system with the describing function in a frequency domain

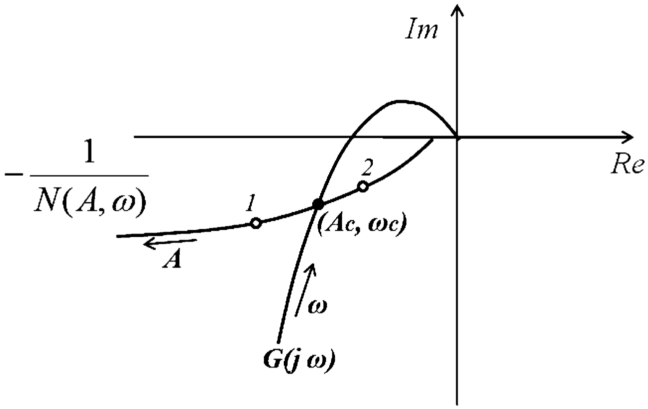

Fig. B.3 Graphical solution of the harmonic balance equation

Now, assume that the perturbation of the limit cycle with the parameters (A_c, ω_c) occurs such that the amplitude A_c of the limit cycle increases slightly. Then the operating point moves slightly outside of the zone encircled by the Nyquist plot (point 1 on the plot Fig. B.3). Assuming the linear plant does not have poles in the right-hand side of the complex plane, then in accordance with the Nyquist criterion of stability (that can be applied to the linearized system in the vicinity of the operating point (A_c, ω_c)), the system is stable, and the amplitude A_c starts decreasing moving the operating point to its original position (A_c, ω_c). Let's perturb the operating point (limit cycle with the parameters (A_c, ω_c)) such that the amplitude of oscillations A_c decreases slightly. Then the operating point moves insight the Nyquist plot (point 2 on the plot Fig. B.3). Applying the Nyquist criterion of stability to the system linearized in the operating point (A_c, ω_c) we can conclude that the linearized system now is unstable and the amplitude starts increasing moving the operating point to its original position (A_c, ω_c). Therefore, for the considered case the limit cycle is stable.

Appendix C
Linear Systems Theory

C.1 Introduction

This appendix provides the necessary background in terms of systems theory that is required for the sliding mode control developments. For further details see [47, 155].

C.1.1 Linear Time-Invariant Systems

Consider the linear system

$$\dot{x}(t) = Ax(t), \tag{C.1}$$

where $A \in \mathbb{R}^{n\times n}$ and the state vector $x \in \mathbb{R}^n$. A necessary and sufficient condition for asymptotic stability is that the eigenvalues of A have negative real parts.

Now consider the system

$$\dot{x}(t) = Ax(t) + Bu(t). \tag{C.2}$$

Essentially a forcing function or control input $u(t)$ has been introduced into the system. The general solution to this state equation may be expressed in the form

$$x(t) = Ve^{\Lambda t}W^T x(0) + V\int_0^t e^{\Lambda(t-\tau)}W^T Bu(\tau)\, d\tau, \tag{C.3}$$

where V is a matrix whose columns consist of the linearly independent right eigenvectors corresponding to the distinct eigenvalues λ_i, $i = 1, \ldots, n$ of the matrix A, $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, and the rows of the matrix W^T consist of the corresponding left eigenvectors, i.e., $V^{-1} = W^T$. Here $x(0)$ represents an arbitrary initial condition. Again the time domain characteristics of the system are determined by the eigenvalues of A. In addition, the associated right eigenvectors determine the "shape" of a given mode. As the solution depends upon a linear combination of

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functions of the form $v_i e^{\lambda_i t}$, appropriate eigenvector entries enable the transient $e^{\lambda_i t}$ to contribute, or not, to a particular state variable. In this way it is seen that the entire eigenstructure, and not just the eigenvalues, are effective in determining the time response of a system.

C.1.2 Controllability and Observability

Consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{C.4}$$

$$y(t) = Cx(t), \tag{C.5}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$. The variables $u(t)$ and $y(t)$ will be referred to as the *inputs* and *outputs*, respectively. The matrices A, B, and C will be termed the *system*, *input distribution*, and *output distribution* matrices, respectively. For convenience the system in Eqs. (C.4) and (C.5) will be referred to as a system triple (A, B, C).

If $T \in \mathbb{R}^{n\times n}$ is nonsingular, then the change of coordinates $x \mapsto Tx$ induces a new system representation with system matrix TAT^{-1}, input distribution matrix TB, and output distribution matrix CT^{-1}, i.e., the triple (TAT^{-1}, TB, CT^{-1}).

Definition C.1. The system is said to be completely controllable if given any initial condition $x(t_0)$ there exists an input function on the finite interval $[t_0, t_1]$ such that $x(t_1) = 0$.

From this definition the following theorem can be proved.

Theorem C.1 *Given any pair (A, B) the following conditions are all equivalent:*

- *(A, B) is completely controllable.*
- *The controllability matrix $[B \;\; AB \;\; A^2B \;\; \ldots \;\; A^{n-1}B\,]$ has full rank.*
- *The matrix $[sI - A \;\; B\,]$ has full rank for all $s \in \mathbb{C}$.*
- *The spectrum of $(A + BF)$ can be assigned arbitrarily by choice of $F \in \mathbb{R}^{m\times n}$.*

The third condition, which is often the most convenient method of establishing controllability, is often referred to as the Popov–Belevitch–Hautus rank test or PBH test.

Definition C.2. The linear system is said to be completely observable if the output function $y(t)$ over some time interval $[t_0, t_1]$ uniquely determines the initial condition $x(t_0)$.

An important duality exists between the notions of controllability and observability which can be summarized as follows.

Theorem C.2 *The pair (A, C) is completely observable if and only if the pair (A^T, C^T) is completely controllable.*

From the theorem above, the results of Theorem C.1 can be modified to provide a list of equivalent statements for observability.

In addition to these concepts there exist two slightly weaker notions: *stabilizability* and *detectability*.

Theorem C.3 *Given any pair (A, B) the following conditions are equivalent:*

- *(A, B) is stabilizable.*
- *The matrix $[sI - A \;\; B]$ has full rank for all $s \in \mathbb{C}_+$.*
- *There exists an $F \in \mathbb{R}^{m\times n}$ such that the eigenvalues of $A + BF$ belong to $\mathbb{C}_-$.*

The notion of detectability can be defined as the dual of stabilizability.

If the pair (A, B) in Eq. (C.4) is not controllable then there exists a change of coordinates $x \mapsto Tx$ so that in the new coordinate system the new system and input distribution matrices have the form

$$TAT^{-1} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \quad TB = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \tag{C.6}$$

where the pair (A_{22}, B_2) is completely controllable. Note that because of the special structure of the canonical form in Eq. (C.6) it follows

$$\lambda(A_{11}) \subset \lambda(A + BF)$$

for any $F \in \mathbb{R}^{m\times n}$. Consequently, the pair (A, B) is stabilizable if and only if A_{11} is stable.

By duality, a similar canonical form exists for pairs (A, C) which are not observable.

C.1.3 Invariant Zeros

Consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{C.7}$$

$$y(t) = Cx(t). \tag{C.8}$$

Assume the initial state is given by $x(0)$. Taking Laplace transforms of the system representation yields

$$\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} X(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} x(0) \\ Y(s) \end{bmatrix}. \tag{C.9}$$

The polynomial system matrix

$$P(s) = \begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} \tag{C.10}$$

is usually referred to as Rosenbrock's system matrix. A necessary and sufficient condition for an input

$$u(t) = u(0)e^{zt} \tag{C.11}$$

to yield rectilinear motion in the state space of the form

$$x(t) = x(0)e^{zt} \tag{C.12}$$

such that the output of the system is identically zero for all time is that z, $x(0)$, and $u(0)$ satisfy

$$P(z)\begin{bmatrix} x(0) \\ u(0) \end{bmatrix} = 0. \tag{C.13}$$

This result defines a set of complex frequencies z which are associated with specific directions $x(0)$ and $u(0)$ in the state and input spaces for which the output of the system is zero. These elements are called *invariant zeros*. It is clear that information regarding the existence of invariant zeros comes from examining the rank of $P(z)$ in Eq. (C.13). For example, in the case of a square system—i.e., systems with equal numbers of inputs and outputs—in order for Eq. (C.13) to have a nonzero solution for $x(0)$ and $u(0)$, $\det(P(z))$ must be zero.

C.1.4 State Feedback Control

For the controllable state-space system represented by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{C.14}$$

a state feedback controller is defined by

$$u(t) = -Kx(t), \tag{C.15}$$

where $K \in \mathbb{R}^{m \times n}$. The state equation of the closed-loop system is given by

$$\dot{x}(t) = (A - BK)x(t). \tag{C.16}$$

As the system is controllable, from Theorem C.1, the closed-loop poles can be allocated to any desired location by appropriate choice of K.

The linear quadratic regulator (LQR) is a particular formulation of the state feedback control problem. Given the state-space system (C.14) with given initial condition $x(0)$, the input signal $u(t)$ is sought which regulates the system state to the origin by minimizing the cost function

$$J = \frac{1}{2}\int_0^{\infty} x(t)^T Qx(t) + u(t)^T Ru(t)\, dt, \tag{C.17}$$

where Q and R are positive definite symmetric matrices which penalize the deviation of the state from the origin and the magnitude of the control signal, respectively. The optimal solution, for any initial state, is given by

$$u(t) = -Kx(t) = -R^{-1}B^T Xx(t), \tag{C.18}$$

where X is the unique positive semidefinite solution ($X \geq 0$), of the algebraic Riccati equation

$$A^T X + XA - XBR^{-1}B^T X + Q = 0. \tag{C.19}$$

For further details see [8].

C.1.5 Static Output Feedback Control

It has been demonstrated in the previous section that it is possible to readily determine a state feedback-based control strategy for a controllable linear system. In practice it may not be feasible to measure all the state variables for a given system. If only a subset of state information is available, the feedback control problem must now consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{C.20}$$

$$y(t) = Cx(t), \tag{C.21}$$

where $y \in \mathbb{R}^p$ denotes an available vector of output measurements. The development of a control law

$$u(t) = -KCx(t) = -Ky(t), \tag{C.22}$$

where $K \in \mathbb{R}^{m\times p}$ is needed. It is important to note that this problem has associated with it further constraints. A sufficient condition to ensure that the eigenvalues of $A - BKC$ may be placed arbitrarily close to desired locations is that the pair (A, B) is controllable, the pair (A, C) is observable, and the dimensionality requirement

$$m + p + 1 \geq n \tag{C.23}$$

is satisfied. These requirements are referred to as the *Kimura–Davison* conditions.

Appendix D
Lyapunov Stability

Earlier the notion of stability was considered using the concept of phase portraits for second-order systems. Now a more general (and more abstract) approach will be taken. For the general nonlinear system

$$\dot{x}(t) = f(t, x), \tag{D.1}$$

where $x \in \mathbb{R}^n$, an *equilibrium point* in the state space is given by vectors x_e satisfying

$$f(t, x_e) = 0 \text{ for all } t > 0. \tag{D.2}$$

In general there may be many values of x_e which satisfy this condition. In terms of analyzing the stability of a particular equilibrium point it is useful to assume the point is at the origin. This can be done without loss of generality since given an equilibrium point x_e the change of variables (actually just a simple translation of the origin) $x \mapsto \tilde{x}$ where $\tilde{x} = x - x_e$ implies the origin is an equilibrium point of

$$\dot{\tilde{x}}(t) = f(t, \tilde{x}) \tag{D.3}$$

since $f(t, \tilde{x} = 0) = f(t, x_e) = 0$.

The solution to Eq. (D.1) will be written as $x(t, x_0)$ where t signifies the evolution of the state with respect to time and x_0 represents the initial conditions—usually the value of the state at $t = 0$. Note, it is usually very difficult if not impossible to obtain an analytic expression for $x(t, x_0)$ for general nonlinear systems.

Definition D.1. The origin of Eq. (D.1) is said to be stable if given any $\epsilon > 0$ there exists a $\delta > 0$ such that if $\|x_0\| < \delta$ then $\|x(t, x_0)\| < \epsilon$ for all $t > 0$.

Less formally the definition means that by starting close enough to the equilibrium point, the solution will always remain arbitrarily close to it.

Definition D.2. The origin of Eq. (D.1) is said to be asymptotically stable if it is stable and the solution $x(t, x_0) \to 0$ as $t \to \infty$.

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Less formally this definition means that by starting close enough to the equilibrium point, the solution will always remain arbitrarily close, but in addition, the trajectory will always move towards the equilibrium point.

D.1 Local Results

Consider a domain $\mathcal{D}$ in the state space which contains a neighborhood of the origin $\{x \in \mathbb{R}^n : \|x\| < r\}$ where r is a positive scalar. In two dimensions this represents a circle centered at the origin of radius r; in three dimensions $\mathcal{D}$ represents a sphere of radius r.

Definition D.3. A function $V : \mathcal{D} \mapsto \mathbb{R}$ is positive definite in the domain $\mathcal{D}$ if:

- $V(x) \geq 0$ for all $x \in \mathcal{D}$.
- $V(x) = 0$ implies $x = 0$.

The function is said to be positive semidefinite in the domain $\mathcal{D}$ if only the first condition holds. If $\mathcal{D} = \mathbb{R}^n$ the function is said to be positive definite or positive semidefinite, respectively.

The function $V : \mathbb{R}^2 \mapsto \mathbb{R}$ given by $V(x_1, x_2) = x_1^2 + x_2^2$ is positive definite. This is a special case of a generic class of functions called *quadratic forms*.

One version of Lyapunov's theorem is given below:

Theorem D.1 *Consider the nonlinear system in Eq. (D.1). Suppose in the domain $\mathcal{D}$ there exists a differentiable positive definite function $V : \mathbb{R} \times \mathcal{D} \mapsto \mathbb{R}$ such that*

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0$$

then the origin is stable. Furthermore if $\dot{V} < 0$ for $x \neq 0$ then the origin is asymptotically stable.

D.2 Global Results

To extend these local results to be global, intuitively all that needs to be done is to ensure the conditions of Theorem D.1 hold for $\mathcal{D} = \mathbb{R}^n$. It turns out this is not quite sufficient. A further constraint needs to be included, namely, $V(x) \to \infty$ as $\|x\| \to \infty$. A function V with such a property is termed *radially unbounded*.

Theorem D.2 *Consider the nonlinear system in Eq. (D.1). Suppose there exists a differentiable function $V : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$ which is radially unbounded and positive definite such that*

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0$$

then the origin is globally stable. Furthermore if $\dot{V} < 0$ for $x \neq 0$ then the origin is asymptotically stable.

Quadratic forms are functions of the type $x^T P x$ where $P \in \mathbb{R}^{n \times n}$ is symmetric, i.e., $P^T = P$. Thought of in terms of the components x_i of x, a quadratic form is a weighted sum of all possible products $x_i x_j$: specifically

$$x^T P x = \sum_{i=1}^{n} \sum_{j=i}^{n} p_{ij} x_i x_j$$

Consider a general linear system described by

$$\dot{x}(t) = Ax(t), \tag{D.4}$$

where $A \in \mathbb{R}^{n \times n}$. Consider a Lyapunov candidate of the form

$$V(x) = x^T P x, \tag{D.5}$$

where $P \in \mathbb{R}^{n \times n}$ is some symmetric positive definite matrix. Differentiating Eq. (D.5) with respect to time gives

$$\begin{aligned} \dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T (PA + A^T P) x. \end{aligned} \tag{D.6}$$

In order to prove that the system is asymptotically stable the symmetric matrix in Eq. (D.6) must be negative definite. Consider the matrix equation

$$PA + A^T P = -Q, \tag{D.7}$$

where $Q \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix. Given A and Q, if a symmetric positive definite matrix P exists, solving the so-called *Lyapunov equation* given in Eq. (D.7), then the linear system in Eq. (D.4) will be asymptotically stable.

In fact the following can be shown:

Theorem D.3 *There exists a unique s.p.d matrix P satisfying Eq. (D.7) if and only if the matrix A is stable, i.e., the eigenvalues of A lie in the open left half plane.*

D.2.1 Quadratic Stability

The previous section considered the special case of linear systems. This enabled an analytic expression for the solution to be obtained. For general nonlinear systems this is usually impossible. An approach for studying the stability of differential equations, without the need to obtain an explicit solution, is the method of

Lyapunov. Loosely speaking, if a differentiable function $V : \mathbb{R}^n \mapsto \mathbb{R}$ can be found which is positive except at an equilibrium point and whose total time derivative decreases along the system trajectories, then the equilibrium point is stable. The key point is that this approach obviates the need to solve the nonlinear differential equation when assessing its stability properties.

Unfortunately, no systematic way exists to synthesize Lyapunov functions for nonlinear systems. This section considers the special case when the scalar function $V : \mathbb{R}^n \to \mathbb{R}$ is the quadratic form

$$V(x) = x^T P x, \tag{D.8}$$

where $P \in \mathbb{R}^{n \times n}$ is some symmetric positive definite matrix. By construction the function is nonzero except at the origin. Next, form the function of time

$$V(t) = x(t)^T P x(t), \tag{D.9}$$

where $x(t)$ represents the solution of the differential equation (D.1). Differentiating Eq. (D.9) with respect to time gives

$$\begin{aligned} \dot{V}(t) &= \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) \\ &= 2x(t)^T P \dot{x}(t) \\ &= 2x(t)^T P f(x,t), \end{aligned}$$

where the second equality follows because the quantities are scalars and hence

$$\dot{x}(t)^T P x(t) = \left(\dot{x}(t)^T P x(t)\right)^T = x(t)^T P \dot{x}(t)$$

Definition D.4. The origin of the system (D.1) is said to be quadratically stable if there exist symmetric positive definite matrices $P, Q \in \mathbb{R}^{n \times n}$ such that the total time derivative satisfies

$$\dot{V}(x) = 2x^T P f(x,t) \leq -x^T Q x$$

The inequality above implies $\|x(t)\| < \mathrm{e}^{-\alpha t}$ where $\alpha = \lambda_{min}(P^{-1}Q)$ and hence the origin is asymptotically stable. If $f(x,t) = Ax(t)$ then it is well known that A has stable eigenvalues if and only if, given any symmetric positive definite matrix Q, there exists a unique symmetric positive definite matrix P satisfying the *Lyapunov equation*

$$PA + A^T P = -Q. \tag{D.10}$$

Consequently, any stable linear system is quadratically stable. A symmetric positive definite matrix P satisfying Eq. (D.10) will be referred to as a *Lyapunov matrix* for the matrix A.

Lyapunov theory may also be used as a means of examining the *robustness* of a given linear system; suppose

$$\dot{x}(t) = Ax(t) + \xi(t, x), \tag{D.11}$$

where the matrix A is stable and $\xi(\cdot)$ is an imprecisely known function which represents uncertainty in the system. Let the pair of positive definite matrices (P, Q) satisfy the Lyapunov Eq. (D.10) and define

$$\mu = \frac{\lambda_{min}(Q)}{\lambda_{max}(P)} \tag{D.12}$$

and suppose that the uncertain function satisfies

$$\|\xi(t, x)\| \leq \tfrac{1}{2}\mu\|x(t)\| \tag{D.13}$$

then the system in Eq. (D.11) is stable. This can be established by using $V = x^T P x$ as a Lyapunov function: the derivative along the trajectories satisfies

$$\begin{aligned} \dot{V} &= x(t)^T PAx(t) + x(t)^T A^T Px(t) + 2x(t)^T P\xi(t, x) \\ &= -x(t)^T Qx(t) + 2x(t)^T P\xi(t, x) \\ &\leq -x(t)^T Qx(t) + 2\|Px(t)\|\|\xi(t, x)\|, \end{aligned} \tag{D.14}$$

where the Cauchy–Schwarz inequality (see Appendix A.1.4) has been used to obtain the last inequality. Now

$$\|Px\| = \sqrt{x^T P^2 x} \leq \sqrt{\lambda_{max}(P^2)\|x\|^2} = \lambda_{max}(P)\|x\|, \tag{D.15}$$

where the Rayleigh principle has been used to obtain the middle inequality. Also directly from the Rayleigh principle

$$-x^T Qx \leq -\lambda_{min}(Q)\|x\|^2. \tag{D.16}$$

Thus from the inequality in Eq. (D.14) and using Eqs. (D.15) and (D.16) it follows that

$$\begin{aligned} \dot{V}(t) &\leq -\lambda_{min}(Q)\|x(t)\|^2 + 2\lambda_{max}(P)\|x(t)\|\|\xi(t, x)\| \\ &= -\lambda_{max}(P)\|x(t)\|\left(\mu\|x(t)\| - 2\|\xi(t, x)\|\right) \end{aligned}$$

and therefore if $\xi(\cdot)$ satisfies Eq. (D.13) the Lyapunov derivative is always negative and stability is proved.

In view of the condition (D.13), it is natural to attempt to choose Q in an effort to maximize Eq. (D.12). It can be shown that the maximum is given by

$$\hat{\mu} = \frac{1}{\lambda_{max}(P)} \tag{D.17}$$

when $Q = I$. Furthermore, it can be shown that

$$\hat{\mu} \leq -2 \max \left[\text{Re } \lambda(A)\right] \tag{D.18}$$

with equality if the matrix A is *normal*, i.e., if it has n orthonormal eigenvectors.

When dealing with uncertain systems, it may not be possible to guarantee asymptotic stability. Consider the nonlinear system (D.1) and suppose it is subject to an imprecisely known exogenous signal $\xi(\cdot)$ so that

$$\dot{x}(t) = f(x, t, \xi). \tag{D.19}$$

Let $\mathcal{E} \subset \mathbb{R}^n$ be a bounded set, then the following definition can be made. For further details see [116].

Definition D.5. The solution $x(\cdot)$ to the uncertain system (D.19) is said to be ultimately bounded with respect to the set $\mathcal{E}$ if:

- On any finite interval the solution remains bounded, i.e., if $\|x(t_0)\| < \delta$ then $\|x(t)\| < d(\delta)$ for any $t \in [t_0,\ t_1)$.
- In finite time the solution $x(t)$ enters the bounded set $\mathcal{E}$ and remains there for all subsequent time.

The set $\mathcal{E}$ is usually an acceptably small neighborhood of the origin and the concept is often termed *practical stability*.

For further details see [116].

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