Appendix A
Taylor's Theorem

A.1 Single Variable

The single most important result needed to develop an asymptotic approximation is Taylor's theorem. The single variable version of the theorem is below.

Theorem A.1. *Given a function $f(x)$ assume that its $(n+1)$st derivative $f^{(n+1)}(x)$ is continuous for $x_L < x < x_R$. In this case, if a and x are points in the interval (x_L, x_R) then*

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(a) + \cdots + \frac{1}{n!}(x-a)^n f^{(n)}(a) + R_{n+1}, \tag{A.1}$$

where the remainder is

$$R_{n+1} = \frac{1}{(n+1)!}(x-a)^{n+1} f^{(n+1)}(\eta), \tag{A.2}$$

and η is a point between a and x.

There are different, but equivalent, ways to write the above result. One is

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \cdots + \frac{1}{n!}h^n f^{(n)}(x) + R_{n+1}, \tag{A.3}$$

The requirement here is that x and $x+h$ are points in the interval (x_L, x_R).

A.2 Two Variables

The two-variable version of the expansion in (A.3) is

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$$f(x+h,t+k) = f(x,t) + Df(x,t) + \frac{1}{2}D^2f(x,t) + \cdots + \frac{1}{n!}D^nf(x,t) + R_{n+1}. \tag{A.4}$$

where

$$D = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial t}\,.$$

Writing this out, through quadratic terms, yields

$$\begin{aligned} f(x+h,t+k) = &f(x,t) + hf_x(x,t) + kf_t(x,t) \\ &+ \frac{1}{2}h^2 f_{xx}(x,t) + hkf_{xt}(x,t) + \frac{1}{2}k^2 f_{tt}(x,t) + \cdots\,. \end{aligned}$$

The subscripts in the above expression denote partial differentiation. So, for example,

$$f_{xt} = \frac{\partial^2 f}{\partial x \partial t}\,.$$

It is assumed that the function f has continuous partial derivatives up through order $n+1$.

The above expansion can be expressed in a form similar to the one in (A.1), and the result is

$$\begin{aligned} f(x,t) = &f(a,b) + (x-a)f_x(a,b) + (t-b)f_t(a,b) \\ &+ \frac{1}{2}(x-a)^2 f_{xx}(a,b) + (x-a)(t-b)f_{xt}(a,b) + \frac{1}{2}(t-b)^2 f_{tt}(a,b) \\ &+ \cdots\,. \end{aligned}$$

A.3 Multivariable Versions

For more than two variables it is convenient to use vector notation. In this case (A.4) takes the form

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + Df(\mathbf{x}) + \frac{1}{2}D^2f(\mathbf{x}) + \cdots + \frac{1}{n!}D^nf(\mathbf{x}) + R_{n+1},$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_k)$, $\mathbf{h} = (h_1, h_2, \cdots, h_k)$ and

$$\begin{aligned} D &= \mathbf{h}\cdot\nabla \\ &= h_1\frac{\partial}{\partial x_1} + h_2\frac{\partial}{\partial x_2} + \cdots + h_k\frac{\partial}{\partial x_k}\,. \end{aligned}$$

Writing this out, through quadratic terms, yields

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + \mathbf{h}\cdot\nabla f(\mathbf{x}) + \frac{1}{2}\mathbf{h}^T\mathbf{H}\mathbf{h} + \cdots\,,$$

where **H** is the Hessian and is given as

$$\mathbf{H} = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \dfrac{\partial^2 f}{\partial x_k \partial x_1} \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_k \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_k} & \dfrac{\partial^2 f}{\partial x_2 \partial x_k} & \cdots & \dfrac{\partial^2 f}{\partial x_k^2} \end{pmatrix}.$$

Taylor's theorem can also be extended to vector functions, although the formulas are more involved. To write down the expansion through the linear terms, assume that $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ and $\mathbf{x} = (x_1, x_2, \ldots, x_k)$. In this case,

$$\mathbf{f}(\mathbf{x}+\mathbf{h}) = \mathbf{f}(\mathbf{x}) + (\nabla \mathbf{f})\mathbf{h} + \cdots ,$$

where

$$\nabla \mathbf{f} = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_k} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_k} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \dfrac{\partial f_m}{\partial x_2} & \cdots & \dfrac{\partial f_m}{\partial x_k} \end{pmatrix}.$$

Appendix B
Fourier Analysis

B.1 Fourier Series

It is assumed here that the function $f(x)$ is piecewise continuous for $0 \leq x \leq \ell$. Recall that this means $f(x)$ is continuous on the interval $0 \leq x \leq \ell$ except at a finite number of points within the interval at which the function has a jump discontinuity.

The Fourier sine series for $f(x)$ is defined as

$$S(x) = \sum_{n=1}^{\infty} \beta_n \sin(\lambda_n x), \tag{B.1}$$

where $\lambda_n = n\pi/\ell$ and

$$\beta_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin(\lambda_n x) dx. \tag{B.2}$$

The Fourier cosine series for $f(x)$ is defined as

$$C(x) = \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos(\lambda_n x), \tag{B.3}$$

where

$$\alpha_n = \frac{2}{\ell} \int_0^{\ell} f(x) \cos(\lambda_n x) dx. \tag{B.4}$$

A certain amount of smoothness is required of the function $f(x)$ so the above series are defined. For example, $f(x)$ must be smooth enough that the integrals in (B.2) and (B.4) exist. Certainly assuming $f(x)$ is continuous is enough for the integrals, but, unfortunately, this is not enough to guarantee that the series in (B.1) and (B.3) converge. They will converge, however, if

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$f(x)$ and $f'(x)$ are piecewise continuous. The question naturally arises as to what they converge to, and for this we have the following result.

Theorem B.1. *Assume $f(x)$ and $f'(x)$ are piecewise continuous for $0 \leq x \leq \ell$. On the interval $0 < x < \ell$, the Fourier sine series, and the Fourier cosine series, converge to $f(x)$ at points where the function is continuous, and they converge to $\frac{1}{2}(f(x+) + f(x-))$ at points where the function has a jump discontinuity. At the endpoints, $S(0) = S(\ell) = 0$, while $C(0) = f(0)$ and $C(\ell) = f(\ell)$.*

When using a Fourier series to solve a differential equation one usually needs the expansion of the solution as well as its derivatives. The problem is that it is not always possible to obtain the series for $f'(x)$ by differentiating the series for $f(x)$. For example, given a sine series as in (B.1) one might be tempted to conclude that

$$S'(x) = \sum_{n=1}^{\infty} \beta_n \lambda_n \cos(\lambda_n x).$$

The issue is that the differentiation has resulted in λ_n appearing in the coefficient. As an example, for the function

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 2 & \text{if } 1 < x \leq 2, \end{cases}$$

one finds that

$$\beta_n \lambda_n = \frac{2}{\ell}\left[1 - 2(-1)^n + \cos(n\pi/2)\right].$$

The general term $\beta_n \lambda_n \cos(\lambda_n x)$ of the series does not converge to zero as $n \to \infty$, and this means that the series does not converge. Consequently, additional restrictions must be imposed on $f(x)$ to guarantee convergence. Basically what are needed are conditions that will give us $\beta_n = O(1/n^2)$, and this brings us to the next result.

Theorem B.2. *Assume $f(x)$ is continuous, with $f'(x)$ and $f''(x)$ piecewise continuous, for $0 \leq x \leq \ell$. If $f(x)$ is expanded in a cosine series then the series for $f'(x)$ can be found by differentiating the series for $f(x)$. If $f(x)$ is expanded in a sine series, and if $f(0) = f(\ell) = 0$, then the series for $f'(x)$ can be found by differentiating the series for $f(x)$.*

The question of convergence for integration is much easier to answer. As long as the Fourier series of $f(x)$ converges then the series for the integral of f can be found by simply integrating the series for f.

B.2 Fourier Transform

To derive the formula for the Fourier transform from the Fourier series, it is convenient to use the symmetric interval $-\ell < x < \ell$. Generalizing (B.2) and (B.3), the Fourier series of a continuous function $f(x)$ is

$$f(x) = \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \left[\alpha_n \cos(\lambda_n x) + \beta_n \sin(\lambda_n x)\right],$$

where $\lambda_n = n\pi/\ell$,

$$\alpha_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos(\lambda_n x)dx,$$

and

$$\beta_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin(\lambda_n x)dx.$$

By using the identities $\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$, the Fourier series can be written in exponential form as

$$f(x) = \sum_{n=-\infty}^{\infty} \gamma_n e^{i\lambda_n x},$$

where

$$\gamma_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(\bar{x})e^{-i\lambda_n \bar{x}}d\bar{x}.$$

Combining these two expressions

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2\ell}\int_{-\ell}^{\ell} f(\bar{x})e^{i\lambda_n(x-\bar{x})}d\bar{x}.$$

The sum in the above equation is reminiscent of the Riemann sum used to define integration. To make this more evident, let $\Delta\lambda = \lambda_{n+1} - \lambda_n = \frac{\pi}{\ell}$. With this

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\ell}^{\ell} f(\bar{x})e^{i\lambda_n(x-\bar{x})}d\bar{x}\Delta\lambda.$$

The argument originally used by Fourier is that in the limit of $\ell \to \infty$, the above expression yields

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\bar{x})e^{i\lambda(x-\bar{x})}d\bar{x}d\lambda.$$

Fourier then made the observation that the above equation can be written as $f(x) = \mathcal{F}^{-1}(\mathcal{F}(f))$, where $\mathcal{F}$ is the Fourier transform defined in Section 7.2.5. With this, the Fourier transform was born.

To say that the above derivation is heuristic would be more than generous. However, it is historically correct, and it does show the origin of the Fourier transform and its inverse. The formal proof of the derivation can be found in Weinberger [1995].

Appendix C
Stochastic Differential Equations

The steps used to solve the Langevin equation look routine, and the solutions in (4.85) and (4.86) are not particularly remarkable. However, on closer inspection, the randomness of the forcing function raises some serious mathematical questions. An example of $\mathbf{R}$ is shown in Figure 4.27 using 400 points along the t-axis. As will be discussed in more detail in Section 4.7.1, the value of $\mathbf{R}(t_1)$ is independent of the value of $\mathbf{R}(t_2)$ if $t_1 \neq t_2$. This means that if more than 400 points are used, the graph would appear even more random than in Figure 4.27. The question that immediately arises is whether the non-differentiability of this function causes the differential equation (4.84), or its solution (4.84), to be meaningless. One approach for addressing this issue rests on denial, where the calculations are carried out as if everything is just fine. This is, in fact, what was done to derive (4.85), and this approach almost works. To have it succeed, all that is needed is to make sense of the solution, and then use this to justify the entire process.

The question is, therefore, how to define the integrals in (4.85) and (4.86). The exponentials are not an issue, and so to simplify the discussion we will concentrate on the expression

$$\mathbf{W}(t) = \int_0^t \mathbf{R}(\tau) d\tau. \tag{C.1}$$

The definition of this integral employs the same Riemann sum used in Calculus. With this in mind, we introduce a partition $0 < t_1 < t_2 < \cdots < t_m < t$, where $t_0 = 0$ and $t_{m+1} = t$. For simplicity, it is assumed the points are equally spaced, and so $t_{j+1} - t_j = \Delta t$. Letting s_j be a point from the interval $[t_j, t_{j+1}]$, then we introduce the partial sum

$$\mathbf{S}_m = \sum_{j=0}^{m-1} \mathbf{R}(s_j) \Delta t. \tag{C.2}$$

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The question is, if $\Delta t \to 0$, does $\mathbf{S}_m$ converge? The answer is yes, although convergence is measured in the mean-square sense. Knowing that it converges then the limit of $\mathbf{S}_m$ serves as the definition of the integral in (C.1). This definition preserves most, but not all, of the properties associated with standard integration. In particular, $\mathbf{W}$ is a continuous function of t, and the integral is additive in the sense that if $t_1 < t_2$ then

$$\int_0^{t_2} \mathbf{R}(\tau)d\tau = \int_0^{t_1} \mathbf{R}(\tau)d\tau + \int_{t_1}^{t_2} \mathbf{R}(\tau)d\tau.$$

Moreover, the partial sums in (C.2) provide a method for numerically evaluating the stochastic integrals in (4.85) and (4.86).

Now that integration has been put onto a solid mathematical footing, we turn to the differential equation (4.84). In the case of when $\mathbf{R}$ is smooth, this equation can be integrated to yield

$$\mathbf{v}(t) = \mathbf{v}(0) - \lambda \int_0^t \mathbf{v}(\tau)d\tau + \frac{1}{m}\int_0^t \mathbf{R}(\tau)d\tau. \tag{C.3}$$

For smooth functions this integral equation is equivalent to the differential equation (4.84). This fact is used to explain what happens when a random forcing is used. Specifically, the interpretation of the differential equation (4.84) is that $\mathbf{v}$ satisfies (C.3). It is for this reason that in the subject of stochastic differential equations, (4.84) is conventionally written using differentials as

$$d\mathbf{v} = -\lambda \mathbf{v}dt + \frac{1}{m}\mathbf{R}dt.$$

The implication in using this notation is that the stochastic differential equation is being interpreted as the solution of the associated integral equation. With this viewpoint, (C.1) can be written as $d\mathbf{W} = \mathbf{R}dt$. Those interested in pursuing the theoretical foundation of the stochastic differential equations should consult Oksendal [2003].

Appendix D
Identities

D.1 Trace

In the following, $\mathbf{A}$ and $\mathbf{B}$ are 3×3 matrices, and α and β are scalars.

$$\begin{aligned} \mathrm{tr}(\alpha\mathbf{A} + \beta\mathbf{B}) &= \alpha\, \mathrm{tr}(\mathbf{A}) + \beta\, \mathrm{tr}(\mathbf{B}) \\ \mathrm{tr}(\mathbf{A}\mathbf{B}) &= \mathrm{tr}(\mathbf{B}\mathbf{A}) \\ \mathrm{tr}(\mathbf{A}^T) &= \mathrm{tr}(\mathbf{A}) \end{aligned}$$

If $\mathbf{A}$ is symmetric and $\mathbf{B}$ is skew-symmetric then $\mathrm{tr}(\mathbf{A}\mathbf{B}) = 0$.

D.2 Determinant

In the following, $\mathbf{A}$ and $\mathbf{B}$ are 3×3 matrices, and α and β are scalars.

$$\begin{aligned} \det(\mathbf{A}\mathbf{B}) &= \det(\mathbf{B}\mathbf{A}) = \det(\mathbf{A})\det(\mathbf{B}) \\ \det(\alpha\mathbf{A}) &= \alpha^3\det(\mathbf{A}) \\ \det(\mathbf{A}^T) &= \det(\mathbf{A}) \\ \det(\mathbf{A}^{-1}) &= 1/\det(\mathbf{A}) \\ \det(\mathbf{I}) &= 1 \end{aligned}$$

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D.3 Vector Calculus

In the following, ϕ is a scalar, $\mathbf{u} = (u, v, w)$ is a vector, and $\mathbf{A}(\mathbf{x})$ is a 3×3 matrix. They are all smooth functions of $\mathbf{x} = (x, y, z)$.

$$\begin{aligned}
\nabla \cdot \mathbf{u} &= \mathrm{tr}(\nabla \mathbf{u}) \\
\nabla \cdot (\phi \mathbf{u}) &= \mathbf{u} \cdot \nabla \phi + \phi(\nabla \cdot \mathbf{u}) \\
\nabla \cdot (\mathbf{A}\mathbf{u}) &= \mathbf{u} \cdot (\nabla \cdot \mathbf{A}) + \mathrm{tr}(\mathbf{A}^T \nabla \mathbf{u}) \\
\nabla \cdot (\phi \mathbf{A}) &= \mathbf{A}^T \nabla \phi + \phi(\nabla \cdot \mathbf{A}) \\
(\mathbf{v} \cdot \nabla)\mathbf{u} &= (\nabla \mathbf{u})\mathbf{v} \\
\nabla \times (\nabla \phi) &= \mathbf{0} \\
\nabla \cdot (\nabla \times \mathbf{u}) &= 0
\end{aligned}$$

In the above identities

$$\nabla \mathbf{u} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{pmatrix},$$

and

$$\nabla \cdot \mathbf{A} = \begin{pmatrix} \dfrac{\partial A_{11}}{\partial x} + \dfrac{\partial A_{21}}{\partial y} + \dfrac{\partial A_{31}}{\partial z} \\ \dfrac{\partial A_{12}}{\partial x} + \dfrac{\partial A_{22}}{\partial y} + \dfrac{\partial A_{32}}{\partial z} \\ \dfrac{\partial A_{13}}{\partial x} + \dfrac{\partial A_{23}}{\partial y} + \dfrac{\partial A_{33}}{\partial z} \end{pmatrix}.$$

Appendix E
Equations for a Newtonian Fluid

E.1 Cartesian Coordinates

Letting $\mathbf{v} = (u, v, w)$ and $\mathbf{f} = (f, g, h)$, then for an incompressible Newtonian fluid in Cartesian coordinates:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho f$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \rho g$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \rho h$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

E.2 Cylindrical Coordinates

Letting $\mathbf{v} = (v_r, v_\theta, v_z) = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_z\mathbf{e}_z$ and $\mathbf{f} = f_r\mathbf{e}_r + f_\theta\mathbf{e}_\theta + f_z\mathbf{e}_z$, then for an incompressible Newtonian fluid in cylindrical coordinates:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right)$$
$$= -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho f_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right)$$
$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] + \rho f_\theta$$

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$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)$$
$$=-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]+\rho f_z$$
$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}=0$$

Transformation laws for velocities:

$$u=v_r\cos\theta-v_\theta\sin\theta \qquad v_r=\frac{1}{\sqrt{x^2+y^2}}(xu+yv)$$
$$v=v_r\sin\theta+v_\theta\cos\theta \qquad v_\theta=\frac{1}{\sqrt{x^2+y^2}}(-yu+xv)$$
$$w=v_z \qquad v_z=w$$

Transformation laws for derivatives:

$$\frac{\partial}{\partial x}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \qquad \frac{\partial}{\partial r}=\frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x}+\frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}$$
$$\frac{\partial}{\partial y}=\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \qquad \frac{\partial}{\partial \theta}=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}$$
$$\frac{\partial}{\partial z}=\frac{\partial}{\partial z} \qquad \frac{\partial}{\partial z}=\frac{\partial}{\partial z}$$

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+v_r\frac{\partial}{\partial r}+\frac{v_\theta}{r}\frac{\partial}{\partial \theta}+v_z\frac{\partial}{\partial z}$$

Formulas from vector analysis:

$$\nabla\times\mathbf{v}=\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta}-\frac{\partial v_\theta}{\partial z}\right)\mathbf{e}_r+\left(\frac{\partial v_r}{\partial z}-\frac{\partial v_z}{\partial r}\right)\mathbf{e}_\theta+\frac{1}{r}\left(\frac{\partial(rv_\theta)}{\partial r}-\frac{\partial v_r}{\partial \theta}\right)\mathbf{e}_z$$
$$\nabla^2\phi=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial \theta^2}+\frac{\partial^2\phi}{\partial z^2}$$
$$\nabla\phi=\frac{\partial\phi}{\partial r}\mathbf{e}_r+\frac{1}{r}\frac{\partial\phi}{\partial \theta}\mathbf{e}_\theta+\frac{\partial\phi}{\partial z}\mathbf{e}_z$$
$$\nabla\cdot\mathbf{v}=\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}$$

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