# Part V

# **Appendices**

# A Basic Covering Theory

In this appendix, we provide the main results on coverings of topological spaces needed to develop coverings of Lie groups and manifolds. In particular, this material is needed to show that, for each finite-dimensional Lie algebra $\mathfrak{g}$, there exists a 1-connected Lie group $G$ with Lie algebra $\mathbf{L}(G) = \mathfrak{g}$ which is unique up to isomorphism.

## A.1 The Fundamental Group

To define the notion of a simply connected space, we first have to define its fundamental group. The elements of this group are homotopy classes of loops. The present section develops this concept and provides some of its basic properties.

**Definition A.1.1.** Let $X$ be a topological space, $I := [0,1]$, and $x_0, x_1 \in X$. We write

$$P(X, x_0) := \{\gamma \in C(I, X) \colon \gamma(0) = x_0\}$$

and

$$P(X, x_0, x_1) := \{\gamma \in P(X, x_0) \colon \gamma(1) = x_1\}.$$

We call two paths $\alpha_0, \alpha_1 \in P(X, x_0, x_1)$ *homotopic*, written $\alpha_0 \sim \alpha_1$, if there exists a continuous map

$$H \colon I \times I \to X \quad \text{with} \quad H_0 = \alpha_0, \quad H_1 = \alpha_1$$

(for $H_t(s) := H(t,s)$) and

$$(\forall t \in I) \quad H(t,0) = x_0,\ H(t,1) = x_1.$$

It is easy to show that $\sim$ is an equivalence relation (Exercise A.1.2), called *homotopy*. The homotopy class of $\alpha$ is denoted by $[\alpha]$.

We write $\Omega(X, x_0) := P(X, x_0, x_0)$ for the set of loops based at $x_0$. For $\alpha \in P(X, x_0, x_1)$ and $\beta \in P(X, x_1, x_2)$ we define a product $\alpha * \beta$ in $P(X, x_0, x_2)$ as the concatenation

$$(\alpha * \beta)(t) := \begin{cases} \alpha(2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ \beta(2t-1) & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

J. Hilgert, K.-H. Neeb, *Structure and Geometry of Lie Groups*,
Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-84794-8, 

**Lemma A.1.2.** *If* $\varphi\colon [0,1] \to [0,1]$ *is a continuous map with* $\varphi(0) = 0$ *and* $\varphi(1) = 1$*, then for each* $\alpha \in P(X, x_0, x_1)$ *we have* $\alpha \sim \alpha \circ \varphi$.

*Proof.* Use $H(t,s) := \alpha(ts + (1-t)\varphi(s))$. □

**Proposition A.1.3.** *The following assertions hold:*

(1) $\alpha_1 \sim \alpha_2$ *and* $\beta_1 \sim \beta_2$ *implies* $\alpha_1 * \beta_1 \sim \alpha_2 * \beta_2$*, so that we obtain a well-defined product*

$$[\alpha] * [\beta] := [\alpha * \beta]$$

*of homotopy classes.*

(2) *If* $x$ *also denotes the constant map* $I \to \{x\} \subseteq X$*, then*

$$[x_0] * [\alpha] = [\alpha] = [\alpha] * [x_1] \quad \textit{for} \quad \alpha \in P(X, x_0, x_1).$$

(3) (Associativity) $[\alpha * \beta] * [\gamma] = [\alpha] * [\beta * \gamma]$ *for* $\alpha \in P(X, x_0, x_1)$*,* $\beta \in P(X, x_1, x_2)$ *and* $\gamma \in P(X, x_2, x_3)$.

(4) (Inverse) *For* $\alpha \in P(X, x_0, x_1)$ *and* $\overline{\alpha}(t) := \alpha(1-t)$*, we have*

$$[\alpha] * [\overline{\alpha}] = [x_0].$$

(5) (Functoriality) *For any continuous map* $\varphi\colon X \to Y$ *with* $\varphi(x_0) = y_0$*, we have*

$$(\varphi \circ \alpha) * (\varphi \circ \beta) = \varphi \circ (\alpha * \beta)$$

*and* $\alpha \sim \beta$ *implies* $\varphi \circ \alpha \sim \varphi \circ \beta$.

*Proof.* (1) If $H^\alpha$ is a homotopy from $\alpha_1$ to $\alpha_2$ and $H^\beta$ a homotopy from $\beta_1$ to $\beta_2$, then we put

$$H(t,s) := \begin{cases} H^\alpha(t, 2s) & \text{for } 0 \le s \le \frac{1}{2}, \\ H^\beta(t, 2s-1) & \text{for } \frac{1}{2} \le s \le 1 \end{cases}$$

(cf. Exercise A.1.1).

(2) For the first assertion, we use Lemma A.1.2 and

$$x_0 * \alpha = \alpha \circ \varphi \quad \text{for} \quad \varphi(t) := \begin{cases} 0 & \text{for } 0 \le t \le \frac{1}{2}, \\ 2t-1 & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

For the second, we have

$$\alpha * x_1 = \alpha \circ \varphi \quad \text{for} \quad \varphi(t) := \begin{cases} 2t & \text{for } 0 \le t \le \frac{1}{2}, \\ 1 & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

(3) We have $(\alpha * \beta) * \gamma = (\alpha * (\beta * \gamma)) \circ \varphi$ for

$$\varphi(t) := \begin{cases} 2t & \text{for } 0 \leq t \leq \frac{1}{4}, \\ \frac{1}{4} + t & \text{for } \frac{1}{4} \leq t \leq \frac{1}{2}, \\ \frac{t+1}{2} & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

(4)

$$H(t,s) := \begin{cases} \alpha(2s) & \text{for } s \leq \frac{1-t}{2}, \\ \alpha(1-t) & \text{for } \frac{1-t}{2} \leq s \leq \frac{1+t}{2}, \\ \overline{\alpha}(2s-1) & \text{for } s \geq \frac{1+t}{2}. \end{cases}$$

(5) This assertion is trivial. □

**Definition A.1.4.** From the preceding definition, we derive in particular that the set

$$\pi_1(X, x_0) := \Omega(X, x_0)/\sim$$

of homotopy classes of loops in $x_0$ carries a natural group structure. This group is called the *fundamental group of $X$ with respect to $x_0$*.

A space $X$ is called *simply connected* if $\pi_1(X, x_0)$ vanishes for all $x_0 \in X$. If $X$ is pathwise connected, it suffices to check this for a single $x_0 \in X$ (Exercise A.1.4).

**Lemma A.1.5 (Functoriality of the Fundamental Group).** *If $f\colon X \to Y$ is a continuous map with $f(x_0) = y_0$, then*

$$\pi_1(f)\colon \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad [\gamma] \mapsto [f \circ \gamma]$$

*is a group homomorphism. Moreover, we have*

$$\pi_1(\mathrm{id}_X) = \mathrm{id}_{\pi_1(X,x_0)} \quad \text{and} \quad \pi_1(f \circ g) = \pi_1(f) \circ \pi_1(g).$$

*Proof.* This follows directly from Proposition A.1.3(5). □

**Remark A.1.6.** The map

$$\sigma\colon \pi_1(X, x_0) \times \big(P(X, x_0)/\sim\big) \to P(X, x_0)/\sim, \quad ([\alpha], [\beta]) \mapsto [\alpha * \beta] = [\alpha] * [\beta]$$

defines an action of the group $\pi_1(X, x_0)$ on the set $P(X, x_0)/\sim$ of homotopy classes of paths starting in $x_0$ (Proposition A.1.3).

**Remark A.1.7.** (a) Suppose that the topological space $X$ is contractible, i.e., there exists a continuous map $H\colon I \times X \to X$ and $x_0 \in X$ with $H(0, x) = x$ and $H(1, x) = x_0$ for $x \in X$. Then $\pi_1(X, x_0) = \{[x_0]\}$ is trivial (Exercise).

(b) $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

(c) $\pi_1(\mathbb{R}^n, 0) = \{0\}$ because $\mathbb{R}^n$ is contractible.

More generally, if the open subset $\Omega \subseteq \mathbb{R}^n$ is starlike with respect to $x_0$, then $H(t, x) := x + t(x - x_0)$ yields a contraction to $x_0$, and we conclude that $\pi_1(\Omega, x_0) = \{\mathbf{1}\}$.

(d) If $G \subseteq \mathrm{GL}_n(\mathbb{R})$ is a linear Lie group with a polar decomposition, i.e., for $K := G \cap \mathrm{O}_n(\mathbb{R})$ and $\mathfrak{p} := \mathbf{L}(G) \cap \mathrm{Sym}_n(\mathbb{R})$, the polar map

$$p\colon K \times \mathfrak{p} \to G, \quad (k,x) \mapsto ke^x$$

is a homeomorphism, then the inclusion $K \to G$ induces an isomorphism

$$\pi_1(K, \mathbf{1}) \to \pi_1(G, \mathbf{1})$$

because the vector space $\mathfrak{p}$ is contractible.

The following lemma implies in particular that the fundamental groups of topological groups are always abelian.

**Lemma A.1.8.** *Let $G$ be a topological group and consider the identity element $\mathbf{1}$ as a base point. Then the path space $P(G, \mathbf{1})$ also carries a natural group structure given by the pointwise product $(\alpha \cdot \beta)(t) := \alpha(t)\beta(t)$, and we have*

(1) $\alpha \sim \alpha'$, $\beta \sim \beta'$ *implies* $\alpha \cdot \beta \sim \alpha' \cdot \beta'$, *so that we obtain a well-defined product*

$$[\alpha][\beta] := [\alpha] \cdot [\beta] := [\alpha \cdot \beta]$$

*of homotopy classes, defining a group structure on* $P(G, \mathbf{1})/\sim$.
(2) $\alpha \sim \beta \iff \alpha \cdot \beta^{-1} \sim \mathbf{1}$, *the constant map.*
(3) (Commutativity) $[\alpha] \cdot [\beta] = [\beta] \cdot [\alpha]$ *for* $\alpha, \beta \in \Omega(G, \mathbf{1})$.
(4) (Consistency) $[\alpha] \cdot [\beta] = [\alpha] * [\beta]$ *for* $\alpha \in \Omega(G, \mathbf{1})$, $\beta \in P(G, \mathbf{1})$.

*Proof.* (1) follows by composing homotopies with the multiplication map $m_G$.
(2) follows from (1).
(3)

$$[\alpha][\beta] = [\alpha*\mathbf{1}][\mathbf{1}*\beta] = \big[(\alpha*\mathbf{1})(\mathbf{1}*\beta)\big] = \big[(\mathbf{1}*\beta)(\alpha*\mathbf{1})\big] = [\mathbf{1}*\beta][\alpha*\mathbf{1}] = [\beta][\alpha].$$

(4) $[\alpha][\beta] = [(\alpha * \mathbf{1})(\mathbf{1} * \beta)] = [\alpha * \beta] = [\alpha] * [\beta]$. □

As a consequence of (4), we can calculate the product of homotopy classes as a pointwise product of representatives and obtain:

**Proposition A.1.9 (Hilton's Lemma).** *For each topological group $G$, the fundamental group $\pi_1(G) := \pi_1(G, \mathbf{1})$ is abelian.*

*Proof.* We only have to combine (3) and (4) in Lemma A.1.8 for loops $\alpha, \beta \in \Omega(G, \mathbf{1})$. □

### A.1.1 Exercises for Section A.1

**Exercise A.1.1.** If $f\colon X \to Y$ is a map between topological spaces and $X = X_1 \cup \cdots \cup X_n$ holds with closed subsets $X_1, \ldots, X_n$, then $f$ is continuous if and only if all restrictions $f|_{X_i}$ are continuous.

**Exercise A.1.2.** Show that the homotopy relation on $P(X, x_0, x_1)$ is an equivalence relation.

**Exercise A.1.3.** Show that for $n > 1$ the sphere $\mathbb{S}^n$ is simply connected. For the proof, proceed along the following steps:
(a) Let $\gamma : [0,1] \to \mathbb{S}^n$ be continuous. Then there exists an $m \in \mathbb{N}$ such that $\|\gamma(t) - \gamma(t')\| < \frac{1}{2}$ for $|t - t'| < \frac{1}{m}$.
(b) Define $\widetilde{\alpha} : [0,1] \to \mathbb{R}^{n+1}$ as the piecewise affine curve with $\widetilde{\alpha}(\frac{k}{m}) = \gamma(\frac{k}{m})$ for $k = 0, \ldots, m$. Then $\alpha(t) := \frac{1}{\|\widetilde{\alpha}(t)\|}\widetilde{\alpha}(t)$ defines a continuous curve $\alpha : [0,1] \to \mathbb{S}^n$.
(c) $\alpha \sim \gamma$.
(d) $\alpha$ is not surjective. The image of $\alpha$ is the central projection of a polygonal arc on the sphere.
(e) If $\beta \in \Omega(\mathbb{S}^n, y_0)$ is not surjective, then $\beta \sim y_0$ (it is homotopic to a constant map).
(f) $\pi_1(\mathbb{S}^n, y_0) = \{[y_0]\}$ for $n \geq 2$ and $y_0 \in \mathbb{S}^n$.

**Exercise A.1.4.** Let $X$ be a topological space, $x_0, x_1 \in X$ and $\alpha \in P(X, x_0, x_1)$ a path from $x_0$ to $x_1$. Show that the map

$$C\colon \pi_1(X, x_1) \to \pi_1(X, x_0), \quad [\gamma] \mapsto [\alpha * \gamma * \overline{\alpha}]$$

is an isomorphism of groups. In this sense, the fundamental group does not depend on the base point if $X$ is arcwise connected.

**Exercise A.1.5.** Let $\sigma\colon G \times X \to X$ be a continuous action of the topological group $G$ on the topological space $X$ and $x_0 \in X$. Then the orbit map $\sigma^{x_0} : G \to X, g \mapsto \sigma(g, x_0)$ defines a group homomorphism

$$\pi_1(\sigma^{x_0})\colon \pi_1(G) \to \pi_1(X, x_0).$$

Show that the image of this homomorphism is central, i.e., lies in the center of $\pi_1(X, x_0)$.

## A.2 Coverings

In this section, we discuss the concept of a covering map. Its main application in Lie theory is that it provides, for each connected Lie group $G$, a simply connected covering group $q_G\colon \widetilde{G} \to G$ and hence also a tool to calculate its fundamental group $\pi_1(G) \cong \ker q_G$. In the following chapter, we shall investigate to which extent a Lie group is determined by its Lie algebra and its fundamental group.

**Definition A.2.1.** Let $X$ and $Y$ be topological spaces. A continuous map $q\colon X \to Y$ is called a *covering* if each $y \in Y$ has an open neighborhood $U$ such that $q^{-1}(U)$ is a nonempty disjoint union of open subsets $(V_i)_{i\in I}$, such that for each $i \in I$ the restriction $q|_{V_i}\colon V_i \to U$ is a homeomorphism. We call any such $U$ an *elementary* open subset of $X$.

Note that this condition implies in particular that $q$ is surjective and that the fibers of $q$ are discrete subsets of $X$.

**Examples A.2.2.** (a) The exponential function $\exp\colon \mathbb{C} \to \mathbb{C}^\times,\ z \mapsto e^z$ is a covering map.

(b) The map $q\colon \mathbb{R} \to \mathbb{T},\ x \mapsto e^{ix}$ is a covering.

(c) The power maps $p_k\colon \mathbb{C}^\times \to \mathbb{C}^\times,\ z \mapsto z^k$ are coverings.

(d) If $q\colon G \to H$ is a surjective open morphism of topological groups with discrete kernel, then $q$ is a covering (Exercise A.2.2). All the examples (a)–(c) are of this type.

**Lemma A.2.3 (Lebesgue Number).** *Let $(X, d)$ be a compact metric space and $(U_i)_{i\in I}$ an open cover. Then there exists a positive number $\lambda > 0$, called a Lebesgue number of the covering, such that any subset $S \subseteq X$ with diameter $\leq \lambda$ is contained in some $U_i$.*

*Proof.* Let us assume that such a number $\lambda$ does not exist. Then for each $n \in \mathbb{N}$ there exists a subset $S_n$ of diameter $\leq \frac{1}{n}$ which is not contained in some $U_i$. Pick a point $s_n \in S_n$. The sequence $(s_n)$ has a subsequence converging to some $s \in X$ and $s$ is contained in some $U_i$. Since $U_i$ is open, there exists an $\varepsilon > 0$ with $U_\varepsilon(s) \subseteq U_i$. If $n \in \mathbb{N}$ is such that $\frac{1}{n} < \frac{\varepsilon}{2}$ and $d(s_n, s) < \frac{\varepsilon}{2}$, we arrive at the contradiction $S_n \subseteq U_{\varepsilon/2}(s_n) \subseteq U_\varepsilon(s) \subseteq U_i$. □

**Remark A.2.4.** (1) If $(U_i)_{i\in I}$ is an open cover of the unit interval $[0, 1]$, then there exists an $n > 0$ such that all subsets of the form $\left[\frac{k}{n}, \frac{k+1}{n}\right]$, $k = 0, \ldots, n-1$, are contained in some $U_i$.

(2) If $(U_i)_{i\in I}$ is an open cover of the unit square $[0, 1]^2$, then there exists an $n > 0$ such that all subsets of the form

$$\left[\frac{k}{n}, \frac{k+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right], \quad k, j = 0, \ldots, n-1,$$

are contained in some $U_i$.

**Theorem A.2.5 (Path Lifting Theorem).** *Let $q\colon X \to Y$ be a covering map and $\gamma\colon [0, 1] \to Y$ a path. Let $x_0 \in X$ be such that $q(x_0) = \gamma(0)$. Then there exists a unique path $\widetilde{\gamma}\colon [0, 1] \to X$ such that*

$$q \circ \widetilde{\gamma} = \gamma \quad \textit{and} \quad \widetilde{\gamma}(0) = x_0.$$

*Proof.* Cover $Y$ by elementary open sets $U_i, i \in I$. By Lemma A.2.3, there exists an $n \in \mathbb{N}$ such that all sets $\gamma\left(\left[\frac{k}{n}, \frac{k+1}{n}\right]\right)$, $k = 0, \ldots, n-1$, are contained in some $U_i$. We now use induction to construct $\widetilde{\gamma}$. Let $V_0 \subseteq q^{-1}(U_0)$ be an open subset containing $x_0$ for which $q|_{V_0}$ is a homeomorphism onto $U_0$ and define $\widetilde{\gamma}$ on $\left[0, \frac{1}{n}\right]$ by

$$\widetilde{\gamma}(t) := (q|_{V_0})^{-1} \circ \gamma(t).$$

Assume that we have already constructed a continuous lift $\widetilde{\gamma}$ of $\gamma$ on the interval $\left[0, \frac{k}{n}\right]$ and that $k < n$. Then we pick an open subset $V_k \subseteq X$ containing $\widetilde{\gamma}(\frac{k}{n})$ for which $q|_{V_k}$ is a homeomorphism onto some $U_i$ and define $\widetilde{\gamma}$ for $t \in \left[\frac{k}{n}, \frac{k+1}{n}\right]$ by

$$\widetilde{\gamma}(t) := (q|_{V_k})^{-1} \circ \gamma(t).$$

We thus obtain the required lift $\widetilde{\gamma}$ of $\gamma$.

If $\widehat{\gamma}\colon [0,1] \to X$ is any continuous lift of $\gamma$ with $\widehat{\gamma}(0) = x_0$, then $\widehat{\gamma}(\left[0, \frac{1}{n}\right])$ is a connected subset of $q^{-1}(U_0)$ containing $x_0$, hence contained in $V_0$. This shows that $\widetilde{\gamma}$ coincides with $\widehat{\gamma}$ on $\left[0, \frac{1}{n}\right]$. Applying the same argument at each step of the induction, we obtain $\widehat{\gamma} = \widetilde{\gamma}$, so that the lift $\widetilde{\gamma}$ is unique. □

**Theorem A.2.6 (Covering Homotopy Theorem).** *Let $I := [0,1]$ and $q\colon X \to Y$ be a covering map and $H\colon I^2 \to Y$ be a homotopy with fixed endpoints of the paths $\gamma := H_0$ and $\eta := H_1$. For any lift $\widetilde{\gamma}$ of $\gamma$ there exists a unique lift $G\colon I^2 \to X$ of $H$ with $G_0 = \widetilde{\gamma}$. Then $\widetilde{\eta} := G_1$ is the unique lift of $\eta$ starting in the same point as $\widetilde{\gamma}$ and $G$ is a homotopy from $\widetilde{\gamma}$ to $\widetilde{\eta}$. In particular, lifts of homotopic curves in $Y$ starting in the same point are homotopic in $X$.*

*Proof.* Using the Path Lifting Property (Theorem A.2.5), for each $t \in I$ we find a unique continuous lift $I \to X, s \mapsto G(s,t)$, starting in $\widetilde{\gamma}(t)$ with $q(G(s,t)) = H(s,t)$. It remains to show that the map $G\colon I^2 \to X$ obtained in this way is continuous.

So let $s \in I$. Using Lemma A.2.3, we find a natural number $n$ such that for each connected neighborhood $W_s$ of $s$ of diameter $\leq \frac{1}{n}$ and each $i = 0, \ldots, n$, the set $H\big(W_s \times \left[\frac{k}{n}, \frac{k+1}{n}\right]\big)$ is contained in some elementary subset $U_k$ of $Y$. Assuming that $G$ is continuous in $W_s \times \{\frac{k}{n}\}$, $G$ maps this set into a connected subset of $q^{-1}(U_k)$, hence into some open subset $V_k$ for which $q|_{V_k}$ is a homeomorphism onto $U_k$. But then the lift $G$ on $W_s \times \left[\frac{k}{n}, \frac{k+1}{n}\right]$ must be contained in $V_k$, so that it is of the form $(q|_{V_k})^{-1} \circ H$, hence continuous. This means that $G$ is continuous on $U_s \times \left[\frac{k}{n}, \frac{k+1}{n}\right]$. Now an inductive argument shows that $G$ is continuous on $U_s \times I$ and hence on the whole square $I^2$.

Since the fibers of $q$ are discrete and the curves $s \mapsto H(s,0)$ and $s \mapsto H(s,1)$ are constant, the curves $G(s,0)$ and $G(s,1)$ are also constant. Therefore, $\widetilde{\eta}$ is the unique lift of $\eta$ starting in $\widetilde{\gamma}(0) = G(0,0) = G(1,0)$, and $G$ is a homotopy with fixed endpoints from $\widetilde{\gamma}$ to $\widetilde{\eta}$. □

**Corollary A.2.7.** *If $q\colon X \to Y$ is a covering with $q(x_0) = y_0$, then the corresponding homomorphism*

$$\pi_1(q)\colon \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad [\gamma] \mapsto [q \circ \gamma]$$

*is injective.*

*Proof.* If $\gamma, \eta$ are loops in $x_0$ with $[q\circ\gamma] = [q\circ\eta]$, then the Covering Homotopy Theorem A.2.6 implies that $\gamma$ and $\eta$ are homotopic. Therefore, $[\gamma] = [\eta]$ shows that $\pi_1(q)$ is injective. □

**Corollary A.2.8.** *If $Y$ is simply connected and $X$ is arcwise connected, then each covering map $q\colon X \to Y$ is a homeomorphism.*

*Proof.* Since $q$ is an open continuous map, it remains to show that $q$ is injective. So pick $x_0 \in X$ and $y_0 \in Y$ with $q(x_0) = y_0$. If $x \in X$ also satisfies $q(x) = y_0$, then there exists a path $\alpha \in P(X, x_0, x)$ from $x_0$ to $x$. Now $q \circ \alpha$ is a loop in $Y$, hence contractible because $Y$ is simply connected. Now the Covering Homotopy Theorem implies that the unique lift $\alpha$ of $q \circ \alpha$ starting in $x_0$ is a loop, and therefore that $x_0 = x$. This proves that $q$ is injective. □

The following theorem provides a powerful tool from which the preceding corollary easily follows. We recall that a topological space $X$ is called *locally arcwise connected* if each neighborhood $U$ of a point $x \in X$ contains an arcwise connected neighborhood.

**Theorem A.2.9 (Lifting Theorem).** *Assume that $q\colon X \to Y$ is a covering map with $q(x_0) = y_0$, that $W$ is arcwise connected and locally arcwise connected, and that $f\colon W \to Y$ is a given map with $f(w_0) = y_0$. Then a continuous map $g\colon W \to X$ with*

$$g(w_0) = x_0 \quad \text{and} \quad q \circ g = f \tag{A.1}$$

*exists if and only if*

$$\pi_1(f)\big(\pi_1(W, w_0)\big) \subseteq \pi_1(q)\big(\pi_1(X, x_0)\big), \ \text{i.e.,}\ \operatorname{im}\big(\pi_1(f)\big) \subseteq \operatorname{im}\big(\pi_1(q)\big). \tag{A.2}$$

*If $g$ exists, then it is uniquely determined by (A.1). Condition (A.2) is in particular satisfied if $W$ is simply connected.*

*Proof.* If $g$ exists, then $f = q\circ g$ implies that the image of the homomorphism $\pi_1(f) = \pi_1(q) \circ \pi_1(g)$ is contained in the image of $\pi_1(q)$.

Let us, conversely, assume that this condition is satisfied. To define $g$, let $w \in W$ and $\alpha_w\colon I \to W$ be a path from $w_0$ to $w$. Then $f \circ \alpha_w\colon I \to Y$ is a path which has a continuous lift $\beta_w\colon I \to X$ starting in $x_0$. We claim that $\beta_w(1)$ does not depend on the choice of the path $\alpha_w$. Indeed, if $\alpha'_w$ is another path from $w_0$ to $w$, then $\alpha_w * \overline{\alpha'_w}$ is a loop in $w_0$, so that $(f \circ \alpha_w) * (f \circ \overline{\alpha'_w})$ is a loop in $y_0$. In view of (A.2), the homotopy class of this loop is contained in the image of $\pi_1(q)$, so that it has a lift $\eta\colon I \to X$ which is a loop in $x_0$. Since the reverse of the second half $\eta|_{[\frac{1}{2},1]}$ of $\eta$ is a lift of $f \circ \alpha'_w$, starting in $x_0$, it is $\beta'_w$, and we obtain

$$\beta'_w(1) = \eta\left(\frac{1}{2}\right) = \beta_w(1).$$

We now put $g(w) := \beta_w(1)$, and it remains to see that $g$ is continuous. This is where we shall use the assumption that $W$ is locally arcwise connected. Let $w \in W$ and put $y := f(w)$. Further, let $U \subseteq Y$ be an elementary neighborhood of $y$ and $V$ be an arcwise connected neighborhood of $w$ in $W$ such that $f(V) \subseteq U$. Fix a path $\alpha_w$ from $w_0$ to $w$ as before. For any point $w' \in V$, we choose a path $\gamma_{w'}$ from $w$ to $w'$ in $V$, so that $\alpha_w * \gamma_{w'}$ is a path from $w_0$ to $w'$. Let $\widetilde{U} \subseteq X$ be an open subset of $X$ for which $q|_{\widetilde{U}}$ is a homeomorphism onto $U$ and $g(w) \in \widetilde{U}$. Then the uniqueness of lifts implies that

$$\beta_{w'} = \beta_w * \big((q|_{\widetilde{U}})^{-1} \circ (f \circ \gamma_{w'})\big).$$

We conclude that

$$g(w') = (q|_{\widetilde{U}})^{-1}\big(f(w')\big) \in \widetilde{U},$$

hence $g|_V$ is continuous.

Finally, we show that $g$ is unique. In fact, if $h\colon W \to X$ is another lift of $f$ satisfying $h(w_0) = x_0$, then the set $S := \{w \in W\colon g(w) = h(w)\}$ is nonempty and closed. We claim that it is also open. In fact, let $w_1 \in S$ and $U$ be a connected open elementary neighborhood of $f(w_1)$ and $V$ an arcwise connected neighborhood of $w_1$ with $f(V) \subseteq U$. If $\widetilde{U} \subseteq q^{-1}(U)$ is the open subset on which $q$ is a homeomorphism containing $g(w_1) = h(w_1)$, then, since $V$ is arcwise connected, we have that $g(V), h(V) \subseteq \widetilde{U}$, whence $V \subseteq S$. Therefore, $S$ is open, closed and nonempty. Since $W$ is connected this implies that $S = W$, i.e., $g = h$. □

**Corollary A.2.10 (Uniqueness of Simply Connected Coverings).** *Suppose that $Y$ is locally arcwise connected. If $q_1\colon X_1 \to Y$ and $q_2\colon X_2 \to Y$ are two simply connected arcwise connected coverings, then there exists a homeomorphism $\varphi\colon X_1 \to X_2$ with $q_2 \circ \varphi = q_1$.*

*Proof.* Since $Y$ is locally arcwise connected, both covering spaces $X_1$ and $X_2$ also have this property. Pick points $x_1 \in X_1$, $x_2 \in X_2$ with $y := q_1(x_1) = q_2(x_2)$. According to the Lifting Theorem A.2.9, there exists a unique lift $\varphi\colon X_1 \to X_2$ of $q_1$ with $\varphi(x_1) = x_2$. We likewise obtain a unique lift $\psi\colon X_2 \to X_1$ of $q_2$ with $\psi(x_2) = x_1$. Then $\varphi \circ \psi\colon X_1 \to X_1$ is a lift of $\mathrm{id}_Y$ fixing $x_1$, so that the uniqueness of lifts implies that $\varphi \circ \psi = \mathrm{id}_{X_1}$. The same argument yields $\psi \circ \varphi = \mathrm{id}_{X_2}$, so that $\varphi$ is a homeomorphism with the required properties. □

**Definition A.2.11.** A topological space $X$ is called *semilocally simply connected* if each point $x_0 \in X$ has a neighborhood $U$ such that each loop $\alpha \in \Omega(U, x_0)$ is homotopic to $[x_0]$ in $X$, i.e., the natural homomorphism

$$\pi(i_U)\colon \pi_1(U, x_0) \to \pi_1(X, x_0), \quad [\gamma] \mapsto [i_U \circ \gamma]$$

induced by the inclusion map $i_U\colon U \to X$ is trivial.

**Theorem A.2.12 (Existence of simply connected coverings).** *Let $Y$ be arcwise connected and locally arcwise connected. Then $Y$ has a simply connected covering space if and only if $Y$ is semilocally simply connected.*

*Proof.* If $q\colon X \to Y$ is a simply connected covering space and $U \subseteq Y$ is a pathwise connected elementary open subset. Then each loop $\gamma$ in $U$ lifts to a loop $\widetilde{\gamma}$ in $X$, and since $\widetilde{\gamma}$ is homotopic to a constant map in $X$, the same holds for the loop $\gamma = q \circ \widetilde{\gamma}$ in $Y$.

Conversely, let us assume that $Y$ is semilocally simply connected. We choose a base point $y_0 \in Y$ and let

$$\widetilde{Y} := P(Y, y_0)/\sim\, := \bigcup_{y_1 \in Y} P(Y, y_0, y_1)/\sim$$

be the set of homotopy classes of paths starting in $y_0$. We shall provide $\widetilde{Y}$ with a topology such that the map

$$q\colon \widetilde{Y} \to Y, \quad [\gamma] \mapsto \gamma(1)$$

defines a simply connected covering of $Y$.

Let $\mathcal{B}$ denote the set of all arcwise connected open subsets $U \subseteq Y$ for which each loop in $U$ is contractible in $Y$ and note that our assumptions on $Y$ imply that $\mathcal{B}$ is a basis of the topology of $Y$, i.e., each open subset is a union of elements of $\mathcal{B}$. If $\gamma \in P(Y, y_0)$ satisfies $\gamma(1) \in U \in \mathcal{B}$, let

$$U_{[\gamma]} := \big\{[\eta] \in q^{-1}(U)\colon \big(\exists \beta \in C(I, U)\big)\ \eta \sim \gamma * \beta\big\}.$$

We shall now verify several properties of these definitions, culminating in the proof of the theorem.

(1) $[\eta] \in U_{[\gamma]} \Rightarrow U_{[\eta]} = U_{[\gamma]}$.

To prove this, let $[\zeta] \in U_{[\eta]}$. Then $\zeta \sim \eta * \beta$ for some path $\beta$ in $U$. Further, $\eta \sim \gamma * \beta'$ for some path $\beta'$ in $U$. Now $\zeta \sim \gamma * \beta' * \beta$, and $\beta' * \beta$ is a path in $U$, so that $[\zeta] \in U_{[\gamma]}$. This proves $U_{[\eta]} \subseteq U_{[\gamma]}$. We also have $\gamma \sim \eta * \overline{\beta'}$, so that $[\gamma] \in U_{[\eta]}$, and the first part implies that $U_{[\gamma]} \subseteq U_{[\eta]}$.

(2) $q$ maps $U_{[\gamma]}$ injectively onto $U$.

That $q(U_{[\gamma]}) = U$ is clear since $U$ and $Y$ are arcwise connected. To show that it is one-to-one, let $[\eta], [\eta'] \in U_{[\gamma]}$, which we know from (1) is the same as $U_{[\eta]}$. Suppose $\eta(1) = \eta'(1)$. Since $[\eta'] \in U_{[\eta]}$, we have $\eta' \sim \eta * \alpha$ for some loop $\alpha$ in $U$. But then $\alpha$ is contractible in $Y$, so that $\eta' \sim \eta$, i.e., $[\eta'] = [\eta]$.

(3) $U, V \in \mathcal{B}$, $\gamma(1) \in U \subseteq V$, implies $U_{[\gamma]} \subseteq V_{[\gamma]}$.

This is trivial.

(4) The sets $U_{[\gamma]}$ for $U \in \mathcal{B}$ and $[\gamma] \in \widetilde{Y}$ form a basis of a topology on $\widetilde{Y}$.

Suppose $[\gamma] \in U_{[\eta]} \cap V_{[\eta']}$. Let $W \subseteq U \cap V$ be in $\mathcal{B}$ with $\gamma(1) \in W$. Then $[\gamma] \in W_{[\gamma]} \subseteq U_{[\gamma]} \cap V_{[\gamma]} = U_{[\eta]} \cap V_{[\eta']}$.

(5) $q$ is open and continuous.

We have already seen in (2) that $q(U_{[\gamma]}) = U$, and these sets form a basis of the topology on $\widetilde{Y}$, resp., $Y$. Therefore, $q$ is an open map. We also have for $U \in \mathcal{B}$ the relation

$$q^{-1}(U) = \bigcup_{\gamma(1)\in U} U_{[\gamma]},$$

which is open. Hence $q$ is continuous.

(6) $q|_{U_{[\gamma]}}$ is a homeomorphism.

This is because it is bijective, continuous and open.

At this point we have shown that $q\colon \widetilde{Y} \to Y$ is a covering map. It remains to see that $\widetilde{Y}$ is arcwise connected and simply connected.

(7) Let $H\colon I \times I \to Y$ be a continuous map with $H(t,0) = y_0$. Then $h_t(s) := H(t,s)$ defines a path in $Y$ starting in $y_0$. Let $\widetilde{h}(t) := [h_t] \in \widetilde{Y}$. Then $\widetilde{h}$ is a path in $\widetilde{Y}$ covering the path $t \mapsto h_t(1) = H(t,1)$ in $Y$. We claim that $\widetilde{h}$ is continuous. Let $t_0 \in I$. We shall prove continuity at $t_0$. Let $U \in \mathcal{B}$ be a neighborhood of $h_{t_0}(1)$. Then there exists an interval $I_0 \subseteq I$ which is a neighborhood of $t_0$ with $h_t(1) \in U$ for $t \in I_0$. Then $\alpha(s) := H(t_0+s(t-t_0),1)$ is a continuous curve in $U$ with $\alpha(0) = h_{t_0}(1)$ and $\alpha(1) = h_t(1)$, so that $h_{t_0} * \alpha$ is curve with the same endpoint as $h_t$. Applying Exercise A.2.1 to the restriction of $H$ to the interval between $t_0$ and $t$, we see that $h_t \sim h_{t_0} * \alpha$, so that $\widetilde{h}(t) = [h_t] \in U_{[h_{t_0}]}$ for $t \in I_0$. Since $q|_{U_{[h_{t_0}]}}$ is a homeomorphism, $\widetilde{h}$ is continuous at $t_0$.

(8) $\widetilde{Y}$ is arcwise connected.

For $[\gamma] \in \widetilde{Y}$ put $h_t(s) := \gamma(st)$. By (7), this yields a path $\widetilde{\gamma}(t) = [h_t]$ in $\widetilde{Y}$ from $\widetilde{y}_0 := [y_0]$ (the class of the constant path) to the point $[\gamma]$.

(9) $\widetilde{Y}$ is simply connected.

Let $\widetilde{\alpha} \in \Omega(\widetilde{Y}, \widetilde{y}_0)$ be a loop in $\widetilde{Y}$ and $\alpha := q \circ \widetilde{\alpha}$ its image in $Y$. Let $h_t(s) := \alpha(st)$. Then we have the path $\widetilde{h}(t) = [h_t]$ in $\widetilde{Y}$ from (7). This path covers $\alpha$ since $h_t(1) = \alpha(t)$. Further, $\widetilde{h}(0) = \widetilde{y}_0$ is the constant path. Also, by definition, $\widetilde{h}(1) = [\alpha]$. From the uniqueness of lifts we derive that $\widetilde{h} = \widetilde{\alpha}$ is closed, so that $[\alpha] = [y_0]$. Therefore, the homomorphism

$$\pi_1(q)\colon \pi_1(\widetilde{Y}, \widetilde{y}_0) \to \pi_1(Y, y_0)$$

vanishes. Since it is also injective (Corollary A.2.7), $\pi_1(\widetilde{Y}, \widetilde{y}_0)$ is trivial, i.e., $\widetilde{Y}$ is simply connected. □

**Definition A.2.13.** Let $q\colon X \to Y$ be a covering. A homeomorphism $\varphi\colon X \to X$ is called a *deck transformation* of the covering if $q \circ \varphi = q$. This means that $\varphi$ permutes the elements in the fibers of $q$. We write $\mathrm{Deck}(X, q)$ for the group of deck transformations.

**Example A.2.14.** For the covering map $\exp\colon \mathbb{C} \to \mathbb{C}^\times$, the deck transformations have the form

$$\varphi(z) = z + 2\pi i n, \quad n \in \mathbb{Z}.$$

**Proposition A.2.15.** *Let* $q\colon \widetilde{Y} = (P(Y, y_0)/\sim) \to Y$ *be the simply connected covering of* $Y$ *with base point* $\widetilde{y}_0 = [y_0]$. *For each* $[\gamma] \in \pi_1(Y, y_0)$, *we write* $\varphi_{[\gamma]} \in \mathrm{Deck}(\widetilde{Y}, q)$ *for the unique lift of* $\mathrm{id}_Y$ *mapping* $[y_0]$ *to the endpoint* $[\gamma] = \widetilde{\gamma}(1)$ *of the canonical lift* $\widetilde{\gamma}$ *of* $\gamma$ *starting in* $\widetilde{y}_0$. *Then the map*

$$\Phi\colon \pi_1(Y, y_0) \to \mathrm{Deck}(\widetilde{Y}, q), \quad \Phi([\gamma]) = \varphi_{[\gamma]}$$

*is an isomorphism of groups.*

*Proof.* For $\gamma, \eta \in \Omega(Y, y_0)$, the composition $\varphi_{[\gamma]} \circ \varphi_{[\eta]}$ is a deck transformation mapping $\widetilde{y}_0$ to the endpoint of $\varphi_{[\gamma]} \circ \widetilde{\eta}$ which coincides with the endpoint of the lift of $\eta$ starting in $\widetilde{\gamma}(1)$. Hence it also is the endpoint of the lift of the loop $\gamma * \eta$. Therefore, $\Phi$ is a group homomorphism.

To see that $\Phi$ is injective, we note that $\varphi_{[\gamma]} = \mathrm{id}_{\widetilde{Y}}$ implies that $\widetilde{\gamma}(1) = \widetilde{y}_0$, so that $\widetilde{\gamma}$ is a loop, and hence that $[\gamma] = [y_0] = \widetilde{y}_0$.

For the surjectivity, let $\varphi$ be a deck transformation and $y := \varphi(\widetilde{y}_0)$. If $\alpha$ is a path from $\widetilde{y}_0$ to $y$, then $\gamma := q \circ \alpha$ is a loop in $y_0$ with $\alpha = \widetilde{\gamma}$, so that $\varphi_{[\gamma]}(\widetilde{y}_0) = y$, and the uniqueness of lifts implies that $\varphi = \varphi_{[\gamma]}$. □

### A.2.1 Exercises for Section A.2

**Exercise A.2.1.** Let $F\colon I^2 \to X$ be a continuous map with $F(0, s) = x_0$ for $s \in I$ and define

$$\gamma(t) := F(t, 0), \quad \eta(t) := F(t, 1), \quad \alpha(t) := F(1, t), \quad t \in I.$$

Show that $\gamma * \alpha \sim \eta$.

**Exercise A.2.2.** Let $q\colon G \to H$ be an morphism of topological groups with discrete kernel $\Gamma$. Show that:

(1) If $V \subseteq G$ is an open $\mathbf{1}$-neighborhood with $(V^{-1}V) \cap \Gamma = \{\mathbf{1}\}$ and $q$ is open, then $q|_V\colon V \to q(V)$ is a homeomorphism.
(2) If $q$ is open and surjective, then $q$ is a covering.
(3) If $q$ is open and $H$ is connected, then $q$ is surjective, hence a covering.

**Exercise A.2.3.** A map $f\colon X \to Y$ between topological spaces is called a *local homeomorphism* if each point $x \in X$ has an open neighborhood $U$ such that $f|_U\colon U \to f(U)$ is a homeomorphism onto an open subset of $Y$.

(1) Show that each covering map is a local homeomorphism.
(2) Find a surjective local homeomorphism which is not a covering. Can you also find an example where $X$ is connected?

# B Some Multilinear Algebra

In this appendix, we provide some tools from multilinear algebra. Some are needed in Chapter 7 on Lie algebras, where we construct the universal enveloping algebra, and some other tools are needed for the discussion of differential forms on manifolds in Chapter 10. Section B.3 on Clifford algebras plays a crucial role in Chapter 17 on classical groups. Throughout, $\mathbb{K}$ is an arbitrary field of characteristic zero.

## B.1 Tensor Products and Tensor Algebra

Let $V$ and $W$ be vector spaces. A *tensor product* of $V$ and $W$ is a pair $(V \otimes W, \otimes)$ of a vector space $V \otimes W$ and a bilinear map

$$\otimes\colon V \times W \to V \otimes W, \quad (v, w) \mapsto v \otimes w$$

with the following universal property. For each bilinear map $\beta\colon V \times W \to U$ into a vector space $U$, there exists a unique linear map $\widetilde{\beta}\colon V \otimes W \to U$ satisfying

$$\widetilde{\beta}(v \otimes w) = \beta(v, w) \quad \text{for} \quad v \in V, w \in W.$$

Taking $(U, \beta) = (V \otimes W, \otimes)$, we conclude immediately that $\mathrm{id}_{V \otimes W}$ is the unique linear endomorphism of $V \otimes W$ fixing all elements of the form $v \otimes w$.

Before we turn to the existence of tensor products, we discuss their uniqueness. In category theory, one gives a precise meaning to the statement that objects with a universal property are determined up to isomorphism. The following lemma makes this precise for tensor products.

**Lemma B.1.1 (Uniqueness of tensor products).** *If* $(V \otimes W, \otimes)$ *and* $(V \widetilde{\otimes} W, \widetilde{\otimes})$ *are two tensor products of the vector spaces* $V$ *and* $W$*, then there exists a unique linear isomorphism*

$$f\colon V \otimes W \to V \widetilde{\otimes} W \quad \textit{with} \quad f(v \otimes w) = v \widetilde{\otimes} w \quad \textit{for} \quad v \in V, w \in W.$$

*Proof.* Since $\widetilde{\otimes}$ is bilinear, the universal property of $(V \otimes W, \otimes)$ implies the existence of a unique linear map

$$f\colon V \otimes W \to V \widetilde{\otimes} W \quad \text{with} \quad f(v \otimes w) = v \widetilde{\otimes} w \quad \text{for} \quad v \in V, w \in W.$$

J. Hilgert, K.-H. Neeb, *Structure and Geometry of Lie Groups*,
Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-84794-8, 

Similarly, the universal property of $(V\widetilde{\otimes}W, \widetilde{\otimes})$ implies the existence of a linear map

$$g\colon V\widetilde{\otimes}W \to V\otimes W \quad \text{with} \quad g(v\widetilde{\otimes}w) = v\otimes w \quad \text{for} \quad v\in V, w\in W.$$

Then $g\circ f \in \mathrm{End}(V\otimes W)$ is a linear map with $(g\circ f)(v\otimes w) = v\otimes w$ for $v\in V$ and $w\in W$, so that the uniqueness part of the universal property of $(V\otimes W, \otimes)$ yields $g\circ f = \mathrm{id}_{V\otimes W}$. We likewise get $f\circ g = \mathrm{id}_{V\widetilde{\otimes}W}$, showing that $f$ is a linear isomorphism. □

Now we turn to the existence of the tensor product.

**Definition B.1.2.** Let $S$ be a set. We write $F(S) := \mathbb{K}^{(S)}$ for the *free vector space on* $S$. It is the subspace of the cartesian product $\mathbb{K}^S$, the set of all functions $f\colon S\to\mathbb{K}$ for which the set $\{s\in S\colon f(s)\neq 0\}$ is finite.

For $s\in S$, we define $\delta_s(t) := \delta_{st}$, which is 1 for $s=t$, and 0 otherwise. Then $(\delta_s)_{s\in S}$ is a basis for the vector space $F(S)$ and we have a map

$$\delta\colon S\to F(S), \quad s\mapsto \delta_s.$$

Now the pair $(F(S),\delta)$ has the universal property that, for each map $f\colon S\to V$ to a vector space $V$, there exists a unique linear map $\widetilde{f}\colon F(S)\to V$ with $\widetilde{f}\circ\delta = f$.

**Proposition B.1.3 (Existence of tensor products).** *If $V$ and $W$ are vector spaces, then there exists a tensor product $(V\otimes W,\otimes)$.*

*Proof.* In the free vector space $F(V\times W)$ over $V\times W$, we consider the subspace $N$, generated by elements of the form

$$\delta_{(v_1+v_2,w)} - \delta_{(v_1,w)} - \delta_{(v_2,w)}, \quad \delta_{(v,w_1+w_2)} - \delta_{(v,w_1)} - \delta_{(v,w_2)},$$

and

$$\delta_{(\lambda v,w)} - \delta_{(v,\lambda w)}, \quad \lambda\delta_{(v,w)} - \delta_{(\lambda v,w)},$$

for $v,v_1,v_2\in V$, $w,w_1,w_2\in W$, and $\lambda\in\mathbb{K}$. We put

$$V\otimes W := F(V\times W)/N \quad \text{and} \quad v\otimes w := \delta_{(v,w)} + N.$$

The bilinearity of $\otimes$ follows from the definition of $N$. In particular, we have

$$(v_1+v_2)\otimes w = \delta_{(v_1+v_2,w)} + N = \delta_{(v_1,w)} + \delta_{(v_2,w)} + N = v_1\otimes w + v_2\otimes w$$

and

$$(\lambda v)\otimes w = \delta_{(\lambda v,w)} + N = \lambda\delta_{(v,w)} + N = \lambda(v\otimes w).$$

The linearity in the second argument is verified similarly.

To show that $(V \otimes W, \otimes)$ has the required universal property, let $\beta\colon V \times W \to U$ be a bilinear map. We use the universal property of $(F(V \times W), \delta)$ to obtain a linear map

$$\gamma\colon F(V \times W) \to U \quad \text{with} \quad \gamma(\delta_{(v,w)}) = \beta(v, w)$$

for $v \in V, w \in W$. The bilinearity of $\beta$ now implies that $N \subseteq \ker \gamma$, so that $\gamma$ factors through a unique linear map

$$\widetilde{\beta}\colon V \otimes W = F(V \times W)/N \to U \quad \text{with} \quad \widetilde{\beta}(v \otimes w) = \gamma(\delta_{(v,w)}) = \beta(v, w).$$

That $\widetilde{\beta}$ is uniquely determined by this property follows from the fact that the elements of the form $v \otimes w$ generate $V \otimes W$ linearly, which in turn follows from $\delta(V \times W)$ being a linear basis for $F(V \times W)$. □

Tensor products of finitely many factors are defined in a similar fashion as follows.

**Definition B.1.4.** Let $V_1, \ldots, V_k$ be vector spaces. A *tensor product* of $V_1, \ldots, V_k$ is a pair

$$(V_1 \otimes V_2 \otimes \cdots \otimes V_k, \otimes)$$

of a vector space $V_1 \otimes V_2 \otimes \cdots \otimes V_k$ and a $k$-linear map

$$\otimes\colon V_1 \times \cdots \times V_k \to V_1 \otimes V_2 \otimes \cdots \otimes V_k, \quad (v_1, \ldots, v_k) \mapsto v_1 \otimes \cdots \otimes v_k,$$

with the following universal property. For each $k$-linear map

$$\beta\colon V_1 \times \cdots \times V_k \to U$$

into a vector space $U$, there exists a unique linear map $\widetilde{\beta}\colon V_1 \otimes \cdots \otimes V_k \to U$ satisfying

$$\widetilde{\beta}(v_1 \otimes \cdots \otimes v_k) = \beta(v_1, \ldots, v_k) \quad \text{for} \quad v_i \in V_i.$$

For $(U, \beta) = (V_1 \otimes \cdots \otimes V_k, \otimes)$, we conclude immediately that $\mathrm{id}_{V_1 \otimes \cdots \otimes V_k}$ is the unique linear endomorphism of $V_1 \otimes \cdots \otimes V_k$ fixing all elements of the form $v_1 \otimes \cdots \otimes v_k$.

Again, the universal property determines $k$-fold tensor products.

**Lemma B.1.5 (Uniqueness of k-fold tensor products).** *If*

$$(V_1 \otimes \cdots \otimes V_k, \otimes) \quad \textit{and} \quad (V_1 \widetilde{\otimes} \cdots \widetilde{\otimes} V_k, \widetilde{\otimes})$$

*are two tensor products of the vector spaces $V_1, \ldots, V_k$, then there exists a unique linear isomorphism*

$$f\colon V_1 \otimes \cdots \otimes V_k \to V_1 \widetilde{\otimes} \cdots \widetilde{\otimes} V_k \quad \textit{with} \quad f(v_1 \otimes \cdots \otimes v_k) = v_1 \widetilde{\otimes} \cdots \widetilde{\otimes} v_k$$

*for $v_i \in V_i$.*

We omit the simple proof of the uniqueness. The existence is easily reduced to the two-fold case:

**Lemma B.1.6.** *If $V_1, \ldots, V_k$ are vector spaces and $k \geq 2$, then the iterated two-fold tensor product*

$$V_1 \otimes \cdots \otimes V_k := (V_1 \otimes \cdots \otimes V_{k-1}) \otimes V_k$$

*and*

$$v_1 \otimes \cdots \otimes v_k := (v_1 \otimes \cdots \otimes v_{k-1}) \otimes v_k$$

*is a tensor product of $V_1, \ldots, V_k$.*

*Proof.* Since we know already that this is true for $k = 2$, we argue by induction and assume that the assertion holds for $(k-1)$-fold iterated tensor products. In this way, we immediately see that $(v_1 \otimes \cdots \otimes v_{k-1}) \otimes v_k$ is $k$-linear.

To verify the universal property, let $\beta \colon V_1 \times \cdots \times V_k \to U$ be a $k$-linear map. We first use the induction hypothesis to obtain for each $v_k \in V_k$ a unique linear map $\widetilde{\beta}_{v_k} \colon V_1 \otimes \cdots \otimes V_{k-1} \to U$ with

$$\widetilde{\beta}_{v_k}(v_1 \otimes \ldots \otimes v_{k-1}) = \beta(v_1, \ldots, v_{k-1}, v_k) \quad \text{for} \quad v_i \in V_i, i \leq k-1.$$

From the uniqueness of $\widetilde{\beta}_{v_k}$, we further derive that

$$\widetilde{\beta}_{\lambda v_k + \lambda' v_k'} = \lambda \widetilde{\beta}_{v_k} + \lambda' \widetilde{\beta}_{v_k'}$$

for $\lambda, \lambda' \in \mathbb{K}$ and $v_k, v_k' \in V_k$. Hence the map

$$(V_1 \otimes \cdots \otimes V_{k-1}) \times V_k \to U, \quad (x, v_k) \mapsto \widetilde{\beta}_{v_k}(x)$$

is bilinear. Now the universal property of the two-fold tensor product provides a unique linear map

$$\widetilde{\beta} \colon (V_1 \otimes \cdots \otimes V_{k-1}) \otimes V_k \to U$$

with $\widetilde{\beta}((v_1 \otimes \cdots \otimes v_{k-1}) \otimes v_k) = \widetilde{\beta}_{v_k}(v_1 \otimes \cdots \otimes v_{k-1}) = \beta(v_1, \ldots, v_{k-1}, v_k)$. □

**Definition B.1.7 (The tensor algebra of a vector space).** Let $V$ be a $\mathbb{K}$-vector space and $V^{\otimes n}$ the $n$-fold tensor product of $V$ with itself. For $n = 0, 1$, we put $V^{\otimes 0} := \mathbb{K}$ and $V^{\otimes 1} := V$.

We claim that, for $n, m \in \mathbb{N}$, there exists a bilinear map

$$\mu_{n,m} \colon V^{\otimes n} \times V^{\otimes m} \to V^{\otimes (n+m)}$$

with

$$\mu_{n,m}\big((v_1 \otimes \cdots \otimes v_n), (v_{n+1} \otimes \cdots \otimes v_{n+m})\big) = v_1 \otimes \cdots \otimes v_{n+m}$$

for $v_1, \ldots, v_{n+m} \in V$. In fact, for each $x \in V^{\otimes n}$, the map

$$\mu_x : V^m \to V^{\otimes(n+m)}, \quad (w_1, \ldots, w_m) \mapsto x \otimes w_1 \otimes \cdots \otimes w_m$$

is $m$-linear, hence determines a linear map

$$\widetilde{\mu}_x : V^{\otimes m} \to V^{\otimes(n+m)} \quad \text{with} \quad \widetilde{\mu}_x(w_1 \otimes \cdots \otimes w_m) = \mu_x(w_1, \ldots, w_m).$$

Since $\mu_x$ is also linear in $x$, we obtain a unique bilinear map

$$\mu_{n,m} : V^{\otimes n} \times V^{\otimes m} \to V^{\otimes(n+m)}$$

with

$$\begin{aligned} &\mu_{n,m}\big((v_1 \otimes \cdots \otimes v_n), (v_{n+1} \otimes \cdots \otimes v_{n+m})\big) \\ &= \widetilde{\mu}_{(v_1 \otimes \cdots \otimes v_n)}(v_{n+1} \otimes \cdots \otimes v_{n+m}) = v_1 \otimes \cdots \otimes v_n \otimes v_{n+1} \otimes \cdots \otimes v_{n+m}. \end{aligned}$$

We further define bilinear maps

$$\mu_{0,n} : V^{\otimes 0} \times V^{\otimes n} = \mathbb{K} \times V^{\otimes n} \to V^{\otimes n}, \quad (\lambda, v) \mapsto \lambda v$$

and

$$\mu_{n,0} : V^{\otimes n} \otimes V^{\otimes 0} = V^{\otimes n} \times \mathbb{K} \to V^{\otimes n}, \quad (v, \lambda) \mapsto \lambda v.$$

Putting all maps $\mu_{n,k}$, $n, k \in \mathbb{N}_0$, together, we obtain a bilinear multiplication on the vector space

$$\mathcal{T}(V) := \oplus_{n=0}^{\infty} V^{\otimes n}.$$

It is now easy to show that this multiplication is associative and has an identity element $\mathbf{1} \in V^{\otimes 0}$ (Exercise B.1.5). The algebra obtained in this way is called the *tensor algebra of* $V$.

**Lemma B.1.8 (Universal property of the tensor algebra).** *Let $V$ be a vector space and $\eta : V \to \mathcal{T}(V)$ the canonical embedding of $V$ as $V^{\otimes 1}$. Then the pair $(\mathcal{T}(V), \eta)$ has the following property. For any linear map $f : V \to A$ into a unital associative $\mathbb{K}$-algebra $A$, there exists a unique homomorphism $\widetilde{f} : \mathcal{T}(V) \to A$ of unital associative algebras with $\widetilde{f} \circ \eta = f$.*

*Proof.* For the uniqueness of $\widetilde{f}$, we first note that the requirement of being a homomorphism of unital algebras determines $\widetilde{f}$ on $\mathbf{1}$ via $\widetilde{f}(\mathbf{1}) = \mathbf{1}_A$. On $\eta(V) = V^{\otimes 1}$ it is determined by $\widetilde{f} \circ \eta = f$, and on $\mathcal{T}(V)$ it is thus determined since the algebra $\mathcal{T}(V)$ is generated by the subspace $\mathbb{K}\mathbf{1} + V$.

For the existence of $\widetilde{f}$, we note that, for each $n \in \mathbb{N}$, the map

$$V^n \to A, \quad (v_1, \ldots, v_n) \mapsto f(v_1) \cdots f(v_n)$$

is $n$-linear, so that there exists a unique linear map

$$\widetilde{f}_n : V^{\otimes n} \to A \quad \text{with} \quad \widetilde{f}(v_1 \otimes \cdots \otimes v_n) = f(v_1) \cdots f(v_n)$$

for $v_i \in V$. We now combine these linear maps $\widetilde{f}_n$ to a linear map

$$\widetilde{f}\colon \mathcal{T}(V) \to A \quad \text{with} \quad \widetilde{f}_n(\mathbf{1}) = \mathbf{1}_A, \quad \widetilde{f}|_{V^{\otimes n}} = \widetilde{f}_n.$$

Then the construction implies that $\widetilde{f} \circ \eta = f$. That $\widetilde{f}$ is an algebra homomorphism follows from

$$\begin{aligned}\widetilde{f}\big((v_1 \otimes \cdots \otimes v_n) \cdot (w_1 \otimes \cdots \otimes w_m)\big) &= f(v_1)\cdots f(v_n)f(w_1)\cdots f(w_m)\\ &= \widetilde{f}(v_1 \otimes \cdots \otimes v_n)\widetilde{f}(w_1 \otimes \cdots \otimes w_m)\end{aligned}$$

for $v_1, \ldots, v_n, w_1, \ldots, w_m \in V$. □

### B.1.1 Exercises for Section B.1

**Exercise B.1.1.** Let $U$, $V$ and $W$ be finite-dimensional vector spaces. Show that there are isomorphisms:

(i) $U \otimes V \cong V \otimes U$,
(ii) $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$.

**Exercise B.1.2.** The aim of this exercise is to get a more concrete picture of the tensor product of two vector spaces in terms of bases. Let $V$ and $W$ be vector spaces. We consider a basis $B_V = \{e_i : i \in I\}$ for $V$ and a basis $B_W = \{f_j : j \in J\}$ for $W$. Show that:

(i) Each function $f\colon B_V \times B_W \to \mathbb{K}$ has a unique bilinear extension $\widetilde{f}\colon V \times W \to \mathbb{K}$.
(ii) The set $B_V \otimes B_W = \{e_i \otimes f_j : i \in I, j \in J\}$ is a basis for $V \otimes W$.
(iii) Each element $x \in V \otimes W$ has a unique representation as a finite sum $x = \sum_{i\in I} e_i \otimes w_i$ with $w_i \in W$.
(iv) If $V_1$ and $V_2$ are vector spaces, then $(V_1 \oplus V_2) \otimes W \cong (V_1 \otimes W) \oplus (V_2 \otimes W)$.

**Exercise B.1.3.** Let $V := \mathbb{K}^n$ and $W := \mathbb{K}^m$. Show that one can turn the space $M_{n,m}(\mathbb{K})$ of $(n \times m)$-matrices with entries in $\mathbb{K}$ into a tensor product $(\mathbb{K}^n \otimes \mathbb{K}^m, \otimes)$ satisfying

$$e_i \otimes e_j := E_{ij},$$

where $e_1, \ldots, e_n$ denotes the canonical basis vectors in $\mathbb{K}^n$ and $E_{ij}$ is the matrix which has a single nonzero entry in the $i$th row and the $j$th column.

**Exercise B.1.4.** If $V$ and $W$ are finite-dimensional, then the map

$$\Phi\colon V^* \otimes W \to \mathrm{Hom}(V, W), \qquad \Phi(\alpha \otimes w)(v) := \alpha(v)w$$

is a linear isomorphism.

**Exercise B.1.5.** Let $V$ be a vector space and $\mathcal{T}(V) = \bigoplus_{n\in\mathbb{N}_0} V^{\otimes n}$. Show that the multiplication on $\mathcal{T}(V)$ defined by Definition B.1.7 yields an associative $\mathbb{K}$-algebra.

**Exercise B.1.6.** Let $V_i$ and $W_i$ be $\mathbb{K}$-vector spaces (for $i = 1, 2$) and $A \in \mathrm{Hom}_{\mathbb{K}}(V_1, V_2)$, $B \in \mathrm{Hom}_{\mathbb{K}}(W_1, W_2)$. Show that there exists a unique $\mathbb{K}$-linear map $C\colon V_1 \otimes V_2 \to W_1 \otimes W_2$ such that

$$C(v_1 \otimes v_2) = A(v_1) \otimes B(v_2)$$

for all $v_1 \in V_1$ and $v_2 \in V_2$. The map $C$ is usually denoted by $A \otimes B$.

**Exercise B.1.7.** Suppose that $V_1, \ldots, V_k$ are vector spaces and that a group $G$ acts linearly on each of them. Show that

$$g \cdot (v_1 \otimes \cdots \otimes v_k) := g \cdot v_1 \otimes \cdots \otimes g \cdot v_k$$

for $g \in G$ and $v_j \in V_j$ defines a linear action on $V_1 \otimes \cdots \otimes V_k$.

## B.2 Symmetric and Exterior Products

### B.2.1 Symmetric and Exterior Powers

**Definition B.2.1.** Let $V$ be a vector space and $n \geq 2$. We define

$$S^n(V) := V^{\otimes n}/U,$$

where $U$ is the subspace spanned by all elements of the form

$$v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}, \quad \sigma \in S_n.$$

The space $S^n(V)$ is called the *nth symmetric power of* $V$. We put

$$v_1 \vee \cdots \vee v_n := v_1 \otimes \cdots \otimes v_n + U$$

and observe that this product is symmetric in the sense that

$$v_1 \vee \cdots \vee v_n = v_{\sigma(1)} \vee \cdots \vee v_{\sigma(n)}$$

for each $\sigma \in S_n$ and $v_1, \ldots, v_n \in V$. For $n = 1$, we put $S^1(V) := V$ and also $S^0(V) := \mathbb{K}$.

If $X$ and $Y$ are sets, then a map $f\colon X^n \to Y$ is said to be *symmetric* if, for each permutation $\sigma \in S_n$, we have

$$f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \quad \text{for} \quad x \in X^n.$$

**Lemma B.2.2 (Universal property of symmetric powers).** *Let $V$ and $X$ be vector spaces and $f\colon V^n \to X$ be a symmetric $n$-linear map. Then there exists a unique linear map $\widetilde{f}\colon S^n(V) \to X$ with*

$$\widetilde{f}(v_1 \vee \cdots \vee v_n) = f(v_1, \ldots, v_n) \quad \textit{for} \quad v_1, \ldots, v_n \in V.$$

*Proof.* From the universal property of the $n$-fold tensor product $V^{\otimes n}$, we obtain a unique linear map $f_0 \colon V^{\otimes n} \to X$ with

$$f_0(v_1 \otimes \cdots \otimes v_n) = f(v_1, \ldots, v_n) \quad \text{for} \quad v_1, \ldots, v_n \in V.$$

In view of the symmetry of $f$, the linear map $f_0$ vanishes on $U$, hence factors through a linear map $\widetilde{f} \colon S^n(V) \to X$ with the desired property. □

**Definition B.2.3.** Let $V$ and $W$ be $\mathbb{K}$-vector spaces, $n \in \mathbb{N}$, and

$$\text{sgn} \colon S_n \to \{1, -1\}$$

be the *signature homomorphism* mapping all transpositions to $-1$. An $n$-linear map $f \colon V^n \to W$ is called *alternating* if

$$f(v_1, \ldots, v_n) = \text{sgn}(\sigma) f(v_{\sigma(1)}, \ldots, v_{\sigma(n)})$$

holds for all $\sigma \in S_n$ and $v_1, \ldots, v_n \in V$.

We write $\text{Alt}^n(V, W)$ for the set of alternating $n$-linear maps $V^n \to W$. Clearly, sums and scalar multiples of alternating maps are alternating, so that $\text{Alt}^n(V, W)$ carries a natural vector space structure. For $n = 0$, we shall follow the convention that $\text{Alt}^0(V, W) := W$ is the set of constant maps, which are considered to be 0-linear.

**Example B.2.4.** From linear algebra, we know the $n$-linear map

$$\big(\mathbb{K}^n\big)^n \to \mathbb{K}, \quad \det(v_1, \ldots, v_k) := \sum_{\sigma \in S_k} \text{sgn}(\sigma) v_{1,\sigma(1)} \cdots v_{k,\sigma(k)}.$$

Here we identify the space $M_n(\mathbb{K})$ of $(n \times n)$-matrices with entries in $\mathbb{K}$ with the space $(\mathbb{K}^n)^n$ of $n$-tuples of (column) vectors [La93, Sect. XIII.4].

**Definition B.2.5.** Let $V$ be a vector space and $n \geq 2$. We define

$$\Lambda^n(V) := V^{\otimes n}/W,$$

where $W$ is the subspace spanned by the elements of the form

$$v_1 \otimes \cdots \otimes v_n - \text{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}, \quad \sigma \in S_n.$$

The space $\Lambda^n(V)$ is called the *nth exterior power of* $V$. We put

$$v_1 \wedge \cdots \wedge v_n := v_1 \otimes \cdots \otimes v_n + W$$

and note that this product is alternating, i.e.,

$$v_1 \wedge \cdots \wedge v_n = \text{sgn}(\sigma) v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)}$$

for all $\sigma \in S_n$ and $(v_1, \ldots, v_n) \in V^n$. For $n = 2$, this means that

$$v_1 \wedge v_2 = -v_2 \wedge v_1.$$

We also put $\Lambda^1(V) := V$ and $\Lambda^0(V) := \mathbb{K}$.

**Lemma B.2.6 Universal property of the exterior power.** *Let $V$ and $X$ be vector spaces and $f \in \mathrm{Alt}^n(V, X)$. Then there exists a unique linear map $\widetilde{f}\colon \Lambda^n(V) \to X$ with*

$$\widetilde{f}(v_1 \wedge \cdots \wedge v_n) = f(v_1, \ldots, v_n) \quad \textit{for} \quad v_1, \ldots, v_n \in V.$$

*We thus obtain a linear bijection*

$$\mathrm{Alt}^n(V, X) \to \mathrm{Hom}\big(\Lambda^n(V), X\big), \quad f \mapsto \widetilde{f}.$$

*Proof.* The proof is completely analogous to the symmetric case. □

### B.2.2 Symmetric and Exterior Algebra

**Definition B.2.7.** Let $V$ be a vector space and $(\mathcal{T}(V), \eta)$ the tensor algebra of $V$ (cf. Lemma B.1.8). We define the *symmetric algebra* $S(V)$ over $V$ as the quotient $\mathcal{T}(V)/I_s$, where $I_s$ is the ideal generated by the elements $\eta(v) \otimes \eta(w) - \eta(w) \otimes \eta(v)$. We write

$$\eta_s\colon V \to S(V), \quad v \mapsto \eta(v) + I_s$$

for the canonical map induced by $\eta$. The product in $S(V)$ is denoted by $\vee$.

Likewise, we define the *exterior algebra* $\Lambda(V)$ over $V$ as the quotient $\mathcal{T}(V)/I_a$, where $I_a$ is the ideal generated by the elements

$$\eta(v) \otimes \eta(w) + \eta(w) \otimes \eta(v), \quad v, w \in V.$$

We write

$$\eta_a\colon V \to \Lambda(V), \quad v \mapsto \eta(v) + I_a$$

for the canonical map induced by $\eta$. The product in $\Lambda(V)$ is denoted by $\wedge$.

**Lemma B.2.8 (Universal property of the symmetric algebra).** *Let $V$ be a vector space and $(S(V), \eta_s)$ its symmetric algebra. Then $S(V)$ is a commutative unital algebra and for any linear map $f\colon V \to A$ into a unital commutative associative algebra $A$, there exists a unique homomorphism $\widetilde{f}\colon S(V) \to A$ of unital associative algebras with $\widetilde{f} \circ \eta_s = f$.*

*Proof.* Using the universal property of the tensor algebra $\mathcal{T}(V)$, we see that there exists a unique unital algebra homomorphism $\widehat{f}\colon \mathcal{T}(V) \to A$ with $\widehat{f} \circ \eta = f$. Since $A$ is commutative, for any $v, w \in V$, the element $\eta(v) \otimes \eta(w) - \eta(w) \otimes \eta(v)$ is contained in $\ker \widehat{f}$, and therefore, $I_s \subseteq \ker \widehat{f}$ shows that $\widehat{f}$ factors through an algebra homomorphism $\widetilde{f}\colon S(V) \to A$ with $\widetilde{f} \circ \eta_s = f$. The uniqueness of $\widetilde{f}$ follows from the fact that $\mathcal{T}(V)$ is generated, as a unital algebra, by $\eta(V)$, so that $S(V)$ is generated by the image of $\eta_s$. Since the generators $\eta_s(v)$, $v \in V$, of $S(V)$ commute, the algebra $S(V)$ is commutative. □

**Remark B.2.9.** (a) The structure of the symmetric algebra can be made more concrete as follows. Let $\mathcal{T}(V)_k := V^{\otimes k}$ and $U_2 \subseteq \mathcal{T}(V)_2$ the subspace spanned by the commutators $[\eta(v), \eta(w)], v, w \in V$. Then the ideal $I_s$ is of the form

$$I_s = \mathcal{T}(V)U_2\mathcal{T}(V) = \sum_{p,q\in\mathbb{N}_0} \mathcal{T}(V)_p \otimes U_2 \otimes \mathcal{T}(V)_q = \bigoplus_{n=2}^{\infty} I_{s,n},$$

where $I_{s,n} := \sum_{p+q=n-2} \mathcal{T}(V)_p \otimes U_2 \otimes \mathcal{T}(V)_q$. This implies that the symmetric algebra $S(V)$ is a direct sum

$$S(V) = \bigoplus_{n=0}^{\infty} S(V)_n, \quad \text{where} \quad S(V)_n := \mathcal{T}(V)_n / I_{s,n}.$$

Let

$$\mu_n \colon V^n \to S(V)_n, \quad (v_1, \ldots, v_n) \mapsto \eta_s(v_1) \vee \cdots \vee \eta_s(v_n)$$

denote the $n$-fold multiplication map. Since $S(V)$ is commutative, this map is symmetric, hence induces a linear map

$$\widetilde{\mu}_n \colon S^n(V) \to S(V)_n$$

determined by

$$\widetilde{\mu}_n(v_1 \vee \cdots \vee v_n) = \eta_s(v_1) \vee \cdots \vee \eta_s(v_n).$$

On the other hand, it is clear that the subspace $I_{s,n}$ of $V^{\otimes n}$ is contained in the kernel of the quotient map $V^{\otimes n} \to S^n(V)$, so that there exists a linear map $f_n \colon S(V)_n \to S^n(V)$, with

$$f_n\big(\eta_s(v_1) \vee \cdots \vee \eta_s(v_n)\big) = v_1 \vee \cdots \vee v_n.$$

Then $f_n \circ \widetilde{\mu}_n = \mathrm{id}_{S^n(V)}$ and, similarly, $\widetilde{\mu}_n \circ f_n = \mathrm{id}_{S(V)_n}$. This proves that $\widetilde{\mu}_n$ is a linear isomorphism. In the following, we therefore identify $S^n(V)$ with the subspace $S(V)_n$ of the symmetric algebra and write $\eta_s(v)$ simply as $v$.

Note that $S^n(V) \vee S^m(V) \subseteq S^{n+m}(V)$, so that the direct sum

$$S(V) = \bigoplus_{n\in\mathbb{N}} S^n(V)$$

defines the structure of a *graded algebra* on $S(V)$ with $S^0(V) = \mathbb{K}\mathbf{1}$ containing the identity element.

(b) A similar argument applies to the exterior algebra and shows that the ideal $I_a$ has the form $I_a = \bigoplus_{n=2}^{\infty}(I_a \cap V^{\otimes n})$, so that

$$\Lambda(V) = \bigoplus_{n=0}^{\infty} \Lambda(V)_n, \quad \text{where} \quad \Lambda(V)_n := \mathcal{T}(V)_n / I_{a,n}.$$

Let $\mu_n : V^n \to \Lambda(V)_n, (v_1, \ldots, v_n) \mapsto \eta_a(v_1) \wedge \cdots \wedge \eta_a(v_n)$ denote the $n$-fold multiplication map. Then the relation $\eta_a(v_i)\eta_a(v_j) + \eta_a(v_j)\eta_a(v_i) = 0$ and the fact that $S_n$ is generated by transpositions imply that $\mu_n$ is alternating. Hence it induces a linear map $\widetilde{\mu}_n : \Lambda^n(V) \to \Lambda(V)_n$, determined by

$$\widetilde{\mu}_n(v_1 \wedge \cdots \wedge v_n) = \eta_a(v_1) \wedge \cdots \wedge \eta_a(v_n).$$

On the other hand, it is clear that the subspace $I_{a,n}$ of $V^{\otimes n}$ is contained in the kernel of the quotient map $V^{\otimes n} \to \Lambda^n(V)$, so that there is a linear map $f_n : \Lambda(V)_n \to \Lambda^n(V)$ with

$$f_n\big(\eta_a(v_1) \wedge \cdots \wedge \eta_a(v_n)\big) = v_1 \wedge \cdots \wedge v_n.$$

As in the symmetric case, we now see that $\widetilde{\mu}_n$ is a linear isomorphism. In the following, we therefore identify $\Lambda^n(V)$ with the subspace $\Lambda(V)_n$ of the symmetric algebra and write $\eta_a(v)$ simply as $v$.

Each subspace $\Lambda^n(V)$ is spanned by elements of the form $v_1 \wedge \cdots \wedge v_n$, and this implies that for $\alpha \in \Lambda^n(V)$ and $\beta \in \Lambda^m(V)$ we have

$$\alpha \wedge \beta = (-1)^{mn} \beta \wedge \alpha. \tag{B.1}$$

In this sense, the graded algebra $\Lambda(V)$ is *graded commutative*. The even part of this algebra is the subspace

$$\Lambda^{\text{even}}(V) := \bigoplus_{k=0}^{\infty} \Lambda^{2k}(V)$$

which is a central subalgebra, and the odd part is

$$\Lambda^{\text{odd}}(V) := \bigoplus_{k=0}^{\infty} \Lambda^{2k+1}(V).$$

For two elements $\alpha, \beta$ of this subspace we have $\alpha \wedge \beta = -\beta \wedge \alpha$.

**Lemma B.2.10 (Universal property of the exterior algebra).** *Let $V$ be a vector space and $(\Lambda(V), \eta_a)$ be its exterior algebra. Then $\Lambda(V)$ is a graded commutative unital algebra and for any linear map $f : V \to A$ into a unital associative algebra $A$, satisfying*

$$f(v)f(w) = -f(w)f(v) \quad \textit{for} \quad v, w \in V,$$

*there exists a unique homomorphism $\widetilde{f} : \Lambda(V) \to A$ of unital associative algebras with $\widetilde{f} \circ \eta_a = f$.*

*Proof.* Using the universal property of the tensor algebra $\mathcal{T}(V)$, we see that there exists a unique unital algebra homomorphism $\widehat{f} : \mathcal{T}(V) \to A$ with $\widehat{f} \circ \eta = f$. Then we have for $v, w \in V$

$$\widehat{f}\big(\eta(v) \otimes \eta(w) + \eta(w) \otimes \eta(v)\big) = f(v)f(w) + f(w)f(v) = 0.$$

Therefore, $I_a \subseteq \ker \widehat{f}$ shows that $\widehat{f}$ factors through a unital algebra homomorphism $\widetilde{f} \colon \Lambda(V) \to A$ with $\widetilde{f} \circ \eta_a = f$. The uniqueness of $\widetilde{f}$ follows from the fact that $\mathcal{T}(V)$ is generated, as a unital algebra, by $\eta(V)$, so that $\Lambda(V)$ is generated by the image of $\eta_a$. □

### B.2.3 Exterior Algebra and Alternating Maps

Below we shall see how general alternating maps can be expressed in terms of determinants.

**Proposition B.2.11.** *For any $\omega \in \mathrm{Alt}^k(V, W)$ we have:*

(i) *For $b_1, \ldots, b_k \in V$ and linear combinations $v_j = \sum_{i=1}^k a_{ij} b_i$, we have*

$$\omega(v_1, \ldots, v_k) = \det(A)\omega(b_1, \ldots, b_k) \quad \text{and} \quad A := (a_{ij}) \in M_k(\mathbb{K}).$$

(ii) *$\omega(v_1, \ldots, v_k) = 0$ if $v_1, \ldots, v_k$ are linearly dependent.*
(iii) *For $b_1, \ldots, b_n \in V$ and linear combinations $v_j = \sum_{i=1}^n a_{ij} b_i$ we have*

$$\omega(v_1, \ldots, v_k) = \sum_I \det(A_I)\omega(b_{i_1}, \ldots, b_{i_k}),$$

*where $A := (a_{ij}) \in M_{n,k}(\mathbb{K})$, $I = \{i_1, \ldots, i_k\}$ is a $k$-element subset of $\{1, \ldots, n\}$, $1 \leq i_1 < \cdots < i_k \leq n$, and $A_I := (a_{ij})_{i \in I, j=1,\ldots,k} \in M_k(\mathbb{K})$.*

*Proof.* (i) For the following calculation, we note that if $\sigma \colon \{1, \ldots, k\} \to \{1, \ldots, k\}$ is a map which is not bijective, then the alternating property implies that $\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = 0$. We therefore get

$$\begin{aligned}
&\omega(v_1, \ldots, v_k) = \omega\left(\sum_{i=1}^k a_{i1} b_i, \ldots, \sum_{i=1}^k a_{ik} b_i\right) \\
&= \sum_{i_1,\ldots,i_k=1}^k a_{i_1 1} \cdots a_{i_k k} \cdot \omega(b_{i_1}, \ldots, b_{i_k}) \\
&= \sum_{\sigma \in S_k} a_{\sigma(1)1} \cdots a_{\sigma(k)k} \cdot \omega(b_{\sigma(1)}, \ldots, b_{\sigma(k)}) \\
&= \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(k)k} \cdot \omega(b_1, \ldots, b_k) = \det(A) \cdot \omega(b_1, \ldots, b_k).
\end{aligned}$$

(ii) follows immediately from (i) because the linear dependence of $v_1, \ldots, v_k$ implies that $\det A = 0$.

(iii) First, we expand

$$\begin{aligned}\omega(v_1,\ldots,v_k) &= \omega\left(\sum_{i=1}^n a_{i1}b_i,\ldots,\sum_{i=1}^n a_{ik}b_i\right)\\ &= \sum_{i_1,\ldots,i_k=1}^n a_{i_1 1}\cdots a_{i_k k}\cdot\omega(b_{i_1},\ldots,b_{i_k}).\end{aligned}$$

If $|\{i_1,\ldots,i_k\}| < k$, then the alternating property implies that $\omega(b_{i_1},\ldots,b_{i_k}) = 0$ because two entries coincide. If $|\{i_1,\ldots,i_k\}| = k$, there exists a permutation $\sigma \in S_k$ with $i_{\sigma(1)} < \cdots < i_{\sigma(k)}$. We therefore get

$$\begin{aligned}\omega(v_1,\ldots,v_k) &= \sum_{1\le i_1<\cdots<i_k\le n}\ \sum_{\sigma\in S_k} a_{i_{\sigma(1)}1}\cdots a_{i_{\sigma(k)}k}\cdot\omega(b_{i_{\sigma(1)}},\ldots,b_{i_{\sigma(k)}})\\ &= \sum_{1\le i_1<\cdots<i_k\le n}\ \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma)a_{i_{\sigma(1)}1}\cdots a_{i_{\sigma(k)}k}\cdot\omega(b_{i_1},\ldots,b_{i_k})\\ &= \sum_I \det(A_I)\omega(b_{i_1},\ldots,b_{i_k}),\end{aligned}$$

where the sum is to be extended over all $k$-element subsets $I = \{i_1,\ldots,i_k\}$ of $\{1,\ldots,n\}$, where $i_1 < \cdots < i_k$. □

**Corollary B.2.12.** (i) *If* $\dim V < k$, *then* $\mathrm{Alt}^k(V,W) = \{0\}$.
(ii) *Let* $\dim V = n$ *and* $b_1,\ldots,b_n$ *be a basis for* $V$*. Then the map*

$$\Phi\colon\ \mathrm{Alt}^k(V,W)\to W^{\binom{n}{k}},\quad \Phi(\omega) = (\omega(b_{i_1},\ldots,b_{i_k}))_{i_1<\cdots<i_k}$$

*is a linear isomorphism. We obtain in particular* $\dim(\mathrm{Alt}^k(V,\mathbb{K})) = \binom{n}{k}$.
(iii) *If* $\dim V = k$ *and* $b_1,\ldots,b_k$ *is a basis for* $V$*, then the map*

$$\Phi\colon\ \mathrm{Alt}^k(V,W)\to W,\quad \Phi(\omega) = \omega(b_1,\ldots,b_k)$$

*is a linear isomorphism.*

*Proof.* (i) In Proposition B.2.11(i), we may choose $b_k = 0$.

(ii) First, we show that $\Phi$ is injective. So let $\omega \in \mathrm{Alt}^k(V,W)$ with $\Phi(\omega) = 0$. We now write any $k$ elements $v_1,\ldots,v_k \in V$ with respect to the basis elements as $v_j = \sum_{i=1}^n a_{ij}b_i$ and obtain with Proposition B.2.11:

$$\omega(v_1,\ldots,v_k) = \sum_{1\le i_1<\cdots<i_k\le n}\det(A_I)\omega(b_{i_1},\ldots,b_{i_k}) = 0.$$

To see that $\Phi$ is surjective, we pick for each $k$-element subset $I = \{i_1,\ldots,i_k\} \subseteq \{1,\ldots,n\}$ with $1 \le i_1 < \cdots < i_k \le n$ an element $w_I \in W$. Then the tuple $(w_I)$ is a typical element of $W^{\binom{n}{k}}$.

Expressing $k$ elements $v_1,\ldots,v_k$ in terms of the basis elements $b_1,\ldots,b_n$ via $v_j = \sum_{i=1}^n a_{ij}b_i$, we obtain an $(n\times k)$-matrix $A$. We now define an alternating $k$-linear map $\omega \in \mathrm{Alt}^k(V,W)$ by

$$\omega(v_1, \ldots, v_k) := \sum_I \det(A_I) w_I.$$

The $k$-linearity of $\omega$ follows directly from the $k$-linearity of the maps

$$(v_1, \ldots, v_k) \mapsto \det(A_I).$$

For $i_1 < \cdots < i_k$ we further have $\omega(b_{i_1}, \ldots, b_{i_k}) = w_I$ because in this case $A_I \in M_k(\mathbb{K})$ is the identity matrix and all other matrices $A_{I'}$ have some vanishing columns. This implies that $\Phi(\omega) = (w_I)$, and hence that $\Phi$ is surjective.

(iii) is a special case of (ii). □

**Definition B.2.13 (Alternator).** Let $V$ and $W$ be vector spaces. For a $k$-linear map $\omega \colon V^k \to W$, we define a new $k$-linear map by

$$\operatorname{Alt}(\omega)(v_1, \ldots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

Writing

$$\omega^\sigma(v_1, \ldots, v_k) := \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}),$$

we then have

$$\operatorname{Alt}(\omega) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \omega^\sigma.$$

The map $\operatorname{Alt}(\cdot)$ is called the *alternator*. We claim that it turns any $k$-linear map into an alternating $k$-linear map. To see this, we first note that for $\sigma, \pi \in S_k$, we have

$$\begin{aligned}\big(\omega^\sigma\big)^\pi(v_1, \ldots, v_k) &= \big(\omega^\sigma\big)(v_{\pi(1)}, \ldots, v_{\pi(k)}) \\ &= \omega(v_{\pi\sigma(1)}, \ldots, v_{\pi\sigma(k)}) = \omega^{\pi\sigma}(v_1, \ldots, v_k).\end{aligned}$$

This implies that

$$\begin{aligned}\operatorname{Alt}(\omega)^\pi &= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \big(\omega^\sigma\big)^\pi = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \omega^{\pi\sigma} = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}\big(\pi^{-1}\sigma\big) \omega^\sigma \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\pi) \operatorname{sgn}(\sigma) \omega^\sigma = \operatorname{sgn}(\pi) \operatorname{Alt}(\omega) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}\big(\sigma\pi^{-1}\big) \omega^\sigma \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \omega^{\sigma\pi} = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \big(\omega^\pi\big)^\sigma = \operatorname{Alt}\big(\omega^\pi\big).\end{aligned}$$

In particular, we see that $\operatorname{Alt}(\omega)$ is alternating.

**Remark B.2.14.** (a) We observe that if $\omega$ is alternating, then $\omega^\sigma = \operatorname{sgn}(\sigma)\omega$ for each permutation $\sigma$, and therefore,

$$\mathrm{Alt}(\omega) = \frac{1}{k!}\sum_{\sigma\in S_k} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\sigma)\omega = \frac{1}{k!}\sum_{\sigma\in S_k}\omega = \omega.$$

(b) For $k = 2$, we have $\mathrm{Alt}(\omega)(v_1, v_2) = \frac{1}{2}(\omega(v_1, v_2) - \omega(v_2, v_1))$; and for $k = 3$,

$$\mathrm{Alt}(\omega)(v_1, v_2, v_3) = \frac{1}{6}\big(\omega(v_1, v_2, v_3) - \omega(v_2, v_1, v_3) + \omega(v_2, v_3, v_1) - \omega(v_3, v_2, v_1) + \omega(v_3, v_1, v_2) - \omega(v_1, v_3, v_2)\big).$$

**Definition B.2.15.** Let $p, q \in \mathbb{N}_0$. For two multilinear maps

$$\omega_1 \colon V_1 \times \cdots \times V_p \to \mathbb{K} \quad \text{and} \quad \omega_2 \colon V_{p+1} \times \cdots \times V_{p+q} \to \mathbb{K}$$

we define the *tensor product* $\omega_1 \otimes \omega_2 \colon V_1 \times \cdots \times V_{p+q} \to \mathbb{K}$ by

$$(\omega_1 \otimes \omega_2)(v_1, \ldots, v_{p+q}) := \omega_1(v_1, \ldots, v_p)\omega_2(v_{p+1}, \ldots, v_{p+q}).$$

It is clear that $\omega_1 \otimes \omega_2$ is a $(p + q)$-linear map.

For $\lambda \in \mathbb{K}$ (the set of 0-linear maps) and a $p$-linear map $\omega$ as above, we obtain in particular

$$\lambda \otimes \omega := \omega \otimes \lambda := \lambda\omega.$$

For two alternating maps $\alpha \in \mathrm{Alt}^p(V, \mathbb{K})$ and $\beta \in \mathrm{Alt}^q(V, \mathbb{K})$, we define their *exterior product*:

$$\alpha \wedge \beta := \frac{(p+q)!}{p!q!}\mathrm{Alt}(\alpha \otimes \beta) = \frac{1}{p!q!}\sum_{\sigma\in S_{p+q}} \mathrm{sgn}(\sigma)(\alpha \otimes \beta)^\sigma. \tag{B.2}$$

It follows from (B.2) that $\alpha \wedge \beta$ is alternating, so that we obtain a bilinear map

$$\wedge \colon \mathrm{Alt}^p(V, \mathbb{K}) \times \mathrm{Alt}^q(V, \mathbb{K}) \to \mathrm{Alt}^{p+q}(V, \mathbb{K}), \quad (\alpha, \beta) \mapsto \alpha \wedge \beta.$$

On the direct sum

$$\mathrm{Alt}(V, \mathbb{K}) := \bigoplus_{p\in\mathbb{N}_0} \mathrm{Alt}^p(V, \mathbb{K}),$$

we now obtain a bilinear product by putting

$$\Big(\sum_p \alpha_p\Big) \wedge \Big(\sum_q \beta_q\Big) := \sum_{p,q} \alpha_p \wedge \beta_q.$$

As before, we identify $\mathrm{Alt}^0(V, \mathbb{K})$ with $\mathbb{K}$ and obtain

$$\lambda\alpha = \lambda \wedge \alpha = \alpha \wedge \lambda$$

for $\lambda \in \mathrm{Alt}^0(V, \mathbb{K}) = \mathbb{K}$ and $\alpha \in \mathrm{Alt}^p(V, \mathbb{K})$.

We take a closer look at the structure of the algebra $(\mathrm{Alt}(V,\mathbb{K}), \wedge)$.

**Lemma B.2.16.** *For* $\alpha \in \mathrm{Alt}^p(V,\mathbb{K})$*,* $\beta \in \mathrm{Alt}^q(V,\mathbb{K})$ *and* $\gamma \in \mathrm{Alt}^r(V,\mathbb{K})$*, we have*

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma).$$

*In particular, the algebra* $(\mathrm{Alt}(V,\mathbb{K}), \wedge)$ *is associative.*

*Proof.* First, we recall from Definition B.2.13 that for any $n$-linear map $\omega\colon V^n \to W$ and $\pi \in S_n$ we have

$$\mathrm{Alt}\big(\omega^\pi\big) = \mathrm{sgn}(\pi)\,\mathrm{Alt}(\omega). \tag{B.3}$$

We identify $S_{p+q}$ in the natural way with the subgroup of $S_{p+q+r}$ fixing the numbers $p+q+1, \ldots, p+q+r$. We thus obtain

$$\begin{aligned}
(\alpha \wedge \beta) \wedge \gamma &= \frac{(p+q+r)!}{(p+q)!r!}\,\mathrm{Alt}\big((\alpha \wedge \beta) \otimes \gamma\big)\\
&= \frac{(p+q+r)!}{p!q!(p+q)!r!} \sum_{\sigma \in S_{p+q}} \mathrm{sgn}(\sigma)\,\mathrm{Alt}\big((\alpha \otimes \beta)^\sigma \otimes \gamma\big)\\
&= \frac{(p+q+r)!}{p!q!(p+q)!r!} \sum_{\sigma \in S_{p+q}} \mathrm{sgn}(\sigma)\,\mathrm{Alt}\big((\alpha \otimes \beta \otimes \gamma)^\sigma\big)\\
&\overset{(B.3)}{=} \frac{(p+q+r)!}{p!q!(p+q)!r!} \sum_{\sigma \in S_{p+q}} \mathrm{Alt}(\alpha \otimes \beta \otimes \gamma)\\
&= \frac{(p+q+r)!}{p!q!r!}\,\mathrm{Alt}(\alpha \otimes \beta \otimes \gamma) = \frac{(p+q+r)!}{p!q!r!}\,\mathrm{Alt}\big(\alpha \otimes (\beta \otimes \gamma)\big)\\
&= \cdots = \frac{(p+q+r)!}{p!(q+r)!}\,\mathrm{Alt}\big(\alpha \otimes (\beta \wedge \gamma)\big) = \alpha \wedge (\beta \wedge \gamma).
\end{aligned}$$ □

From the associativity asserted in the preceding lemma, it follows that the multiplication in $\mathrm{Alt}(V,\mathbb{K})$ is associative. We may therefore suppress brackets and define

$$\omega_1 \wedge \cdots \wedge \omega_n := \big(\cdots\big((\omega_1 \wedge \omega_2) \wedge \omega_3\big) \cdots \wedge \omega_n\big).$$

**Remark B.2.17.** (a) From the calculation in the preceding proof, we know that for three elements $\alpha_i \in \mathrm{Alt}^{p_i}(V,\mathbb{K})$ the triple product in the associative algebra $\mathrm{Alt}(V,\mathbb{K})$ satisfies

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 = \frac{(p_1+p_2+p_3)!}{p_1!p_2!p_3!}\,\mathrm{Alt}(\alpha_1 \otimes \alpha_2 \otimes \alpha_3).$$

Inductively this leads for $n$ elements $\alpha_i \in \mathrm{Alt}^{p_i}(V,\mathbb{K})$ to

$$\alpha_1 \wedge \cdots \wedge \alpha_n = \frac{(p_1+\cdots+p_n)!}{p_1!\cdots p_n!}\,\mathrm{Alt}(\alpha_1 \otimes \cdots \otimes \alpha_n)$$

(Exercise B.2.2).

(b) For $\alpha_i \in \mathrm{Alt}^1(V, \mathbb{K}) \cong V^*$, we in particular obtain

$$\begin{aligned}&(\alpha_1 \wedge \cdots \wedge \alpha_n)(v_1, \ldots, v_n) = n! \,\mathrm{Alt}(\alpha_1 \otimes \cdots \otimes \alpha_n)(v_1, \ldots, v_n)\\ = &\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \alpha_1(v_{\sigma(1)}) \cdots \alpha_n(v_{\sigma(n)}) = \det\big(\alpha_i(v_j)\big).\end{aligned}$$

**Proposition B.2.18.** *The exterior algebra is graded commutative, i.e., for* $\alpha \in \mathrm{Alt}^p(V, \mathbb{K})$ *and* $\beta \in \mathrm{Alt}^q(V, \mathbb{K})$ *we have*

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha.$$

*Proof.* Let $\sigma \in S_{p+q}$ denote the permutation defined by

$$\sigma(i) := \begin{cases} i + p & \text{for } 1 \leq i \leq q, \\ i - q & \text{for } q + 1 \leq i \leq p + q \end{cases}$$

which moves the first $q$ elements to the last $q$ positions. Then we have

$$\begin{aligned}(\beta \otimes \alpha)^\sigma(v_1, \ldots, v_{p+q}) &= (\beta \otimes \alpha)(v_{\sigma(1)}, \ldots, v_{\sigma(p+q)})\\ = \beta(v_{p+1}, \ldots, v_{p+q}) \alpha(v_1, \ldots, v_p) &= (\alpha \otimes \beta)(v_1, \ldots, v_{p+q}).\end{aligned}$$

This leads to

$$\begin{aligned}\alpha \wedge \beta &= \frac{(p+q)!}{p!q!} \mathrm{Alt}(\alpha \otimes \beta) = \frac{(p+q)!}{p!q!} \mathrm{Alt}\big((\beta \otimes \alpha)^\sigma\big)\\ &= \mathrm{sgn}(\sigma) \frac{(p+q)!}{p!q!} \mathrm{Alt}(\beta \otimes \alpha) = \mathrm{sgn}(\sigma)(\beta \wedge \alpha).\end{aligned}$$

On the other hand, $\mathrm{sgn}(\sigma) = (-1)^F$, where

$$\begin{aligned}F &:= \big|\big\{(i,j) \in \{1, \ldots, p+q\} \colon i < j, \sigma(j) < \sigma(i)\big\}\big|\\ &= \big|\big\{(i,j) \in \{1, \ldots, p+q\} \colon i \leq q, j > q\big\}\big| = pq\end{aligned}$$

is the number of inversions of $\sigma$. Putting everything together, the lemma follows. □

**Corollary B.2.19.** *If* $\alpha \in \mathrm{Alt}^p(V, \mathbb{K})$ *and* $p$ *is odd, then* $\alpha \wedge \alpha = 0$.

*Proof.* In view of Proposition B.2.18, we have $\alpha \wedge \alpha = (-1)^{p^2} \alpha \wedge \alpha = -\alpha \wedge \alpha$, which leads to $\alpha \wedge \alpha = 0$. □

**Corollary B.2.20.** *If* $\alpha_1, \ldots, \alpha_k \in V^* = \mathrm{Alt}^1(V, \mathbb{K})$ *and* $\beta_j = \sum_{i=1}^k a_{ij} \alpha_i$, *then*

$$\beta_1 \wedge \cdots \wedge \beta_k = \det(A) \cdot \alpha_1 \wedge \cdots \wedge \alpha_k \quad \textit{for} \quad A = (a_{ij}) \in M_k(\mathbb{K}).$$

*Proof.* The $k$-fold multiplication map

$$\Phi\colon (V^*)^k \to \mathrm{Alt}^k(V,\mathbb{K}), \quad (\gamma_1,\ldots,\gamma_k) \mapsto \gamma_1 \wedge \cdots \wedge \gamma_k$$

is alternating by Proposition B.2.18 because $S_k$ is generated by transpositions. Hence the assertion follows from Proposition B.2.11. □

**Corollary B.2.21.** *If* $\dim V = n$, $b_1,\ldots,b_n$ *is a basis for* $V$, *and* $b_1^*,\ldots,b_n^*$ *the dual basis for* $V^*$, *then the products*

$$b_I^* := b_{i_1}^* \wedge \cdots \wedge b_{i_k}^*, \quad I = (i_1,\ldots,i_k), \quad 1 \leq i_1 < \cdots < i_k \leq n,$$

*form a basis for* $\mathrm{Alt}^k(V,\mathbb{K})$.

*Proof.* For $J = (j_1,\ldots,j_k)$ with $j_1 < \cdots < j_k$, we get with Remark B.2.17(b)

$$b_I^*(b_{j_1},\ldots,b_{j_k}) = \det\big(b_{i_l}^*(b_{j_m})_{l,m=1,\ldots,k}\big) = \begin{cases} 1 & \text{for } I = J, \\ 0 & \text{for } I \neq J. \end{cases}$$

If follows in particular that the elements $b_I$ are linearly independent, and since $\dim \mathrm{Alt}^k(V,\mathbb{K}) = \binom{n}{k}$ (Corollary B.2.12), the assertion follows. □

**Remark B.2.22.** (a) From Corollary B.2.12 it follows in particular that

$$\dim \mathrm{Alt}(V,\mathbb{K}) = \sum_{k=0}^{\dim V} \binom{\dim V}{k} = 2^{\dim V}$$

if $V$ is finite-dimensional.

(b) If $V$ is infinite-dimensional, then it has an infinite basis $(b_i)_{i\in I}$ (this requires Zorn's Lemma). In addition, the set $I$ carries a linear order $\leq$ (this requires the Well Ordering Theorem), and for each $k$-element subset $J = \{j_1,\ldots,j_k\} \subseteq I$ with $j_1 < \cdots < j_k$, we thus obtain an element

$$b_J^* := b_{j_1}^* \wedge \cdots \wedge b_{j_k}^*.$$

Applying the $b_J^*$ to $k$-tuples of basis elements shows that they are linearly independent, so that for each $k > 0$ the space $\mathrm{Alt}^k(V,\mathbb{K})$ is infinite-dimensional.

**Definition B.2.23.** Let $\varphi\colon V_1 \to V_2$ be a linear map and $W$ a vector space. For each $p$-linear map $\alpha\colon V_2^p \to W$ we define its *pull-back by* $\varphi$:

$$(\varphi^*\alpha)(v_1,\ldots,v_p) := \alpha\big(\varphi(v_1),\ldots,\varphi(v_p)\big)$$

for $v_1,\ldots,v_p \in V_1$. It is clear that $\varphi^*\alpha$ is a $p$-linear map $V_1^p \to W$ and that $\varphi^*\alpha$ is alternating if $\alpha$ has this property.

**Remark B.2.24.** If $\varphi\colon V_1 \to V_2$ and $\psi\colon V_2 \to V_3$ are linear maps and $\alpha\colon V_3^p \to W$ is $p$-linear, then

$$(\psi \circ \varphi)^*\alpha = \varphi^*(\psi^*\alpha).$$

**Proposition B.2.25.** *Let $\varphi\colon V_1 \to V_2$ be a linear map. Then the pull-back map*

$$\varphi^*\colon \operatorname{Alt}(V_2,\mathbb{K}) \to \operatorname{Alt}(V_1,\mathbb{K})$$

*is a homomorphism of algebras with unit.*

*Proof.* For $\alpha \in \operatorname{Alt}^p(V_2,\mathbb{K})$ and $\beta \in \operatorname{Alt}^q(V_2,\mathbb{K})$ we have

$$\begin{aligned}\varphi^*(\alpha\wedge\beta) &= \frac{(p+q)!}{p!q!}\varphi^*\big(\operatorname{Alt}(\alpha\otimes\beta)\big) = \frac{(p+q)!}{p!q!}\operatorname{Alt}\big(\varphi^*(\alpha\otimes\beta)\big)\\ &= \frac{(p+q)!}{p!q!}\operatorname{Alt}(\varphi^*\alpha\otimes\varphi^*\beta) = \varphi^*\alpha\wedge\varphi^*\beta.\end{aligned}$$

□

**Remark B.2.26.** The results in this section remain valid for alternating forms with values in any commutative algebra $A$. Then

$$\operatorname{Alt}(V,A) = \bigoplus_{p\in\mathbb{N}_0}\operatorname{Alt}^p(V,A)$$

also carries an associative, graded commutative algebra structure defined by

$$\alpha\wedge\beta := \frac{(p+q)!}{p!q!}\operatorname{Alt}(\alpha\otimes\beta),$$

where

$$(\alpha\otimes\beta)(v_1,\ldots,v_{p+q}) := \alpha(v_1,\ldots,v_p)\cdot\beta(v_{p+1},\ldots,v_{p+q})$$

for $\alpha\in\operatorname{Alt}^p(V,A)$, $\beta\in\operatorname{Alt}^q(V,A)$.

This applies in particular to the 2-dimensional real algebra $A=\mathbb{C}$.

### B.2.4 Orientations on Vector Spaces

Throughout this subsection, all vector spaces are real and finite-dimensional.

**Definition B.2.27.** (a) Let $V$ be an $n$-dimensional real vector space. Then $\operatorname{Alt}^n(V,\mathbb{R})$ is one-dimensional. Any nonzero element $\mu$ of this space is called a *volume form* on $V$.

(b) We define an equivalence relation on the set $\operatorname{Alt}^n(V,\mathbb{R})\setminus\{0\}$ of volume forms setting $\mu_1\sim\mu_2$ if there exists a $\lambda>0$ with $\mu_2=\lambda\mu_1$. We write $[\mu]$ for the equivalence class of $\mu$. These equivalence classes are called *orientations of* $V$. If $O=[\mu]$ is an orientation, then we write $-O:=[-\mu]$ for the *opposite orientation.*

An *oriented vector space* is a pair $(V,O)$, where $V$ is a finite-dimensional real vector space and $O=[\mu]$ an orientation on $V$.

(c) An ordered basis $(b_1,\ldots,b_n)$ for $(V,[\mu])$ is said to be *positively oriented* if $\mu(b_1,\ldots,b_n)>0$, and *negatively oriented* otherwise.

(d) An invertible linear map $\varphi\colon (V,[\mu_V]) \to (W,[\mu_W])$ between oriented vector spaces is called *orientation preserving* if $[\varphi^*\mu_W] = [\mu_V]$. Otherwise $\varphi$ is called *orientation reversing*.

(e) We endow $\mathbb{R}^n$ with the canonical orientation, defined by the determinant form

$$\mu(x_1,\ldots,x_n) := \det(x_1,\ldots,x_n) = \det(x_{ij})_{i,j=1,\ldots,n}.$$

**Remark B.2.28.** (a) If $B := (b_1,\ldots,b_n)$ is a basis for $V$, then Corollary B.2.21 implies that we obtain a volume form by

$$\mu_B := b_1^* \wedge \cdots \wedge b_n^*,$$

and since $\mu_B(b_1,\ldots,b_n) = \det(b_i^*(b_j)) = \det(\mathbf{1}) = 1$, the basis $B$ is positively oriented with respect to the orientation $[\mu_B]$. We call $[\mu_B]$ the *orientation defined by the basis* $B$.

(b) The terminology "volume form" corresponds to the interpretation of $\mu(v_1,\ldots,v_n)$ as an "oriented" volume of the flat

$$[0,1]v_1 + \cdots + [0,1]v_n$$

generated by the $n$-tuple $(v_1,\ldots,v_n)$. Note that $\mu_B(v_1,\ldots,v_n) = \det(b_i^*(v_j))$.

**Lemma B.2.29.** *If $\mu_V$ is a volume form on $V$ and $\varphi \in \mathrm{End}(V)$, then*

$$\varphi^*\mu_V = \det(\varphi)\mu_V.$$

*In particular, $\varphi$ is orientation preserving if and only if $\det(\varphi) > 0$.*

*Proof.* Let $B = (b_1,\ldots,b_n)$ be a positively oriented basis for $V$ and $A = (a_{ij})$ the matrix of $\varphi$ with respect to $B$, i.e., $\varphi(b_j) = \sum_i a_{ij}b_i$. Then

$$\begin{aligned}(\varphi^*\mu_V)(b_1,\ldots,b_n) &= \mu_V\big(\varphi(b_1),\ldots,\varphi(b_n)\big) = \det(A)\mu_V(b_1,\ldots,b_n)\\ &= \det(\varphi)\mu_V(b_1,\ldots,b_n)\end{aligned}$$

follows from Proposition B.2.11(i), and this implies the assertion. □

**Example B.2.30.** (a) If $V = \mathbb{R}^2$ and $\varphi \in \mathrm{GL}(V)$ is a reflection in a line, then $\det(\varphi) < 0$ implies that $\varphi$ is orientation reversing. The same holds for the reflection in a hyperplane in $\mathbb{R}^n$.

(b) Rotations of $\mathbb{R}^3$ around an axis are orientation preserving.

(c) In $V = \mathbb{C}$, considered as a real vector space, we have the natural basis $B = (1, i)$. A corresponding volume form is given by

$$\mu(z,w) := \mathrm{Im}(\overline{z}w) = \mathrm{Re}\, z\, \mathrm{Im}\, w - \mathrm{Im}\, z\, \mathrm{Re}\, w$$

because $\mu(1,i) = \mathrm{Im}(i) = 1 > 0$.

Each complex linear map $\varphi\colon \mathbb{C} \to \mathbb{C}$ is given by multiplication with some complex number $x + iy$, and the corresponding matrix with respect to $B$ is

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix},$$

so that $\det(\varphi) = x^2 + y^2 > 0$ whenever $\varphi \neq 0$. We conclude that each nonzero complex linear map $V \to V$ is orientation preserving.

**Proposition B.2.31 (Real vs. complex determinant).** *Let $V$ be a complex vector space, viewed as a real one, and $\varphi\colon V \to V$ a complex linear map. Then $\det_{\mathbb{R}}(\varphi) = |\det_{\mathbb{C}}(\varphi)|^2$. In particular, each invertible complex linear map is orientation preserving.*

*Proof.* Let $B_{\mathbb{C}} = (b_1, \ldots, b_n)$ be a complex basis for $V$, so that

$$B = (b_1, \ldots, b_n, ib_1, \ldots, ib_n)$$

is a real basis for $V$. Further, let $b_j^* \in \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, $j = 1, \ldots, n$, denote the complex dual basis. In $\mathrm{Alt}^{2n}(V, \mathbb{C}) \cong \mathbb{C}$, we then consider the element

$$\mu := b_1^* \wedge \cdots \wedge b_n^* \wedge \overline{b_1^*} \wedge \cdots \wedge \overline{b_n^*}.$$

That $\mu$ is nonzero follows from

$$\mu(b_1, \ldots, b_n, ib_1, \ldots, ib_n) = \det \begin{pmatrix} \mathbf{1}_n & i\mathbf{1}_n \\ \mathbf{1}_n & -i\mathbf{1}_n \end{pmatrix} = \det \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}^n = (-2i)^n \neq 0.$$

If $A = (a_{ij}) \in M_n(\mathbb{C})$ is the matrix of $\varphi$ with respect to $B_{\mathbb{C}}$, then we have

$$\varphi^* b_j^* = \sum_{k=1}^n a_{jk} b_k^* \quad \text{and} \quad \varphi^* \overline{b_j^*} = \sum_{k=1}^n \overline{a_{jk}}\, \overline{b_k^*}.$$

As in the proof of Lemma B.2.29, we now see that

$$\varphi^*(b_1^* \wedge \cdots \wedge b_n^*) = \det_{\mathbb{C}}(A) \cdot b_1^* \wedge \cdots \wedge b_n^*$$

and

$$\varphi^*(\overline{b_1^*} \wedge \cdots \wedge \overline{b_n^*}) = \overline{\det_{\mathbb{C}}(A)} \cdot \overline{b_1^*} \wedge \cdots \wedge \overline{b_n^*},$$

which leads with Proposition B.2.25 and Lemma B.2.29 to

$$\det_{\mathbb{R}}(\varphi)\mu = \varphi^*\mu = \det_{\mathbb{C}}(A)\overline{\det_{\mathbb{C}}(A)}\mu = |\det_{\mathbb{C}}(A)|^2\mu = |\det_{\mathbb{C}}(\varphi)|^2\mu. \qquad \square$$

### B.2.5 Exercises for Section B.2

**Exercise B.2.1.** Fix $n \in \mathbb{N}$. Show that:

(1) For each matrix $A \in M_n(\mathbb{K})$, we obtain a bilinear map

$$\beta_A\colon \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}, \quad \beta_A(x, y) := \sum_{i,j=1}^n a_{ij} x_i y_j.$$

(2) $A$ can be recovered from $\beta_A$ via $a_{ij} = \beta_A(e_i, e_j)$.
(3) Each bilinear map $\beta\colon \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ is of the form $\beta = \beta_A$ for a unique matrix $A \in M_n(\mathbb{R})$.
(4) $\beta_{A^\top}(x, y) = \beta_A(y, x)$.
(5) $\beta_A$ is skew-symmetric if and only if $A$ is such.

**Exercise B.2.2.** Show that for $\alpha_i \in \mathrm{Alt}^{p_i}(V, \mathbb{K}), i = 1, \ldots, n$, the exterior product satisfies

$$\alpha_1 \wedge \cdots \wedge \alpha_n = \frac{(p_1 + \cdots + p_n)!}{p_1! \cdots p_n!} \mathrm{Alt}(\alpha_1 \otimes \cdots \otimes \alpha_n).$$

**Exercise B.2.3.** Show that $(\mathrm{Alt}(V, \mathbb{K}), \wedge)$ is an exterior algebra over $V^*$.

## B.3 Clifford Algebras, Pin and Spin Groups

A *quadratic space* is a pair $(V, \beta)$, where $V$ is a vector space and $\beta\colon V \times V \to \mathbb{K}$ is a symmetric bilinear form. We write $q(v) := \beta(v, v)$ for the corresponding quadratic form. In this section, $\mathbb{K}$ can be any field of characteristic $\neq 2$.

**Definition B.3.1.** A *Clifford algebra* for $(V, \beta)$ is a pair $(C, \iota)$ of a unital associative algebra $C$ and a linear map $\iota\colon V \to C$ satisfying

$$\iota(x)\iota(y) + \iota(y)\iota(x) = 2\beta(x, y)\mathbf{1} \quad \text{for} \quad x, y \in V \tag{B.4}$$

and the universal property that for each linear map $f\colon V \to A$, $A$ a unital algebra, satisfying

$$f(x)f(y) + f(y)f(x) = 2\beta(x, y)\mathbf{1} \quad \text{for} \quad x, y \in V, \tag{B.5}$$

there exists a unique algebra homomorphism $\widetilde{f}\colon C \to A$ with $\widetilde{f} \circ \iota = f$.

**Remark B.3.2.** If $V$ is a vector space over a field $\mathbb{K}$ of characteristic $\neq 2$, then $\beta$ can be reconstructed from $q$ via

$$\beta(x, y) = \frac{1}{2}\big(q(x + y) - q(x) - q(y)\big).$$

Accordingly, the relation (B.5) is equivalent to

$$f(x)^2 = q(x)\mathbf{1} \quad \text{for all} \quad x \in V. \tag{B.6}$$

**Proposition B.3.3.** *For each quadratic space $(V, \beta)$, there exists a Clifford algebra $(C, \iota)$. It is unique up to isomorphism in the sense that for any other Clifford algebra $(C', \iota')$ of $(V, \beta)$, there exists an algebra isomorphism $\varphi\colon C \to C'$ with $\varphi \circ \iota = \iota'$.*

*Proof.* (**Uniqueness**) As usual, the uniqueness follows from the universal property. If $(C, \iota)$ and $(C', \iota')$ are Clifford algebras for $(V, \beta)$, there exist uniquely determined unital algebra morphisms $f : C \to C'$ with $f \circ \iota = \iota'$ and $f' : C' \to C$ with $f' \circ \iota' = \iota$. Then $f' \circ f : C \to C$ is an algebra endomorphism with $(f' \circ f) \circ \iota = \iota$, so that the uniqueness in the universal property of $(C, \iota)$ implies that $f' \circ f = \mathrm{id}_C$. We likewise obtain $f \circ f' = \mathrm{id}_{C'}$, so that $f$ is an isomorphism of unital algebras.

(**Existence**) Let $\mathcal{T}(V)$ be the tensor algebra of $V$ (Definition B.1.7) and consider the subset

$$M := \{x \otimes x - \beta(x, x) : x \in V\}.$$

We write $J_M$ for the ideal generated by $M$. Then

$$C := \mathcal{T}(V)/J_M$$

is a unital associative algebra and

$$\iota : V \to C, \quad \iota(v) := v + J_M$$

is a linear map satisfying

$$\iota(x)^2 = x \otimes x + J_M = \beta(x, x)\mathbf{1}.$$

To verify the universal property, let $f : V \to A$ be a linear map into the unital algebra $A$, satisfying (B.5). In view of the universal property of $\mathcal{T}(V)$ (Lemma B.1.8), there exists a unital algebra homomorphism $\widehat{f} : \mathcal{T}(V) \to A$ with $\widehat{f}(x) = f(x)$ for all $x \in V$. Then $M \subseteq \ker \widehat{f}$ by (B.5), and since $\ker \widehat{f}$ is an ideal of $\mathcal{T}(V)$, we also have $J_M \subseteq \ker \widehat{f}$, so that $\widehat{f}$ factors through an algebra homomorphism

$$\widetilde{f} : C \to A \quad \text{with} \quad \widetilde{f} \circ \iota = f.$$

To see that $\widetilde{f}$ is unique, it suffices to note that $\iota(V)$ and $\mathbf{1}$ generate $C$ as an associative algebra because $V$ and $\mathbf{1}$ generate $\mathcal{T}(V)$. $\square$

**Definition B.3.4.** Justified by the existence and uniqueness assertion of the preceding proposition, we write $(\mathrm{Cl}(V, \beta), \iota)$ for a Clifford algebra of $(V, \beta)$.

For $\mathbb{K} = \mathbb{R}$, we consider on $\mathbb{R}^{p+q}$ the nondegenerate symmetric bilinear forms

$$\beta_{p,q}(x, y) := -\sum_{j=1}^{p} x_j y_j + \sum_{j=p+1}^{p+q} x_j y_j$$

and write

$$C_{p,q} := \mathrm{Cl}(\mathbb{R}^{p+q}, \beta_{p,q})$$

for the corresponding real Clifford algebras. For the negative definite form $\beta_n := \beta_{n,0}$, we also put $C_n := C_{n,0}$.

**Examples B.3.5.** (a) If $\beta = 0$, then $\mathrm{Cl}(V,\beta) \cong \Lambda(V)$ is the exterior algebra, over $V$ (Lemma B.2.10).

(b) We discuss the Clifford algebras associated to one-dimensional vector spaces. Let $0 \neq e \in V$ be a basis element. Then $\mathrm{Cl}(V,\beta)$ is generated as an associative algebra by $\mathbf{1}$ and $x := \iota(e)$, which satisfies $x^2 = a := \beta(e,e)$, so that $\mathrm{Cl}(V,\beta) \cong \mathbb{K}[x]/(x^2 - a)$. We also write $\mathrm{Cl}(\mathbb{K}, a)$ for this Clifford algebra.

For $\beta = 0$, we thus obtain the ring $\mathbb{K}[\varepsilon] = \mathbb{K}\mathbf{1} + \mathbb{K}\varepsilon$ of dual numbers over $\mathbb{K}$, defined by the relation $\varepsilon^2 = 0$.

For $\beta \neq 0$, two cases occur. If $a$ is a square, i.e., $a = b^2$ for some $b \in \mathbb{K}$, then $f := b^{-1}e$ is a basis element of $V$ satisfying $\beta(f,f) = 1$, so that $\mathrm{Cl}(V,\beta) \cong \mathbb{K}[x]/(x^2 - 1)$. For $c_1 := \frac{1}{2}(\mathbf{1} + x)$ and $c_2 := \frac{1}{2}(\mathbf{1} - x)$ we then obtain two idempotents in $\mathrm{Cl}(V,\beta)$ satisfying $c_1 c_2 = 0$ and $c_1 + c_2 = \mathbf{1}$, which leads to $\mathrm{Cl}(V,\beta) \cong \mathbb{K} \oplus \mathbb{K}$, as an associative algebra.

If $a$ is not a square in $\mathbb{K}$, then $\mathrm{Cl}(V,\beta) \cong \mathbb{K}[x]/(x^2 - a)$ is a field. In fact, if $\lambda_g(h) = gh$ denotes left multiplication in this algebra, then the norm function

$$N\colon \mathrm{Cl}(V,\beta) \to \mathbb{K}, \quad r + sx \mapsto r^2 - as^2 = \det(\lambda_{r+sx})$$

is multiplicative and nonzero on nonzero elements, which easily leads to

$$(r + sx)^{-1} = N(r + sx)^{-1}(r - sx) \quad \text{for} \quad \alpha, \beta \neq 0.$$

(c) For $\mathbb{K} = \mathbb{R}$ and $n = 1$, we have

$$C_1 \cong \mathbb{R}[x]/\big(x^2 + 1\big) \cong \mathbb{C}$$

(cf. Definition B.3.4), and $C_{0,1} \cong \mathbb{R}[x]/(x^2 - 1) \cong \mathbb{R} \oplus \mathbb{R}$.

The key to a systematic understanding of Clifford algebras is the understanding of how the Clifford algebra of an orthogonal direct sum $(V_1 \oplus V_2, \beta_1 \oplus \beta_2)$ of two quadratic spaces can be described in terms of the Clifford algebras $\mathrm{Cl}(V_1, \beta_1)$ and $\mathrm{Cl}(V_2, \beta_1)$. First, we have to observe that Clifford algebras are 2-graded:

**Lemma B.3.6.** *There exists a unique involutive automorphism* $\omega$ *of* $\mathrm{Cl}(V,\beta)$ *with* $\omega \circ \iota = -\iota$.

*Proof.* The map $-\iota\colon V \to \mathrm{Cl}(V,\beta)$ also satisfies

$$\big(-\iota(x)\big)^2 = \iota(x)^2 = \beta(x,x)\mathbf{1} \quad \text{for} \quad x \in V.$$

Therefore, the universal property of $\mathrm{Cl}(V,\beta)$ implies the existence of a homomorphism $\omega\colon \mathrm{Cl}(V,\beta) \to \mathrm{Cl}(V,\beta)$ of unital algebras with $\omega \circ \iota = -\iota$. Then $\omega^2 \circ \iota = \iota$ and the uniqueness part in the universal property imply that $\omega^2 = \mathrm{id}_{\mathrm{Cl}(V,\beta)}$. □

**Definition B.3.7.** (a) Let $\mathbb{Z}/2 = \{\overline{0}, \overline{1}\}$, considered as an abelian group. The eigenspaces

$$\mathrm{Cl}(V,\beta)_{\overline{0}} := \ker(\omega - \mathbf{1}) \quad \text{and} \quad \mathrm{Cl}(V,\beta)_{\overline{1}} := \ker(\omega + \mathbf{1})$$

of the involution $\omega$ define a 2-grading on $\mathrm{Cl}(V,\beta)$, i.e.,

$$\mathrm{Cl}(V,\beta) = \mathrm{Cl}(V,\beta)_{\overline{0}} \oplus \mathrm{Cl}(V,\beta)_{\overline{1}}$$

and

$$\mathrm{Cl}(V,\beta)_a \,\mathrm{Cl}(V,\beta)_b \subseteq \mathrm{Cl}(V,\beta)_{a+b}, \quad a,b \in \mathbb{Z}/2.$$

The involution $\omega$ is also called the *grading automorphism* of $\mathrm{Cl}(V,\beta)$.

Since $\mathrm{Cl}(V,\beta)$ is generated by $\iota(V)$ as a unital algebra, it is spanned by elements of the form[1] $\iota(v_1)\cdots\iota(v_k)$. On these elements we have

$$\omega\big(\iota(v_1)\cdots\iota(v_k)\big) = (-1)^k \iota(v_1)\cdots\iota(v_k),$$

so that

$$\mathrm{Cl}(V,\beta)_{\overline{0}} = \mathrm{span}\big\{\iota(v_1)\cdots\iota(v_k)\colon v_i \in V, k \in 2\mathbb{N}_0\big\}$$

and

$$\mathrm{Cl}(V,\beta)_{\overline{1}} = \mathrm{span}\big\{\iota(v_1)\cdots\iota(v_k)\colon v_i \in V, k \in 2\mathbb{N}_0 + 1\big\}.$$

(b) If $A = A_{\overline{0}} \oplus A_{\overline{1}}$ and $B = B_{\overline{0}} \oplus B_{\overline{1}}$ are $\mathbb{Z}/2$-graded algebras, then we define their *graded tensor product* as the vector space $C := A \otimes B$, endowed with the multiplication defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg(b)\deg(a')} aa' \otimes bb'$$

for homogeneous elements $a, a' \in A$ and $b, b' \in B$. It is easy to verify that we thus obtain an associative $\mathbb{Z}/2$-graded algebra, denoted $A \widehat{\otimes} B$.

**Proposition B.3.8.** *For an orthogonal direct sum* $(V,\beta) = (V_1,\beta_1) \oplus (V_2,\beta_2)$ *of quadratic spaces, we have*

$$\mathrm{Cl}(V_1 \oplus V_2, \beta_1 \oplus \beta_2) \cong \mathrm{Cl}(V_1,\beta_1) \widehat{\otimes} \mathrm{Cl}(V_2,\beta_2).$$

*Proof.* In view of the uniqueness assertion of Proposition B.3.3, it suffices to show that the pair $(C,\iota)$ with $C := \mathrm{Cl}(V_1,\beta_1) \widehat{\otimes} \mathrm{Cl}(V_2,\beta_2)$ and

$$\iota\colon V_1 \oplus V_2 \to C, \quad v_1 \oplus v_2 \mapsto \iota_1(v_1) \otimes \mathbf{1} + \mathbf{1} \otimes \iota_2(v_2)$$

defines a Clifford algebra for $(V_1 \oplus V_2, \beta_1 \oplus \beta_2)$. First, we observe that

$$\begin{aligned}
&\big(\iota_1(v_1) \otimes \mathbf{1} + \mathbf{1} \otimes \iota_2(v_2)\big)^2 \\
&= \iota_1(v_1)^2 \otimes \mathbf{1} + \iota_1(v_1) \otimes \iota_2(v_2) + \big(\mathbf{1} \otimes \iota_2(v_2)\big)\big(\iota_1(v_1) \otimes \mathbf{1}\big) + \mathbf{1} \otimes \iota_2(v_2)^2 \\
&= \beta_1(v_1,v_1)\mathbf{1} \otimes \mathbf{1} + \iota_1(v_1) \otimes \iota_2(v_2) - \iota_1(v_1) \otimes \iota_2(v_2) + \beta_2(v_2,v_2)\mathbf{1} \otimes \mathbf{1} \\
&= \big(\beta_1(v_1,v_1) + \beta_2(v_2,v_2)\big)\mathbf{1} \otimes \mathbf{1}.
\end{aligned}$$

[1] By convention, the empty product corresponding to $k = 0$ is $\mathbf{1}$.

To verify the universal property, let $f\colon V_1 \oplus V_2 \to A$ be a linear map into a unital associative algebra satisfying

$$f(v_1 \oplus v_2)^2 = \big(\beta_1(v_1, v_1) + \beta_2(v_2, v_2)\big)\mathbf{1}.$$

Then the universal property of the Clifford algebras $(\mathrm{Cl}(V_j, \beta_j), \iota_j)$ implies the existence of unique algebra homomorphisms $\widetilde{f}_j\colon \mathrm{Cl}(V_j, \beta_j) \to A$ with $\widetilde{f}_j \circ \iota_j = \iota|_{V_j}$. We combine these two maps to a linear map

$$\widetilde{f}\colon C \to A, \quad c_1 \otimes c_2 \mapsto \widetilde{f}_1(c_1)\widetilde{f}_2(c_2).$$

For $v_1 \in V_1$ and $v_2 \in V_2$, we have

$$f(v_1, 0)f(0, v_2) = -f(0, v_2)f(v_1, 0).$$

From Definition B.3.7, we therefore derive that

$$\widetilde{f}_1(a_1)\widetilde{f}_2(a_2) = (-1)^{\deg(a_1)\deg(a_2)}\widetilde{f}_2(a_2)\widetilde{f}_1(a_1)$$

holds for homogeneous elements $a_j \in \mathrm{Cl}(V_j, \beta_j)$. For homogeneous elements $a_j, a_j' \in \mathrm{Cl}(V_j, \beta_j)$, we thus obtain

$$\begin{aligned}
\widetilde{f}\big((a_1 \otimes a_2)(a_1' \otimes a_2')\big) &= (-1)^{\deg(a_2)\deg(a_1')}\widetilde{f}(a_1a_1' \otimes a_2a_2') \\
&= (-1)^{\deg(a_2)\deg(a_1')}\widetilde{f}_1(a_1a_1')\widetilde{f}_2(a_2a_2') \\
&= (-1)^{\deg(a_2)\deg(a_1')}\widetilde{f}_1(a_1)\widetilde{f}_1(a_1')\widetilde{f}_2(a_2)\widetilde{f}_2(a_2') \\
&= \widetilde{f}_1(a_1)\widetilde{f}_2(a_2)\widetilde{f}_1(a_1')\widetilde{f}_2(a_2') = \widetilde{f}(a_1 \otimes a_2)\widetilde{f}(a_1' \otimes a_2'),
\end{aligned}$$

and therefore $\widetilde{f}$ is an algebra homomorphism. It remains to show that $\widetilde{f}$ is uniquely determined. But this follows from the uniqueness of $\widetilde{f}_1, \widetilde{f}_2$ and the fact that $C$ is generated by $\mathbf{1} \otimes \mathrm{Cl}(V_2, \beta_2)$ and $\mathrm{Cl}(V_1, \beta_1) \otimes \mathbf{1}$. □

The preceding proposition has a number of interesting consequences:

**Corollary B.3.9.** *If $(V, \beta)$ is a quadratic space, then the following assertions hold:*

(i) *If* $\dim V < \infty$*, then* $\dim \mathrm{Cl}(V, \beta) = 2^{\dim V}$.
(ii) *If $v_1, \ldots, v_n \in V$ is an orthogonal basis, then the ordered products*

$$\iota(v_{i_1}) \cdots \iota(v_{i_k}), \quad 1 \leq i_1 < \cdots < i_k \leq n$$

*form a basis for* $\mathrm{Cl}(V, \beta)$.
(iii) *The structure map $\iota\colon V \to \mathrm{Cl}(V, \beta)$ is injective.*

*Proof.* (i) First, we recall that $V$ possesses an orthogonal basis $v_1, \ldots, v_n$. This can be shown by induction on $\dim V$: We may w.l.o.g. assume that $\dim V > 0$ and that $\beta \neq 0$. Then there exists some $v_1 \in V$ with $\beta(v_1, v_1) \neq 0$ and $V = \mathbb{K}v_1 \oplus v_1^\perp$ is an orthogonal decomposition, so that the induction hypothesis implies the existence of an orthogonal basis $v_2, \ldots, v_n$ in $v_1^\perp$.

Now $V = \mathbb{K}v_1 \oplus \cdots \oplus \mathbb{K}v_n$ is an orthogonal decomposition, so that Proposition B.3.8 implies that

$$\mathrm{Cl}(V, \beta) \cong \mathrm{Cl}\big(\mathbb{K}, \beta(v_1, v_1)\big) \widehat{\otimes} \cdots \widehat{\otimes} \mathrm{Cl}\big(\mathbb{K}, \beta(v_n, v_n)\big).$$

Since we know already that Clifford algebras of one-dimensional quadratic spaces are two-dimensional (Examples B.3.5), (i) follows.

(ii) follows immediately from $\mathrm{Cl}(\mathbb{K}v_j, \beta(v_j, v_j)) = \mathbb{K}\mathbf{1} \oplus \mathbb{K}\iota(v_j)$ and the tensor product decomposition in (i).

(iii) Let $0 \neq v \in V$. We have to show that $\iota(v) \neq 0$. We claim that there exists a subspace $V_1 \ni v$ of $V$ with $\dim V_1 \leq 2$ and a subspace $V_2$, such that $V$ is the orthogonal direct sum of $V_1$ and $V_2$. In fact, if $\beta(v, V) = \{0\}$, then we put $V_1 := \mathbb{K}v$ and let $V_2$ be any complementary hyperplane in $V$. If $\beta(V, v) \neq \{0\}$ and $\beta(v, v) = 0$, then we choose some $w \in V$ with $\beta(v, w) \neq 0$ and put $V_1 := \mathbb{K}v + \mathbb{K}w$ and $V_2 := V_1^\perp$. If $\beta(v, v) \neq 0$, we put $V_1 := \mathbb{K}v$ and $V_2 := V_1^\perp$. This proves our claim. Now we apply Proposition B.3.8 to obtain

$$\mathrm{Cl}(V, \beta) \cong \mathrm{Cl}(V_1, \beta_1) \widehat{\otimes} \mathrm{Cl}(V_2, \beta_2),$$

where $\beta_j := \beta|_{V_j \times V_j}$, and $\iota(v) \neq 0$ follows from (ii) because $V_1$ possesses an orthogonal basis. □

### B.3.1 The Clifford Group

In the following, we simply write $v$ for $\iota(v)$, which is justified by the injectivity of $\iota$. We also write $C := \mathrm{Cl}(V, \beta)$ for the Clifford algebra of $(V, \beta)$.

**Definition B.3.10.** The *twisted adjoint action* of the unit group $\mathrm{Cl}(V, \beta)^\times$ on $\mathrm{Cl}(V, \beta)$ is defined by

$$\mathrm{Ad}(a)x := \omega(a)xa^{-1}.$$

This defines a representation because $\omega$ is an algebra isomorphism. We define the *Clifford group* as the stabilizer of the subspace $V$ of $\mathrm{Cl}(V, \beta)$:

$$\Gamma(V, \beta) := \big\{a \in \mathrm{Cl}(V, \beta)^\times \colon \mathrm{Ad}(a)V = V\big\}$$

and thus obtain a representation $\Phi\colon \Gamma(V, \beta) \to \mathrm{GL}(V), \Phi(a) := \mathrm{Ad}(a)|_V$.

**Lemma B.3.11.** *There exists a unique antiautomorphism $x \mapsto x^*$ of $\mathrm{Cl}(V, \beta)$ satisfying $v^* = -v$ for each $v \in V$. This involution commutes with $\omega$.*

*Proof.* Let $\mathrm{Cl}(V,\beta)^{\mathrm{op}}$ be the opposite algebra endowed with the product $x\sharp y := yx$. Then

$$f\colon V \to \mathrm{Cl}(V,\beta)^{\mathrm{op}}, \quad f(v) := -v$$

satisfies $f(v)\sharp f(v) = (-v)^2 = v^2 = \beta(v,v)\mathbf{1}$, so that the universal property of $\mathrm{Cl}(V,\beta)$ implies the existence of a unital algebra homomorphism $\widetilde{f}\colon \mathrm{Cl}(V,\beta) \to \mathrm{Cl}(V,\beta)^{\mathrm{op}}$ with $\widetilde{f}(v) = -v$ for $v \in V$. This means that, considered as a linear endomorphism of $\mathrm{Cl}(V,\beta)$, $\widetilde{f}(xy) = \widetilde{f}(y)\widetilde{f}(x)$ for $x,y \in \mathrm{Cl}(V,\beta)$, and thus $\widetilde{f}^2$ is an algebra endomorphism with $\widetilde{f}^2(v) = v$ for each $v \in V$, hence $\widetilde{f}^2 = \mathrm{id}_{\mathrm{Cl}(V,\beta)}$. Therefore, $x^* := \widetilde{f}(x)$ defines an involutive antiautomorphism of $\mathrm{Cl}(V,\beta)$.

To see that $*$ commutes with $\omega$, we note that $x \mapsto (\omega(x^*))^*$ is an algebra automorphism fixing $V$ pointwise, hence equal to the identity. We conclude that $\omega(x^*) = x^*$ for $x \in \mathrm{Cl}(V,\beta)$. ☐

**Examples B.3.12.** (a) On $C_1 = \mathrm{Cl}(\mathbb{R},-1) \cong \mathbb{C}$ (cf. Definition B.3.4) we have

$$(x+iy)^* = \omega(x+iy) = x - iy.$$

Therefore, the adjoint action is given by

$$\mathrm{Ad}(z)w = \overline{z}wz^{-1} = \overline{z}z^{-1}\cdot w,$$

which immediately shows that the Clifford group is

$$\Gamma(\mathbb{R},-1) = \left\{ z \in \mathbb{C}^\times : \overline{z}z^{-1} = \frac{\overline{z}^2}{|z|^2} \in \mathbb{R} \right\} = \mathbb{R}^\times \mathbf{1} \,\dot\cup\, \mathbb{R}^\times i.$$

(b) The four-dimensional Clifford algebra $C_2$ has the linear basis

$$\mathbf{1}, \quad I := e_1, \quad J := e_2, \quad \text{and} \quad K := e_1e_2,$$

satisfying

$$I^2 = J^2 = K^2 = -\mathbf{1} \quad \text{and} \quad K = IJ = -JI,$$

so that $C_2 \cong \mathbb{H}$ as associative algebras. Here $V = \mathbb{R}I + \mathbb{R}J$ implies that the involution $*$ satisfies

$$(\alpha\mathbf{1} + \beta I + \gamma J + \delta K)^* = \alpha\mathbf{1} - \beta I - \gamma J - \delta K,$$

which is the canonical involution on $\mathbb{H}$.

As $I$ and $J$ are odd and $\mathbf{1}, K$ are even elements of $\mathbb{H}$, it follows from $KIK^{-1} = -I$ and $KJK^{-1} = -J$ that

$$\omega(z) = KzK^{-1} \quad \text{for} \quad z \in \mathbb{H}.$$

Therefore,

$$\mathrm{Ad}(g)v = \omega(g)vg^{-1} = KgK^{-1}vg^{-1}.$$

As $KV = K\,\mathrm{span}\{I,J\} = \mathrm{span}\{I,J\} = V$, the condition $g \in \Gamma(V,\beta)$ is equivalent to $gVg^{-1} \subseteq V$. We write $g = \|g\| \cdot u$ with $\|u\| = 1$, where the norm is the one on the quaternions. Observing that conjugation with $u$ preserves the natural scalar product on $\mathbb{H}$, we derive from $V^\perp = \mathbb{R}\mathbf{1} + \mathbb{R}K$ that $c_u(V) = V$ is equivalent to $c_u(\mathbb{R}\mathbf{1} + \mathbb{R}K) = \mathbb{R}\mathbf{1} + \mathbb{R}K$. In the subalgebra $\mathbb{R}\mathbf{1} + \mathbb{R}K$, the identity is fixed by $c_u$ and $c_u(K)^2 = -\mathbf{1}$. Hence $c_u(V) = V$ is equivalent to $c_u(K) \in \pm K$, i.e., $uKu^{-1}K^{-1} = \pm\mathbf{1}$, which can also be written as $KuK^{-1} = \pm u$. Therefore,

$$\Gamma(V,\beta) = \mathbb{R}^\times \cdot \big(\{\alpha\mathbf{1} + \delta K \colon \alpha^2 + \delta^2 = 1\} \dot\cup \{\beta I + \gamma J \colon \beta^2 + \gamma^2 = 1\}\big).$$

(c) If $\dim V = 1$ and $\mathrm{Cl}(V,\beta) = \mathrm{Cl}(\mathbb{K},a) = \mathbb{K}\mathbf{1} + \mathbb{K}x$ with $x^2 = a$, then $\omega(x) = -x = x^*$. For $a \neq 0$ and $g \in \mathrm{Cl}(V,\beta)$ invertible,

$$\mathrm{Ad}(g)x = \omega(g)xg^{-1} \in \mathbb{K}x$$

is equivalent to $\omega(g) \in \mathbb{K}g$, i.e., $g$ is an eigenvector of $\omega$. Therefore,

$$\Gamma(V,\beta) = \mathbb{K}^\times\mathbf{1} \dot\cup \mathbb{K}^\times x \quad \text{if} \quad a \neq 0.$$

For $a = 0$, we have $\omega(g)x = xg$ for any $g \in \mathrm{Cl}(\mathbb{K},0)$, so that $\Gamma(V,\beta) = \mathbb{K}^\times\mathbf{1} + \mathbb{K}x = \mathrm{Cl}(\mathbb{K},a)^\times$.

**Lemma B.3.13.** *The Clifford group $\Gamma(V,\beta)$ is invariant under $\omega$ and $*$.*

*Proof.* Let $g \in \Gamma(V,\beta)$ and $v \in V$. Then $\mathrm{Ad}(g)v = \omega(g)vg^{-1} \in V$ implies that

$$V \ni \mathrm{Ad}(g)v = -\omega\big(\mathrm{Ad}(g)v\big) = -g\omega(v)\omega(g)^{-1} = \mathrm{Ad}\big(\omega(g)\big)v,$$

which leads to $\omega(g) \in \Gamma(V,\beta)$. We likewise obtain

$$V \ni \mathrm{Ad}(g)v = -\big(\mathrm{Ad}(g)v\big)^* = -(g^*)^{-1}v^*\omega(g^*) = \mathrm{Ad}\big(\omega(g^*)^{-1}\big)v \in V,$$

which shows that $\omega(g^*) \in \Gamma(V,\beta)$, and hence that $g^* \in \Gamma(V,\beta)$. □

**Proposition B.3.14.** *If $(V,\beta)$ is nondegenerate and finite-dimensional, then the kernel of the representation $\Phi\colon \Gamma(V,\beta) \to \mathrm{GL}(V)$ is $\mathbb{K}^\times\mathbf{1}$.*

*Proof.* For $\lambda \in \mathbb{K}^\times$ and $v \in V$, we clearly have $\Phi(\lambda)v = \lambda v\lambda^{-1} = v$, so that $\mathbb{K}^\times\mathbf{1} \subseteq \ker\Phi$.

For the converse, we argue by induction on $\dim V$. If $\dim V = 1$ and $\mathrm{Cl}(V,\beta) = \mathbb{K}\mathbf{1} \oplus \mathbb{K}v$, then Example B.3.12(c) shows that $\Gamma(V,\beta) = \mathbb{K}^\times\mathbf{1} \dot\cup \mathbb{K}^\times v$ and $\ker\Phi = \{g \in \Gamma(V,\beta) \colon \omega(g) = g\} = \mathbb{K}^\times\mathbf{1}$. Now we assume that $\dim V > 1$. Let $g \in \ker\Phi$ and write it as $g = g_+ + g_-$, according to the $\mathbb{Z}/2$-grading of $\mathrm{Cl}(V,\beta)$. Then $\Phi(g)v = v$ is equivalent to $\omega(g)v = vg$, and decomposing into homogeneous summands leads to

$$g_+v = vg_+ \quad \text{and} \quad g_-v = -vg_- \quad \text{for all} \quad v \in V.$$

Let $v_1 \in V$ be a nonisotropic vector. Then $V = \mathbb{K}v_1 \oplus V_1$ with $V_1 := v_1^\perp$ is an orthogonal decomposition, so that $\mathrm{Cl}(V,\beta)$ decomposes accordingly as

$$\mathrm{Cl}(V,\beta) \cong \mathrm{Cl}\big(\mathbb{K}, \beta(v_1,v_1)\big) \widehat{\otimes} \mathrm{Cl}(V_1,\beta_1) \cong (\mathbb{K}\mathbf{1} + \mathbb{K}v_1) \widehat{\otimes} \mathrm{Cl}(V_1,\beta_1),$$

where $\beta_1 = \beta|_{V_1 \times V_1}$. The grading is given by

$$\mathrm{Cl}(V,\beta)_{\overline{0}} = \mathbf{1} \otimes \mathrm{Cl}(V_1,\beta_1)_{\overline{0}} \oplus v_1 \otimes \mathrm{Cl}(V_1,\beta_1)_{\overline{1}}$$

and

$$\mathrm{Cl}(V,\beta)_{\overline{1}} = \mathbf{1} \otimes \mathrm{Cl}(V_1,\beta_1)_{\overline{1}} \oplus v_1 \otimes \mathrm{Cl}(V_1,\beta_1)_{\overline{0}}$$

(cf. Proposition B.3.8). Accordingly, we write

$$g_\pm = \mathbf{1} \otimes a_\pm + v_1 \otimes b_\mp.$$

As $\mathbf{1} \otimes a_+$ commutes with $v_1 \otimes \mathbf{1}$ and $v_1 \otimes b_-$ anticommutes with $v_1 \otimes \mathbf{1}$, from $g_+ v_1 = v_1 g_+$ we obtain

$$0 = v_1^2 \otimes b_- = \beta(v_1,v_1)\mathbf{1} \otimes b_-,$$

which leads to $b_- = 0$. Likewise $g_- v_1 = -v_1 g_-$ leads to $b_+ = 0$, so that $g = \mathbf{1} \otimes (a_+ + a_-) \in \mathrm{Cl}(V_1,\beta_1)$, and our induction hypothesis implies that $g \in \mathbb{K}^\times \mathbf{1}$. ◻

**Corollary B.3.15.** *For $g \in \Gamma(V,\beta)$, we have $gg^* \in \mathbb{K}^\times \mathbf{1}$ and*

$$N \colon \Gamma(V,\beta) \to \mathbb{K}^\times, \quad gg^* = N(g)\mathbf{1},$$

*defines a group homomorphism, called the* norm homomorphism*, satisfying*

$$N\big(\omega(g)\big) = N(g) \quad \text{and} \quad N\big(\mathrm{Ad}(g)h\big) = N(h) \quad \text{for} \quad g,h \in \Gamma(V,\beta).$$

*Proof.* Since $\Gamma(V,\beta)$ is invariant under $*$, we have $gg^* \in \Gamma(V,\beta)$. In view of the preceding proposition, $gg^* \in \mathbb{K}^\times \mathbf{1}$ will follow if we can show that $gg^* \in \ker \Phi$.

For $x \in \mathrm{Cl}(V,\beta)$, we put $S(x) := \omega(x^*)$ and note that this defines an involutive antiautomorphism fixing $V$ pointwise (Lemma B.3.11). Since $\Gamma(V,\beta)$ is invariant under $*$, we have to show that $\Phi(g^{-1}) = \Phi(g^*)$ for $g \in \Gamma(V,\beta)$. For $g \in \Gamma(V,\beta)$ and $v \in V$, the element $\Phi(g^*)v = \omega(g^*)v(g^{-1})^* = S(g)v(g^{-1})^* \in V$ is fixed by $S$, which leads to

$$\Phi(g^*)v = S\big(S(g)v\big(g^{-1}\big)^*\big) = S\big(\big(g^{-1}\big)^*\big)vg = \omega\big(g^{-1}\big)vg = \Phi\big(g^{-1}\big)v,$$

i.e., $\Phi(g^*) = \Phi(g^{-1})$. This proves that $gg^* \in \ker \Phi = \mathbb{K}^\times \mathbf{1}$, so that $N(g)$ is defined. To see that $N$ is a group homomorphism, we calculate

$$N(gh)\mathbf{1} = ghh^*g^* = g\big(N(h)\mathbf{1}\big)g^* = N(h)gg^* = N(h)N(g)\mathbf{1}.$$

Applying $\omega$ to $gg^* = N(g)\mathbf{1}$, we obtain $N(\omega(g))\mathbf{1} = \omega(g)\omega(g)^* = N(g)\mathbf{1}$, so that $N(\omega(g)) = N(g)$, and this further implies that $N(\mathrm{Ad}(g)h) = N(\omega(g)hg^{-1}) = N(h)$. ◻

**Theorem B.3.16.** *If* $(V, \beta)$ *is nondegenerate and finite-dimensional, then* $\mathrm{im}\,(\Phi) = \mathrm{O}(V, \beta)$, *so that* $\Phi$ *defines a short exact sequence*

$$\mathbf{1} \to \mathbb{K}^\times \hookrightarrow \Gamma(V, \beta) \xrightarrow{\Phi} \mathrm{O}(V, \beta) \to \mathbf{1}, \tag{B.7}$$

*where* $\Phi(v) = \sigma_v$ *is the reflection in* $v^\perp$ *for each nonisotropic element* $v \in V \subseteq \mathrm{Cl}(V, \beta)$.

*Proof.* For $v \in V$ we have

$$vv^* = -v^2 = -\beta(v, v)\mathbf{1}, \tag{B.8}$$

and for $g \in \Gamma(V, \beta)$ we have

$$\begin{aligned}\big(\Phi(g)v\big)\big(\Phi(g)v\big)^* &= \omega(g)vg^{-1}\big(\omega(g)vg^{-1}\big)^* = \omega(g)vg^{-1}\big(g^{-1}\big)^*(-v)\omega(g)^* \\ &= -N\big(g^{-1}\big)\beta(v, v)\omega(gg^*) = -N\big(g^{-1}\big)\beta(v, v)\omega\big(N(g)\mathbf{1}\big) = -\beta(v, v)\mathbf{1}.\end{aligned}$$

In view of (B.8), this proves that $\Phi(g) \in \mathrm{O}(V, \beta)$.

To identify the image of $\Phi$, we observe that for any nonisotropic element $v \in V$ we have $\omega(v) = -v$ and $v^{-1} = \beta(v, v)^{-1}v$, so that

$$\mathrm{Ad}(v)x = -\beta(v, v)^{-1}vxv = \beta(v, v)^{-1}v\big(vx - 2\beta(v, x)\mathbf{1}\big) = x - 2\frac{\beta(v, x)}{\beta(v, v)}v,$$

which is the orthogonal reflection $\sigma_v$ in the hyperplane $v^\perp$. We conclude in particular that $\Gamma(V, \beta)$ contains all nonisotropic elements of $V$, considered as a subspace of $\mathrm{Cl}(V, \beta)$, and that $\mathrm{Im}(\Phi)$ contains all orthogonal reflections.

If $(V, \beta)$ is finite-dimensional and nondegenerate, then $\mathrm{O}(V, \beta)$ is generated by reflections (Exercise B.3.1), so that this argument shows that $\mathrm{Im}(\Phi) = \mathrm{O}(V, \beta)$. □

**Remark B.3.17.** If $\beta = 0$, then $\mathrm{Cl}(V, \beta) \cong \Lambda(V)$ is the exterior algebra of $V$, and $\Lambda(V)^\times = \mathbb{K}^\times\mathbf{1} \oplus \sum_{k \geq 1} \Lambda^k(V)$. Since $\Lambda(V)$ is graded commutative (see Lemma B.2.10), the even part is central and any two odd elements anticommute. This shows that for each invertible element $g = g_+ + g_-$, with $g_+$ even and $g_-$ odd, we have

$$g_+v = vg_+ \quad \text{and} \quad g_-v = -vg_-$$

for all $v \in V$. This means that $\omega(g)v = vg$, so that $\Lambda(V)^\times = \Gamma(V, \beta) = \ker \Phi$.

**Remark B.3.18.** The proof of Theorem B.3.16 has several interesting consequences:

(a) As the image of $\Phi$ is generated by orthogonal reflections, it follows that $\Gamma(V, \beta)$ is generated by $\mathbb{K}^\times\mathbf{1} = \ker \Phi$ and the set

$$V^\times := \big\{v \in V : \beta(v, v) \neq 0\big\}$$

of nonisotropic vectors in $V$. For $\lambda \in \mathbb{K}^\times$ and $v \in V^\times$, the relation $\lambda\mathbf{1} = (\lambda v)v^{-1}$ implies that $\Gamma(V,\beta)$ is actually generated by $V^\times$. It follows in particular that $\omega(g) = \pm g$ for each $g \in \Gamma(V,\beta)$ and that

$$\omega(g) = \det\big(\Phi(g)\big)g \quad \text{for} \quad g \in \Gamma(V,\beta).$$

(b) From $N(v) = -\beta(v,v)$ for $v \in V^\times$ it follows that the image $N(\Gamma(V,\beta))$ of $\Gamma(V,\beta)$ under $N$ is the subgroup generated by the square classes represented by $-\beta$.

**Remark B.3.19.** Alternatively, one can introduce the Clifford group more directly as the subgroup $\Gamma'$ of $\mathrm{Cl}(V,\beta)$ generated by the subset $V^\times$ of nonisotropic vectors. Since $\mathrm{Ad}(v)V = V$ for each such element and, as we have seen in the proof of Theorem B.3.16, $\mathrm{Ad}(v)|_V$ is the orthogonal reflection in the hyperplane $v^\perp$, we directly obtain a homomorphism $\Phi\colon \Gamma' \to \mathrm{O}(V,\beta)$ whose image is the subgroup generated by all reflections, hence all of $\mathrm{O}(V,\beta)$ if $(V,\beta)$ is nondegenerate. For any $v \in V^\times$ and $\lambda \in \mathbb{K}^\times$, we have $\lambda\mathbf{1} = v^{-1}\cdot(\lambda v) \in \Gamma'$, so that $\Gamma'$ contains $\mathbb{K}^\times$, which immediately leads to the short exact sequence

$$\mathbf{1} \to \mathbb{K}^\times \to \Gamma' \to \mathrm{O}(V,\beta) \to \mathbf{1}.$$

However, the advantage of the approach via the Clifford group is that it is specified by equations, hence in particular closed if $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.

### B.3.2 Pin and Spin Groups

**Definition B.3.20.** We define the *pin group*[2]

$$\mathrm{Pin}(V,\beta) := \big\{g \in \Gamma(V,\beta)\colon N(g) = 1\big\} = \ker\big(N\colon \Gamma(V,\beta) \to \mathbb{K}^\times\big)$$

and the *spin group*

$$\mathrm{Spin}(V,\beta) := \mathrm{Pin}(V,\beta) \cap \Phi^{-1}\big(\mathrm{SO}(V,\beta)\big).$$

Note that

$$\ker\Phi \cap \mathrm{Pin}(V,\beta) = \big\{\lambda\mathbf{1}\colon \lambda \in \mathbb{K}^\times, \lambda^2 = 1\big\} = \{\pm\mathbf{1}\}$$

consists of 2 elements and that

$$\omega(g) = g \quad \text{for} \quad g \in \mathrm{Spin}(V,\beta) \tag{B.9}$$

(cf. Remark B.3.18). We also write

$$\mathrm{Pin}_{p,q}(\mathbb{K}) := \mathrm{Pin}\big(\mathbb{K}^{p+q},\beta_{p,q}\big) \quad \text{and} \quad \mathrm{Spin}_{p,q}(\mathbb{K}) := \mathrm{Spin}\big(\mathbb{K}^{p+q},\beta_{p,q}\big)$$

for $\beta_{p,q}(x,y) = -x_1y_1 - \cdots - x_py_p + x_{p+1}y_{p+1} + \cdots + x_{p+q}y_{p+q}$. For $q = 0$, we put

$$\mathrm{Pin}_n(\mathbb{K}) := \mathrm{Pin}\big(\mathbb{K}^n,\beta_{n,0}\big) \quad \text{and} \quad \mathrm{Spin}_n(\mathbb{K}) := \mathrm{Spin}\big(\mathbb{K}^n,\beta_{n,0}\big).$$

[2] The name "pin" is due to J. P. Serre; see [ABS64] for the first occurrence of these groups.

**Remark B.3.21.** Our definition of the pin group follows [ABS64], but one finds slightly different definitions in the literature, all of which lead to the same spin group. For instance, Scharlau [Sch85] uses a different homomorphism

$$\widetilde{N}\colon \Gamma(V,\beta) \to \mathbb{K}^\times,$$

defined by $g\omega(g)^* = \widetilde{N}(g)\mathbf{1}$. To see how $\widetilde{N}$ differs from $N$, we note that the group $\Gamma(V,\beta)$ decomposes into the two subsets

$$\Gamma(V,\beta)_\pm := \{\gamma \in \Gamma(V,\beta)\colon \omega(\gamma) = \pm\gamma\}$$

and that $\varepsilon\colon \Gamma(V,\beta) \to \{\pm 1\},\ \omega(\gamma) = \varepsilon(\gamma)\gamma$ defines a group homomorphism satisfying

$$\widetilde{N} = N \cdot \varepsilon.$$

Accordingly, the pin group $\{\gamma \in \Gamma(V,\beta)\colon \widetilde{N}(\gamma) = 1\}$ defined in [Sch85] is the union of

$$\mathrm{Pin}(V,\beta)_+ = \mathrm{Spin}(V,\beta) \quad \text{and} \quad \{\gamma \in \Gamma(V,\beta)_-\colon N(\gamma) = -1\}.$$

We also note that $\varepsilon(\gamma) = \det(\Phi(\gamma))$ follows from the fact that it holds on the generators, the nonisotropic elements of $V$. Therefore, $\Gamma(V,\beta)_+ = \Phi^{-1}(\mathrm{SO}(V,\beta))$, and both homomorphisms $N$ and $\widetilde{N}$ lead to the same spin group

$$\mathrm{Spin}(V,\beta) = \ker(N|_{\Gamma(V,\beta)_+}) = \ker(\widetilde{N}|_{\Gamma(V,\beta)_+}).$$

**Remark B.3.22.** (a) If $\mathbb{K} = \mathbb{R}$ and $\beta$ is negative definite, then $N(v) = -\beta(v,v) > 0$ for each $0 \neq v \in V$, and each nonzero vector $v$ has a multiple $v'$ normalized by $\beta(v',v') = -1$. Then $v' \in \mathrm{Pin}(V,\beta)$, and this implies that the restriction

$$\Phi\colon\ \mathrm{Pin}(V,\beta) \to \mathrm{O}(V,\beta)$$

is still surjective, which leads to the exact sequence

$$\mathbf{1} \to \{\pm\mathbf{1}\} \to \mathrm{Pin}(V,\beta) \to \mathrm{O}(V,\beta) \to \mathbf{1}$$

and, accordingly, to the exact sequence

$$\mathbf{1} \to \{\pm\mathbf{1}\} \to \mathrm{Spin}(V,\beta) \to \mathrm{SO}(V,\beta) \to \mathbf{1}.$$

(b) If $\mathbb{K} = \mathbb{R}$ and $\beta$ is positive definite, then $N(\Gamma(V,\beta)_-) \subseteq \mathbb{R}^\times_-$ implies that $\mathrm{Pin}(V,\beta) \subseteq \Gamma(V,\beta)_+$, and hence that $\mathrm{Pin}(V,\beta) = \mathrm{Spin}(V,\beta)$. However, the alternative definition of the pin group, based on $\widetilde{N}$, yields a larger group (cf. Remark B.3.21).

(c) In general, the homomorphism

$$\Phi\colon\ \mathrm{Spin}(V,\beta) \to \mathrm{SO}(V,\beta)$$

is not surjective. In fact, if $\mathbb{K} = \mathbb{R}$ and $\beta$ is indefinite, then pick $v_1, v_2 \in V$ with $\beta(v_1, v_1) = 1 = -\beta(v_2, v_2)$. Now the product $g := \sigma_{v_1}\sigma_{v_2}$ of the corresponding orthogonal reflections is an element of $\mathrm{SO}(V, \beta)$, and in $\Gamma(V, \beta)$ we have

$$N(v_1 v_2) = N(v_1)N(v_2) = \beta(v_1, v_1)\beta(v_2, v_2) = -1.$$

Then $\Phi(v_1 v_2) = g$, and for any element $\gamma \in \Phi^{-1}(g)$, we have $\gamma = \lambda v_1 v_2$, $\lambda \in \mathbb{K}^\times$, and therefore $N(\gamma) = -\lambda^2 < 0$. This implies that $\gamma \notin \mathrm{Spin}(V, \beta)$, and hence that $\Phi(\mathrm{Spin}(V, \beta))$ is a proper subgroup of $\mathrm{SO}(V, \beta)$.

(d) Suppose that $\mathbb{K} = \mathbb{C}$ and $(V, \beta) = (\mathbb{C}^n, \beta)$ with the standard form $\beta(z, w) = \sum_{j=1}^n z_j w_j$. Since every complex number is a square, $N(v) = -\beta(v, v)$ implies that each nonisotropic vector $v \in \mathbb{C}^n$ has a scalar multiple $\widetilde{v}$ with $N(\widetilde{v}) = 1$. This proves that the homomorphism

$$\Phi\colon \mathrm{Pin}_n(\mathbb{C}) \to \mathrm{O}_n(\mathbb{C})$$

is surjective, which leads to the exact sequences

$$\mathbf{1} \to \{\pm\mathbf{1}\} \to \mathrm{Pin}_n(\mathbb{C}) \to \mathrm{O}_n(\mathbb{C}) \to \mathbf{1}$$

and

$$\mathbf{1} \to \{\pm\mathbf{1}\} \to \mathrm{Spin}_n(\mathbb{C}) \to \mathrm{SO}_n(\mathbb{C}) \to \mathbf{1}.$$

**Example B.3.23.** We recall from Example B.3.12(a) that for $n = 1$ we have $C_1 \cong \mathbb{C}$, $\omega(z) = \overline{z}$, and

$$\Gamma(V, \beta) = \mathbb{R}^\times \cup i\mathbb{R}^\times.$$

From $N(z) = |z|^2$ we derive that

$$\mathrm{Pin}_1(\mathbb{R}) = \big\{z \in \Gamma(V, \beta)\colon |z| = 1\big\} = \{\pm 1, \pm i\} \quad \text{and} \quad \mathrm{Spin}_1(\mathbb{R}) = \{\pm 1\}.$$

**Example B.3.24.** For $n = 2$, we have $C_2 \cong \mathbb{H}$ and we recall from Example B.3.12(b) that

$$\Gamma(V, \beta) = \big\{\alpha\mathbf{1} + \delta K\colon \alpha^2 + \delta^2 > 0\big\}\dot{\cup}\big\{\beta I + \gamma J\colon \beta^2 + \gamma^2 > 0\big\}.$$

Further, $N(x) = \|x\|^2$ follows from $xx^* = \|x\|^2\mathbf{1}$. This implies that

$$\mathrm{Pin}_2(\mathbb{R}) = \big\{\alpha\mathbf{1} + \delta K\colon \alpha^2 + \delta^2 = 1\big\}\dot{\cup}\big\{\beta I + \gamma J\colon \beta^2 + \gamma^2 = 1\big\}$$

is a union of two circles and

$$\mathrm{Spin}_2(\mathbb{R}) = \big\{\alpha\mathbf{1} + \delta K\colon \alpha^2 + \delta^2 = 1\big\} = \mathrm{Pin}_2(\mathbb{R})_0 \cong \mathbb{T}.$$

The complement of the identity component in $\mathrm{Pin}_2(\mathbb{R})$ is the set

$$\big\{\beta I + \gamma J\colon \beta^2 + \gamma^2 = 1\big\},$$

and for all these elements we have $(\beta I + \gamma J)^2 = -\mathbf{1}$, so that the short exact sequence

$$\mathbf{1} \to \mathrm{Spin}_2(\mathbb{R}) \to \mathrm{Pin}_2(\mathbb{R}) \to \pi_0(\mathrm{Pin}_2(\mathbb{R})) \cong \mathbb{Z}/2 \to \mathbf{1}$$

does not split.

**Remark B.3.25.** For the negative definite form $\beta_n$ on $\mathbb{R}^n$, we have the short exact sequence

$$\mathbf{1} \to \mathrm{Spin}_n(\mathbb{R}) \to \mathrm{Pin}_n(\mathbb{R}) \to \{\pm 1\} \to \mathbf{1},$$

so that it makes sense to ask when this sequence splits, i.e., if there exists an involution $\tau \in \mathrm{Pin}_n(\mathbb{R})$, not contained in $\mathrm{Spin}_n(\mathbb{R})$.

In the preceding two examples, we have seen that this is not the case for $n = 1, 2$. However, it is true for $n > 2$, which can be seen as follows. Let $e_1, \ldots, e_n$ be an orthonormal basis for $\mathbb{R}^n$, so that $\beta_n(e_i, e_j) = -\delta_{ij}$. For $n \geq 3$, we put $\tau := e_1e_2e_3$. As $N(e_j) = -\beta(e_j, e_j) = 1$ for each $j$, we have $\tau \in \mathrm{Pin}_n(\mathbb{R}) \setminus \mathrm{Spin}_n(\mathbb{R})$. Moreover,

$$\tau^2 = e_1e_2e_3e_1e_2e_3 = e_1^2e_2e_3e_2e_3 = e_2e_3^2e_2 = \mathbf{1},$$

so that $\tau$ is an involution in $\mathrm{Pin}_n(\mathbb{R}) \setminus \mathrm{Spin}_n(\mathbb{R})$ (cf. Exercise B.3.5).

### B.3.3 Exercises for Section B.3

**Exercise B.3.1.** Show that if $(V, \beta)$ is a nondegenerate finite-dimensional quadratic space, then the orthogonal group $\mathrm{O}(V, \beta)$ is generated by orthogonal reflections.

**Exercise B.3.2.** Show that if $\beta$ is a positive definite form on $\mathbb{R}^2$, then $C_{0,2} \cong \mathrm{Cl}(\mathbb{R}^2, \beta) \cong M_2(\mathbb{R})$.

**Exercise B.3.3.** Establish isomorphisms

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}, \quad \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong M_2(\mathbb{C}), \quad \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong M_4(\mathbb{R}).$$

**Exercise B.3.4.** (a) Write $C'_n := C_{0,n}$ for the Clifford algebra of $\mathbb{R}^n$, endowed with the positive definite form $\beta_{0,n}$. Establish isomorphisms

$$C_{n+2} \cong C'_n \otimes_{\mathbb{R}} C_2 \quad \text{and} \quad C'_{n+2} \cong C_n \otimes_{\mathbb{R}} C'_2.$$

Hint: Let $e_1, \ldots, e_{n+2}$ be an orthonormal basis for $\mathbb{R}^{n+2}$, so that $e_i^2 = -\mathbf{1}$ in $C_{n+2}$. Then for the first isomorphism map $e_i \mapsto e'_i \otimes e_1e_2$ for $i = 1, \ldots, n$ and map $e_{n+1}$ and $e_{n+2}$ on $\mathbf{1} \otimes e_1$ and $\mathbf{1} \otimes e_2$, respectively.

(b) Show that $C_{n+4} \cong C_n \otimes_{\mathbb{R}} M_2(\mathbb{H}) \cong M_2(C_n) \otimes_{\mathbb{R}} \mathbb{H}$.

(c) Prove that $C_{n+8} \cong C_n \otimes_{\mathbb{R}} M_{16}(\mathbb{R}) \cong M_{16}(C_n)$. This is called the *periodicity property*.

**Exercise B.3.5.** Let $D_n = \mathrm{Cl}(\mathbb{C}^n, \beta_n)$ with $\beta_n(z, w) = \sum_{j=1}^n z_jw_j$. Show that

$$D_1 \cong \mathbb{C} \oplus \mathbb{C}, \quad D_2 \cong M_2(\mathbb{C}),$$

and establish the periodicity property

$$D_{n+2} \cong D_n \otimes_{\mathbb{C}} M_2(\mathbb{C}) \cong M_2(D_n), \quad n \geq 1.$$

**Exercise B.3.6.** In the unit group $C_n^\times$ of the Clifford algebra $C_n$ associated to the negative definite form $\beta_n$ on $\mathbb{R}^n$, we consider the subgroup $F$, generated by the canonical basis vectors $e_1, \ldots, e_n$. Show that:

(a) $\Phi(F) \cong (\mathbb{Z}/2)^n$.
(b) $\Phi$ induces the short exact sequence $\mathbf{1} \to \mathbb{Z}/2 \to F \to (\mathbb{Z}/2)^n \to \mathbf{1}$.
(c) $F$ is 2-step nilpotent, i.e., $C^3(F) = \{\mathbf{1}\}$, with $C^2(F) = F' = \{\pm\mathbf{1}\}$ (for $n > 1$). Here $C^n(F)$ denotes the central series for the group $F$.
(d) Each element $f \in F$ can be written as $f = \pm e_{i_1} \cdots e_{i_k}$ with $1 \leq i_1 < \cdots < i_k \leq n$. It satisfies $f^2 = (-\mathbf{1})^{\binom{k+1}{2}}$. Conclude that for $n \leq 2$ all elements $\neq \pm\mathbf{1}$ are of order 4.

**Exercise B.3.7.** Consider the decomposition

$$\mathrm{Cl}(V_1 \oplus V_2, \beta_1 \oplus \beta_2) \cong \mathrm{Cl}(V_1, \beta_1) \widehat{\otimes} \mathrm{Cl}(V_2, \beta_2)$$

from Proposition B.3.8, where $\beta_1$ and $\beta_2$ are non-degenerate. Show that the subgroups $\mathrm{Spin}(V_1, \beta_1)$ and $\mathrm{Spin}(V_2, \beta_2)$ commute.

# C Some Functional Analysis

## C.1 Bounded Operators

**Definition C.1.1.** Let $X$ and $Y$ be normed spaces and $A\colon X \to Y$ be a linear map, also called an operator in this context. We define the (*operator*) *norm of* $A$ by

$$\|A\| := \sup\big\{\|Ax\|\colon x \in X, \|x\| \leq 1\big\} \in [0, \infty].$$

The linear operator $A$ is said to be *bounded* if $\|A\| < \infty$. We write $B(X, Y)$ for the set of bounded linear operators from $X$ to $Y$. It is easy to see that $(B(X, Y), \|\cdot\|)$ is a normed space. For $X = Y$ we simply write $B(X) := B(X, X)$.

**Remark C.1.2.** Note that boundedness of an operator $A$ does not mean that its range $A(X)$ is a bounded subset of $Y$. If $B_X$ denotes the closed unit ball in $X$, then $A$ is bounded if and only if $A(B_X)$ is a bounded subset of $Y$ and $\|A\|$ is the radius of the smallest ball in $Y$ centered at 0 containing $A(B_X)$.

**Lemma C.1.3.** *For a linear map* $A\colon X \to Y$ *between normed spaces, the following are equivalent*

(1) $A$ *is continuous.*
(2) $A$ *is continuous at* 0.
(3) $A$ *is bounded.*

**Definition C.1.4.** Let $X$ and $Y$ be Banach spaces. Then a linear map $A\colon X \to Y$ is called *compact* if the image of every bounded sequence in $X$ under $A$ has a convergent subsequence. Since convergent sequences are in particular bounded, one easily shows that any compact operator is continuous (Exercise C.3.3).

To determine whether a given operator is compact, one needs some tools to determine under which circumstances bounded sequences in Banach spaces possess convergent subsequences. One such tool is the following theorem:

J. Hilgert, K.-H. Neeb, *Structure and Geometry of Lie Groups*,
Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-84794-8, 

**Theorem C.1.5 (Ascoli's Theorem).** *Let $X$ be a compact space, $C(X)$ be the Banach space of all continuous functions $f\colon X \to \mathbb{K}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, endowed with the sup-norm*

$$\|f\| := \sup\{|f(x)|\colon x \in X\}$$

*and $M \subseteq C(X)$ a subset satisfying the following conditions:*

(a) *$M$ is pointwise bounded, i.e., $\sup\{|f(x)|\colon f \in M\} < \infty$ for each $x \in X$.*
(b) *$M$ is equicontinuous, i.e., for each $\varepsilon > 0$ and each $x \in X$ there exists a neighborhood $U_x$ with*

$$|f(x) - f(y)| \leq \varepsilon \quad \text{for} \quad f \in M, y \in U_x.$$

*Then each sequence in $M$ possesses a convergent subsequence.*

*Proof.* Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $M$. For each $k \in \mathbb{N}$ we find, due to (b), points $x_1^k, \ldots, x_{m_k}^k$ in $X$ and neighborhoods $V_1^k, \ldots, V_{m_k}^k$ of these points such that $X \subseteq \bigcup_{i=1}^{m_k} V_i^k$ and

$$|f(x) - f(x_i^k)| < \frac{1}{k} \quad \text{for} \quad f \in M, x \in V_i^k, i = 1, \ldots, m_k.$$

We order the countable set $\{x_i^k : k \in \mathbb{N}, i = 1, \ldots, m_k\}$ as follows to a sequence $(y_m)_{m\in\mathbb{N}}$:

$$x_1^1, \ldots, x_{m_1}^1, x_1^2, \ldots, x_{m_2}^2, \ldots$$

For each $y_m$, the set $\{f_n(y_m) : n \in \mathbb{N}\} \subseteq \mathbb{K}$ is bounded, hence contains a subsequence $f_n^1$, converging at $y_1$. This sequence has a subsequence $f_n^2$, converging at $y_2$, etc. The sequence $(f_n^n)_{n\in\mathbb{N}}$ is a subsequence of the original sequence, converging on the set $\{y_m : m \in \mathbb{N}\}$. To simplify notation, we may now assume that the sequence $f_n$ converges pointwise on this set.

Next we show that the sequence $(f_n)$ converges pointwise. Pick $x \in X$. In view of the completeness of $\mathbb{K}$, it suffices to show that the sequence $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence. So let $\varepsilon > 0$. Then there exists a $k \in \mathbb{N}$ with $\frac{3}{k} < \varepsilon$ and $y_m$ such that

$$|f_n(x) - f_n(y_m)| < \frac{1}{k} \quad \text{for} \quad n \in \mathbb{N}.$$

We choose $n_0 \in \mathbb{N}$ such that

$$|f_n(y_m) - f_{n'}(y_m)| < \frac{1}{k} \quad \text{for} \quad n, n' > n_0.$$

Then

$$\begin{aligned} |f_n(x) - f_{n'}(x)| &\leq |f_n(x) - f_n(y_m)| + |f_n(y_m) - f_{n'}(y_m)| \\ &\quad + |f_{n'}(y_m) - f_{n'}(x)| \\ &\leq \frac{3}{k} \leq \varepsilon. \end{aligned}$$

Let $F(x) := \lim_{n\to\infty} f_n(x)$. It remains to show that $f_n$ converges uniformly to $F$. Let $\varepsilon > 0$ and choose $k \in \mathbb{N}$ with $\frac{3}{k} < \varepsilon$. We pick $n_0 \in \mathbb{N}$ so large that

$$\left|f_n\left(x_i^k\right) - F\left(x_i^k\right)\right| \le \frac{1}{k} \qquad \text{for} \qquad n \ge n_0, i = 1, \ldots, m_k.$$

Since each element $x \in X$ is contained in one of the sets $V_i^k$,

$$\left|f_n(x)-F(x)\right| \le \left|f_n(x)-f_n\left(x_i^k\right)\right|+\left|f_n\left(x_i^k\right)-F\left(x_i^k\right)\right|+\left|F\left(x_i^k\right)-F(x)\right| \le \frac{3}{k} \le \varepsilon,$$

because $|F(x_i^k) - F(x)| = \lim_{n\to\infty} |f_n(x_i^k) - f_n(x)| \le \frac{1}{k}$. This proves that $f_n$ converges uniformly to $F$, and the proof is complete. □

## C.2 Hilbert Spaces

**Definition C.2.1.** A Banach space $X$ is called a *Hilbert space* if there exists a sesquilinear positive definite hermitian form $\langle\cdot,\cdot\rangle$ on $X$ with $\|v\|^2 = \langle v, v\rangle$ for each $v \in X$.

**Lemma C.2.2.** *Let $E$ be a closed subspace of the Hilbert space $\mathcal{H}$ and $E^\perp := \left\{x \in \mathcal{H} \colon \langle x, E\rangle = \{0\}\right\}$. Then $\mathcal{H} = E \oplus E^\perp$.*

*Proof.* Clearly, $E \cap E^\perp = \{0\}$, so we have to show that each $x \in \mathcal{H}$ can be written as a sum of an element $x_0 \in E$ and an element $x_1 \in E^\perp$. The idea is to find $x_0$ as the point in $E$ minimizing the distance to $x$.

Pick $x_n \in E$ with $\|x_n - x\| \to d := \inf\{\|y - x\| \colon y \in E\}$. In the Parallelogram Equation (Exercise C.3.1)

$$\|x_n + x_m - 2x\|^2 + \|x_n - x_m\|^2 = 2\|x_n - x\|^2 + 2\|x_m - x\|^2,$$

the right-hand side is arbitrarily close to $4d^2$ if $n$ and $m$ are large enough. On the other hand, $\frac{1}{2}(x_n + x_m) \in E$ implies that

$$\|x_n + x_m - 2x\|^2 = 4\left\|\frac{1}{2}(x_n + x_m) - x\right\|^2 \ge 4d^2.$$

Therefore, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $E$, hence converges to some $x_0 \in E$ with $\|x - x_0\| = d$. For $y \in E$, the function

$$\|x + \lambda y - x_0\|^2 = \|x - x_0\|^2 + |\lambda|^2\|x_0\|^2 + 2\,\mathrm{Re}\,\lambda\langle x - x_0, y\rangle$$

is minimal at $\lambda = 0$, which implies that $x - x_0 \in E^\perp$ (Exercise C.3.2). Therefore, $x = x_0 + (x - x_0) \in E + E^\perp$. □

## C.3 Compact Symmetric Operators on Hilbert Spaces

**Definition C.3.1.** Let $\mathcal{H}$ be a complex Hilbert space. A bounded operator $A$ on $\mathcal{H}$ is said to be *symmetric* if

$$\langle Av, w\rangle = \langle v, Aw\rangle \quad \text{for} \quad v, w \in \mathcal{H}.$$

**Theorem C.3.2.** *For a compact symmetric operator $A\colon \mathcal{H} \to \mathcal{H}$ and $\mathcal{H}_\lambda := \ker(A - \lambda\mathbf{1})$, the following assertions hold:*

(1) $\|A\| = \sup\{|\langle Ax, x\rangle| \colon x \in \mathcal{H}, \|x\| = 1\}$.
(2) $\|A\|$ *or* $-\|A\|$ *is an eigenvalue of* $A$.
(3) $\bigoplus_{|\lambda|>\varepsilon} \mathcal{H}_\lambda$ *is finite-dimensional for every* $\varepsilon > 0$.
(4) $\bigoplus_{\lambda\in\mathbb{R}} \mathcal{H}_\lambda$ *is dense in* $\mathcal{H}$.

*Proof.* (1) Let $M := \sup\{|\langle Ax, x\rangle| \colon x \in \mathcal{H}, \|x\| = 1\}$. Since

$$\langle Ax, x\rangle \leq \|Ax\|\|x\| \leq \|A\|\|x\|^2$$

follows from the Cauchy–Schwarz inequality, $M \leq \|A\|$.

It remains to verify $\|A\| \leq M$. For $A = 0$ there is nothing to show. So pick $x \in \mathcal{H}$ with $\|x\| = 1$ and $Ax \neq 0$ and put $y := \frac{1}{\|Ax\|}Ax$. Then $\langle Ax, y\rangle = \|Ax\| = \langle x, Ay\rangle$, leads to

$$4\|Ax\| = \big\langle A(x+y), x+y\big\rangle - \big\langle A(x-y), x-y\big\rangle \leq M\big(\|x+y\|^2 + \|x-y\|^2\big) = 4M.$$

(2) In view of (1), there exists a sequence $x_n$ of unit vectors with $|\langle Ax_n, x_n\rangle| \to \|A\|$. Passing to a subsequence, we may assume that the sequence $\langle Ax_n, x_n\rangle$ converges in $\mathbb{R}$. If $A = 0$, the assertion is trivial, so that we may assume $A \neq 0$. Then either $\langle Ax_n, x_n\rangle \to \|A\|$ or $\langle Ax_n, x_n\rangle \to -\|A\|$. We assume the former, the other case is treated in a similar fashion. Now

$$\begin{aligned} 0 &\leq \big\|Ax_n - \|A\|x_n\big\|^2 = \|Ax_n\|^2 - 2\|A\|\langle Ax_n, x_n\rangle + \|A\|^2 \\ &\leq 2\|A\|\big(\|A\| - \langle Ax_n, x_n\rangle\big) \to 0. \end{aligned}$$

Because of the compactness of $A$, we may assume that $Ax_n$ converges to some $x \in \mathcal{H}$. From the above calculation, we infer that $\|A\|x_n \to x$ and in particular $\|x\| = \|A\| > 0$. For $y := \frac{1}{\|A\|}x$, we now find $x_n \to y$, and therefore

$$Ay - \|A\|y = \lim_{n\to\infty} Ax_n - \|A\|x_n = 0,$$

so that $\|A\|$ is an eigenvalue of $A$.

(3) Let $\lambda \neq 0$ be an eigenvalue of $A$. Then $A|_{\mathcal{H}_\lambda}$ is a compact operator. Therefore, the identical map on $\mathcal{H}_\lambda$ is compact, and thus $\dim \mathcal{H}_\lambda < \infty$. In fact, every infinite-dimensional Hilbert space $\mathcal{H}$ contains an orthonormal sequence $(e_n)_{n\in\mathbb{N}}$. Then $(e_n)$ is bounded, but $\|e_n - e_m\|^2 = 2$ for $n \neq m$ implies that it contains no convergent subsequence.

Next we observe that if $x \in \mathcal{H}_\lambda$ and $y \in \mathcal{H}_\mu$ with $\mu \neq \lambda$, then

$$(\lambda - \mu)\langle x, y\rangle = \langle Ax, y\rangle - \langle x, Ay\rangle = 0$$

implies that eigenspaces for different eigenvalues are orthogonal. If there are infinitely many different eigenvalues $\lambda_n$ with $|\lambda_n| > \varepsilon$, then we pick unit vectors $x_n \in \mathcal{H}_{\lambda_n}$ and observe that the sequence $(x_n)$ is bounded, but the sequence $Ax_n = \lambda_n x_n$ has no convergent subsequence because $\|Ax_n - Ax_m\|^2 = \lambda_n^2 + \lambda_m^2 \geq 2\varepsilon^2$. This contradicts the compactness of $A$.

(4) If $\lambda$ is an eigenvector of $A$ and $x$ a corresponding unit eigenvector, then $\lambda = \langle Ax, x\rangle \in \mathbb{R}$. Let $E := \overline{\bigoplus_{\lambda\in\mathbb{R}} \mathcal{H}_\lambda}$. Then $E$ is $A$-invariant, and for $y \in E^\perp$ and $x \in E$ the relation $\langle Ay, x\rangle = \langle y, Ax\rangle = 0$ implies that $E^\perp$ is also $A$-invariant. If $E^\perp \neq \{0\}$, then $AE^\perp \neq \{0\}$ because $\mathcal{H}_0 \subseteq E$. Further, (2) implies the existence of an eigenvector in $E^\perp$, which is a contradiction. We conclude that $E^\perp = \{0\}$, and therefore $E = \mathcal{H}$ follows from Lemma C.2.2. □

### C.3.1 Exercises for Section C.3

**Exercise C.3.1.** Show that in each Hilbert space $\mathcal{H}$, we have for $x, y \in \mathcal{H}$ the Parallelogram Law

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

**Exercise C.3.2.** We consider a function $f\colon \mathbb{C} \to \mathbb{R}$, given by

$$f(z) = a + 2\,\mathrm{Re}(zb) + c^2|z|^2, \qquad a, c \in \mathbb{R}, b \in \mathbb{C}.$$

Show that $b = 0$ if and only if $f$ attains a minimal value at $z = 0$.

**Exercise C.3.3.** Show that each compact operator $A\colon X \to Y$ between Banach spaces is continuous.

# D Hints to Exercises

**Exercise 2.1.1:**

(b) If $w \in W$ is written as a finite sum $w = \sum_\lambda v_\lambda$ with $Av_\lambda = \lambda v_\lambda$, then each $v_\lambda$ is contained in $W$. This can be proved by induction on the length of the sum.

(e) Suppose that $\sum_{i=1}^n v_{\lambda_i} = 0$ with $v_{\lambda_i} \in V_{\lambda_i}(\mathcal{A})$ and show by induction on $n$ that all summands vanish.

**Exercise 2.1.6:** The verification of the convexity is easy. To see that $\mathrm{Pd}_n(\mathbb{K})$ is open, show first that for each $r > 0$ we have $B_r(r\mathbf{1}) = rB_1(\mathbf{1}) \subseteq \mathrm{Pd}_n(\mathbb{K})$ (here $B_r(x)$ denotes the open ball of radius $r$ around $x$) by considering the eigenvalues and using that for $A \in \mathrm{Herm}_n(\mathbb{K})$ we have

$$\|A\| = \max\big\{|\lambda| \colon \det(A - \lambda\mathbf{1}) = 0\big\}.$$

Now observe that $\mathrm{Pd}_n(\mathbb{K}) = \bigcup_{r>0} B_r(r\mathbf{1})$ because for $A \in \mathrm{Pd}_n(\mathbb{K})$ with maximal eigenvalue $r$ we have $A \in B_r(r\mathbf{1})$.

**Exercise 2.1.8:** Show that for $g \in \mathrm{O}_n(\mathbb{C})$ with polar decomposition $g = up$ both components $u$ and $p$ are contained in $\mathrm{O}_n(\mathbb{C})$. Compare the polar decomposition of $(g^\top)^{-1}$ and $g$. Why is $(p^\top)^{-1} \in \mathrm{Pd}_n(\mathbb{C})$?

**Exercise 2.1.9:** For metric spaces, compactness is equivalent to sequential compactness, which means that every sequence has a convergent subsequence.

**Exercise 2.1.10:** The hermitian form $b(x,y) := \langle A.x, y\rangle$ satisfies the *polarization identity*

$$b(x,y) = \tfrac{1}{4}\big(b(x+y,x+y) - b(x-y,x-y) + ib(x+iy,x+iy) - ib(x-iy,x-iy)\big)$$

for $\mathbb{K} = \mathbb{C}$, and for $\mathbb{K} = \mathbb{R}$ we have

$$b(x,y) = \tfrac{1}{4}\big(b(x+y,x+y) - b(x-y,x-y)\big).$$

**Exercise 2.1.11(2):** Write $A = B + iC$ with $B, C$ hermitian and use Exercise 2.1.10 to show that $C = 0$ if (2) holds.

**Exercise 2.1.13(c):** The subalgebra $\mathbb{K}[A] := \mathrm{span}\{A^k \colon k \in \mathbb{N}_0\} \subseteq \mathrm{End}(V)$ is isomorphic to $\mathbb{K}^n$ with pointwise multiplication and the basis vectors $e_j \in \mathbb{K}^n$ (which are idempotents of this algebra) correspond to the projections onto the eigenspace of $A$.

J. Hilgert, K.-H. Neeb, *Structure and Geometry of Lie Groups*,
Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-84794-8, 

**Exercise 2.1.15:** Interpret invertible $(n \times n)$-matrices as bases of $\mathbb{R}^n$. Use the Gram–Schmidt algorithm to see that $\mu$ is surjective and that it has a continuous inverse map.

**Exercise 2.1.16:** Argue as in the proof of Proposition 2.1.10.

**Exercise 2.2.1:**

(a) Use induction on $\dim V$. If $\beta \neq 0$, then there exists $v_1 \in V$ with $\beta(v_1, v_1) = 1$ (polarization identity). Now proceed with the space

$$v_1^\perp := \{v \in V \colon \beta(v_1, v) = 0\}.$$

(b) Consider the symmetric bilinear form $\beta(x, y) = x^\top By$.

**Exercise 2.2.3:** Pick $v_1 \in V \setminus \{0\}$ and find $w_1 \in V$ with $\beta(v_1, w_1) = 1$. Then consider the restriction $\beta_1$ of $\beta$ to the subspace

$$V_1 := \{v_1, w_1\}^\perp = \{x \in V \colon \beta(x, v_1) = \beta(x, v_2) = 0\}$$

and argue by induction. Why is $\beta_1$ nondegenerate?

**Exercise 2.2.5(2):** Exercise 2.2.4.

**Exercise 2.2.6:** Use the polarization identity $\beta(v, w) = \frac{1}{4}(q(v+w)-q(v-w))$.

**Exercise 2.2.10(b):** Consider the homomorphism

$$\varphi \colon \mathbb{K}^\times \to \mathrm{GL}_n(\mathbb{K}), \quad \lambda \mapsto \mathrm{diag}(\lambda, 1, \ldots, 1).$$

**Exercise 2.2.13(ii):** Consider the characteristic polynomial.

**Exercise 2.2.15(ii):** For $v_\varphi \in V$, take $\alpha_\varphi(w) := \beta(v_\varphi, w)$.

**Exercise 2.3.4:** Consider the eigenvectors with respect to left multiplication.

**Exercise 3.1.1(a):** Choose a basis in each space $X_j$ and expand $\beta$ accordingly.

**Exercise 3.1.3:** Use Exercise 3.1.2(b).

**Exercise 3.2.2:** For $x := \mathbf{1} - g$ the *Neumann series* $y := \sum_{n=0}^\infty x^n$ converges in $M_n(\mathbb{K})$. Show that $y$ is an inverse of $g$.

**Exercise 3.2.5:**

(a) Existence (Jordan normal form), Uniqueness (what can you say about nilpotent diagonalizable matrices?)

(c) If $A$ commutes with $X$, it preserves the generalized eigenspaces of $X$ (Exercise 2.1.1), and this implies that it commutes with $X_s$, which is diagonalizable and whose eigenspaces are the generalized eigenspaces of $X$.

**Exercise 3.2.6(a):** Existence: Put $g_u := \mathbf{1} + g_s^{-1} g_n$.

**Exercise 3.2.7:** Choose a matrix $g \in \mathrm{GL}_n(\mathbb{C})$ for which $A' := gAg^{-1}$ is in Jordan normal form $A' = D + N$ ($D$ diagonal and $N$ strictly upper triangular). Then show that the boundedness of $e^{\mathbb{R}A}$ implies $N = 0$ and the boundedness of the subset $e^{\mathbb{R}D}$.

**Exercise 3.2.8:** Jordan decomposition.

**Exercise 3.2.9:**

(b) Use the Jordan normal form to derive some information on the eigenvalues of matrices of the form $e^x$ which is not satisfied by all elements of $\mathrm{GL}_2(\mathbb{R})_+$. (Either the spectrum is contained in the positive axis or its consists of two mutually conjugate complex numbers.) The matrix $g := \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$ is not contained in the image of exp.

(c) $e^X$ commutes with $X$.

**Exercise 3.2.10:** Use Proposition 3.2.1 and Exercise 3.2.7.

**Exercise 3.2.12:** Use Exercise 3.2.11.

**Exercise 3.3.1:** Write $x \in \mathbb{K}^n$ as a sum $x = \sum_j x_j$, where $Ax_j = \lambda_j x_j$ and calculate $\|Ax\|^2$ in these terms.

**Exercise 3.3.2:** Use the multiplicative Jordan decomposition: Each $g \in \mathrm{GL}_n(\mathbb{C})$ can be written in a unique way as $g = du$ with $d$ diagonalizable and $u$ unipotent with $du = ud$; see also Proposition 3.3.3.

**Exercise 4.1.2(3):** $(\mathbf{1} + tE_{ij})^{-1} = \mathbf{1} - tE_{ij}$.

**Exercise 4.1.3:** $\operatorname{ad} X = \lambda_X - \rho_X$ and both summands commute.

**Exercise 4.1.4:** (1) implies $\exp X \exp Y \exp -X = \exp(e^{\operatorname{ad} X}.Y) = \exp Y$. Now conclude that $e^{\operatorname{ad} X}.Y = Y$ (Proposition 3.3.3) and then use Exercise 4.1.3 and Corollary 3.3.4.

**Exercise 4.1.6:** Use the linearity of the integral to see that every linear functional vanishing on $F$ vanishes on $I_t$. Why does this imply the assertion?

**Exercise 4.1.7:** If $\|\cdot\|_1$ is any norm on $\mathfrak{g}$, then the continuity of the bracket implies that $\|[x,y]\|_1 \leq C\|x\|_1\|y\|_1$. Modify $\|\cdot\|_1$ to obtain $\|\cdot\|$.

**Exercise 4.1.8:** Show that

$$\|x * y\| \leq \|x\| + e^{\|x\|}\|y\| \sum_{k>0} \frac{1}{k+1}\left(e^{\|x\|+\|y\|} - 1\right)^k.$$

**Exercise 5.1.1(iii):** If $A$ is commutative with unit $\mathbf{1}$, then $\operatorname{der}(A_L) = \operatorname{End}(A)$, but every derivation $D$ of $A$ satisfies $D\mathbf{1} = 0$.

**Exercise 5.1.7:** Consider the matrix units $E_{ij}$ with a single nonzero entry 1 in position $(i,j)$ and calculate their brackets.

**Exercise 5.2.1:** Write $V = \bigoplus_{i=1}^n V_i$ with $V_i := V_{\lambda_i}(x)$ for the generalized eigenspace decomposition of $x$ and write accordingly each $A \in \operatorname{End}(V)$ as a matrix $A = (A_{ij})$ with $A_{ij} \in \operatorname{Hom}(V_j, V_i)$. Then show that $\operatorname{ad} x - (\lambda_i - \lambda_j)\mathbf{1}$ acts nilpotently on each space $\operatorname{Hom}(V_j, V_i)$ (cf. Exercise 4.1.3).

**Exercise 5.2.5:** Show that $C^{2m+1}(\mathfrak{a} + \mathfrak{b}) \subseteq C^{m+1}(\mathfrak{a}) + C^{m+1}(\mathfrak{b})$ for each $m \in \mathbb{N}$ by writing each $(2m+1)$-fold bracket of elements of $\mathfrak{a} + \mathfrak{b}$ as a sum of iterated brackets of elements of $\mathfrak{a}$, resp., $\mathfrak{b}$.

**Exercise 5.2.7:** Use a suitable induction.

**Exercise 5.3.2:** Use Exercise 2.1.1(b) to find invariant complements.

**Exercise 5.3.5:** Use Exercises 5.3.2, 5.3.3, and 5.3.4.

**Exercise 5.3.6:** Proceed by induction on the degree of $f$. If $f = g \cdot h$, then either $g(\lambda_1) = 0$ or $h(\lambda_1) = 0$. Then split off one factor $X - \lambda_1$.

**Exercise 5.3.8:** Use the Fitting decomposition (Lemma 5.3.11) to find an $A$-invariant complement of $V^0(A)$ on which $A$ is invertible.

**Exercise 5.4.6:** Consider the polynomials $f_i(t) := \prod_{j \neq i} \frac{t - x_j}{x_i - x_j}$ of degree $n - 1$.

**Exercise 5.5.1:** Example 5.4.2(iv) and Theorem 5.5.11.

**Exercise 5.5.5(ii):** Corollary 5.2.7 and Exercise 5.1.9.

**Exercise 5.5.6:** Use Exercise 5.5.5 to see that $\mathrm{ad}\, x\, \mathrm{ad}\, y$ is nilpotent for $x \in \mathfrak{g}$ and $y \in \mathfrak{n}$.

**Exercise 5.5.7:** Proceed by the following steps: Let $\mathfrak{a} := [\mathfrak{g}, \mathfrak{g}]^\perp$.

(i) $[\mathfrak{a}, \mathfrak{a}] \subseteq \mathfrak{g}^\perp$.
(ii) Apply the Cartan criterion 5.4.20 to $[\mathfrak{a}, \mathfrak{a}]$ in order to show that $\mathfrak{a}$ is a solvable ideal.
(iii) $\mathfrak{a} \subseteq \mathfrak{rad}(\mathfrak{g})$.
(iv) $[\mathfrak{rad}(\mathfrak{g}), \mathfrak{g}] \subseteq \mathfrak{g}^\perp$ (Corollary 5.4.15 and Exercise 5.5.6).
(v) $\mathfrak{rad}(\mathfrak{g}) \subseteq \mathfrak{a}$.

**Exercise 5.5.8:** Use that $\mathfrak{gl}(V)$ is abelian.

**Exercise 5.5.10:**

(i) Consider the eigenspaces of the linear endomorphism $\varphi \in \mathrm{End}(\mathfrak{g})$ defined by $\kappa(x, y) = \kappa(\varphi(x), y)$ for all $x, y \in \mathfrak{g}$. They are ideals in $\mathfrak{g}$.
(ii) Consider a simple Lie algebra over $\mathbb{C}$, viewed as a real Lie algebra. It is simple, but its complexification is not.

**Exercise 5.6.2:** Use a Levi decomposition of $\mathfrak{g}$.

**Exercise 6.1.2:** Consider the Taylor expansion of $p$ in some point $x \in U$.

**Exercise 6.1.3:** For $p(x), p(y) \neq 0$, consider the affine line $x + \mathbb{C}(y - x)$ spanned by $x$ and $y$ and show that it intersects $p^{-1}(\mathbb{C}^\times)$ in a connected set.

**Exercise 6.1.4:** Show that the Taylor expansion of $p$ vanishes if $p$ vanishes on $V$.

**Exercise 6.2.1:** Proceed along the following steps:

(i) If $V$ is generated by $v_0 \in V_\lambda(h)$, then there exists a basis $(v_0, \ldots, v_n)$ of $V$ with

$$h \cdot v_k = (\lambda + k) v_k \quad \text{and} \quad e \cdot v_k = \begin{cases} v_{k+1} & \text{if } k < n, \\ 0 & \text{if } k = n. \end{cases}$$

We write $V(\lambda, n)$ for the $(n+1)$-dimensional $\mathfrak{g}$-module, defined by these relations.

(ii) If $k \leq n$, then $V(\lambda + k, n - k)$ is a submodule of $V(\lambda, n)$.

(iii) Use Lie's Theorem to show that each simple finite-dimensional $\mathfrak{g}$-module is isomorphic to some $V(\lambda, 0)$.

(iv) For each finite-dimensional representation $(\rho, V)$ of $\mathfrak{g}$, the operator $\rho(e)$ is nilpotent and for each $n$ the subspaces $\ker\big(\rho(e)^n\big)$ and $\operatorname{im}\big(\rho(e)^n\big)$ are invariant under $\rho(h)$, hence $\mathfrak{g}$-submodules.

(v) Show that each finite-dimensional representation $(\rho, V)$ for which $\rho(h)$ is diagonalizable is a direct sum of modules of the form $V(\lambda, n)$. Hint: Derive a Jordan normal form of $\rho(e)$, adapted to the eigenspace decomposition of $\rho(h)$.

**Exercise 7.1.1:**

(ii) Let $p, q, z, h$ be the basis in Example 5.1.19, then put

$$\beta(ap + bq + cz + dh, a'p + b'q + c'z + d'h) = aa' + bb' + cd' + c'd.$$

(v) With (iii) and (iv), conclude that $\Delta$ lies in the center of the associative subalgebra of $\operatorname{End}(C^\infty(\mathbb{R}^n))$ generated by the angular momentum operators.

**Exercise 7.2.2:** Consider the semidirect product $\mathfrak{g} = \mathbb{K}[X] \rtimes_M \mathbb{K}$, where $\mathbb{K}[X]$ is considered as an abelian Lie algebra and $Mf(X) := Xf(X)$ is the multiplication with $X$.

**Exercise 7.4.3(i):** Universal property of $\mathcal{U}(\mathfrak{g})$.

**Exercise 7.5.1(b):** If $\dim V = \infty$, then $V$ contains a copy of the polynomial algebra $\mathbb{K}[X]$. Then consider the operators

$$P(f) = f' \quad \text{and} \quad Q(f) = Xf.$$

If $\operatorname{char}(\mathbb{K})$ divides $n := \dim V < \infty$, then we think of $V$ as $\mathbb{K}[X]/(X^n)$. Since $P(X^n) = nX^{n-1} = 0$, both operators $P$ and $Q$ preserve the ideal $(X^n)$ and induce operators on $\mathbb{K}[X]/(X^n)$ with $[P, Q] = \mathbf{1}$.

**Exercise 7.5.5(ii):** Show that the $\mathfrak{sl}_2(\mathbb{R})$-module $\mathbb{R}^2 \otimes \mathbb{R}^2$ is isomorphic to the $\mathfrak{sl}_2(\mathbb{R})$-module $\mathfrak{gl}_2(\mathbb{R})$ with $x \cdot y := [x, y]$.

**Exercise 8.2.1:** Consider the two charts from Remark 8.2.2(b) and the chart $(\zeta, W)$ with $\zeta(x) = x$ and $W = ]1, 2[$.

**Exercise 8.2.2:** Add all $n$-dimensional charts which are $C^k$-compatible with the atlas.

**Exercise 8.2.18(2):** $\mathtt{d}f(x)y = y^\top Bx + x^\top By = z$ can be solved with the Ansatz $y := \frac{1}{2}xz$.

**Exercise 8.3.1(5):** Use (4) to separate points in different tangent spaces by disjoint open sets.

**Exercise 9.1.4(4):** Apply the Inverse Function Theorem to the map

$$\Phi\colon G \times G \to G \times G, (x, y) \mapsto (x, xy).$$

**Exercise 9.1.5(2):** For $a \in A^\times$ we have $\lambda_{a^{-1}} = \lambda_a^{-1}$.

**Exercise 9.2.9:** Apply the Uniqueness Lemma to functions of the form $f \circ \lambda_x$, $x \in G$.

**Exercise 9.2.10:** Apply the Uniqueness Lemma to functions of the form $f \circ \lambda_x$, $x \in G$.

**Exercise 9.2.11:** Every skew-symmetric matrix $x \in \mathfrak{su}_2(\mathbb{C})$ is conjugate to a diagonal matrix $\lambda\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

**Exercise 9.2.13:** Use Exercise 9.2.12.

**Exercise 9.3.3(b):** Let $g \in \overline{H}$ and $U$ an open **1**-neighborhood in $G$ for which $U \cap H$ is closed. Show that:

(1) $g \in HU^{-1}$, i.e., $g = hu^{-1}$ with $h \in H$, $u \in U$.
(2) $\overline{H}$ is a subgroup of $G$.
(3) $u \in \overline{H} \cap U = \overline{H \cap U} = H \cap U$.
(4) $g \in H$.

**Exercise 9.3.4:** Use induction on $\dim \operatorname{span} D$.

(1) Show that $D$ is closed.
(2) Show that we may w.l.o.g. assume that $\operatorname{span} D = \mathbb{R}^n$.
(3) Every compact subset $C \subseteq \mathbb{R}^n$ intersects $D$ in a finite subset.
(4) Assume that $\operatorname{span} D = \mathbb{R}^n$ and assume that there exists a basis $f_1, \ldots, f_n$ of $\mathbb{R}^n$, contained in $D$, such that the hyper-plane $F := \operatorname{span}\{f_1, \ldots, f_{n-1}\}$ satisfies $F \cap D = \mathbb{Z}f_1 + \cdots + \mathbb{Z}f_{n-1}$. Show that

$$\delta := \inf\left\{\lambda_n > 0\colon (\exists \lambda_1, \ldots, \lambda_{n-1} \in \mathbb{R})\ \sum_{i=1}^n \lambda_i f_i \in D\right\} > 0.$$

Hint: It suffices to assume $0 \leq \lambda_i \leq 1$ for $i = 1, \ldots, n$ and to observe (4).
(5) Apply induction on $n$ to find $f_1, \ldots, f_n$ as in (4) and pick $f_n' := \sum_{i=1}^n \lambda_i f_i \in D$ with $\lambda_n = \delta$. Show that $D = \mathbb{Z}f_1 + \cdots + \mathbb{Z}f_{n-1} + \mathbb{Z}f_n'$.

**Exercise 9.3.6:**

(1) Use Zorn's Lemma to reduce the situation to the case where $G$ is generated by $H$ and one additional element.
(2) Extend $\mathrm{id}_D\colon D \to D$ to a homomorphism $f\colon G \to D$ and define $H := \ker f$.

**Exercise 9.4.4:** For $\gamma \in \Gamma$, consider the map $G \to \Gamma, g \mapsto g\gamma g^{-1}$.

**Exercise 9.4.6:** $\alpha$ is the unique lifting of $\gamma : [0,1] \to \mathrm{SL}_2(\mathbb{R}), t \mapsto e^{t2\pi u}$. If $\alpha$ is not injective and $\alpha(n) = \mathbf{1}$, then $q \circ \alpha|_{[0,n]}$ homotopic to the zero map, contradicting to the relation $[q \circ \alpha|_{[0,n]}] = n[\gamma]$ in $\pi_1(\mathrm{SL}_2(\mathbb{R})) \cong \mathbb{Z}$.

**Exercise 9.5.2(3):** Consider $q_G \colon \mathbb{R} \to \mathbb{T}, x \mapsto e^{2\pi ix}$.

**Exercise 9.6.1:** Show first that for each $y \in K_Y$ there exists an open subset $U_y$ of $Y$ with $K_X \times U_y \subseteq V$, then use the compactness of $Y$.

**Exercise 10.2.1:** S Note that $\widetilde{\omega}(X) = \omega \circ X$ if we consider $\omega$ as a function on $TM$.

**Exercise 10.2.2:** Use Theorems 8.4.18 and 10.2.23, as well as the fact that the double dual of a finite-dimensional vector space is isomorphic to this vector space.

**Exercise 10.2.5:** See Remark 8.3.25 for the case $\mathcal{T}^{0,1}(T_pM)$.

**Exercise 10.2.9(1):** For each $t \in \mathbb{R}$, the map $\Phi_t$ is affine and the translation part is $\frac{e^{tA}-\mathbf{1}}{A}b$.

**Exercise 10.4.3:** If $0 \leq f \in C_c(G)$, then $\mu(fh^2) = 0$, too. Now, for any $g \in G$, use Corollary 10.3.28 to find a function $0 \leq f \in C_c(G)$ with $f(g) = 1$. Then use Remark 10.3.34.

**Exercise 11.2.1(iii):** Identify $\mathbb{C}$ with $\mathbb{R}^2$ and define the homomorphism $\beta \colon G \to \mathrm{Mot}_2(\mathbb{R})$ by $\alpha(z,t)(x) := e^{it}x + z$.

**Exercise 12.1.4:** Pick a nonzero element $x \in \mathfrak{g}$ and note that $\mathbb{C}x$ is a Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{C})$. Then use the corresponding root decomposition and the complex conjugation with respect to the real form $\mathfrak{g}$ of $\mathfrak{sl}_2(\mathbb{C})$.

**Exercise 12.2.2(ii):** Every element of the Lie algebra $\mathfrak{g}$ is represented by a skew-symmetric matrix $E$ and $E^2$ is negative semi-definite. Now apply (i).

**Exercise 12.2.4:** Lemma 12.2.1 and Exercise 12.2.3.

**Exercise 12.2.5:** Main Theorem on Maximal Tori and Exercise 12.2.3.

**Exercise 12.2.9(c):** Lemma 12.1.14.

**Exercise 12.2.10:**

(b) Consider a limit point $g$ of the sequence $\exp(nx)$ with $\exp(m_k x) \to g$. Then $\exp((m_{k+1} - m_k)x) \to \mathbf{1}$.
(c) For $n \in \mathbb{N}$ and $\exp(n_k x) \to \mathbf{1}$, we have $\exp((n_k - n)x) \to \exp(-nx)$.

**Exercise 12.3.1:**

(i) Show that $\|g_i v - gv\|^2 = 2\|v\|^2 - 2\,\mathrm{Re}\langle g_i v, gv\rangle$.
(ii) Show first that $g_i \to g$ and $h_i \to h$ implies that $g_i^{-1}h_i \to g^{-1}h$ holds weakly.

**Exercise 12.4.1:** If $G = U \cup V$ is a decomposition into disjoint open sets, then $UH = U$, $VH = V$, and the images of these sets give a decomposition of $G/H$ into disjoint open sets.

**Exercise 13.1.2:** Write $e^{\operatorname{ad} x} y = \cosh(\operatorname{ad} x)y + \sinh(\operatorname{ad} x)y$ and show that $\sinh(\operatorname{ad} x)y = 0$ implies $[y, x] = 0$. Use that $\operatorname{ad} x$ is diagonalizable with real eigenvalues.

**Exercise 13.1.3:** The critical implication is to see that if $\mathfrak{g}$ is complex and semisimple, then any real solvable ideal $\mathfrak{r}$ is trivial, which follows from the solvability of the complex ideal $\mathfrak{r} + i\mathfrak{r}$.

**Exercise 13.1.4:** Consider the generalized eigenspaces of elements $A \in \operatorname{End}_S(V)$ and $\ker(A - \lambda\mathbf{1})^n$ for $n \in \mathbb{N}$.

**Exercise 13.2.1:** Choose a complex basis $B = (b_1, \ldots, b_n)$ for $V$ to write it as $W \oplus iW$ for $W := \operatorname{span}_{\mathbb{R}} B$. Then $A^{\mathbb{R}}$ can be represented by a block matrix

$$\begin{pmatrix} C & -D \\ D & C \end{pmatrix} \in M_2\big(M_n(\mathbb{R})\big).$$

Now verify that $\operatorname{tr}_{\mathbb{R}}(A^{\mathbb{R}}) = 2\operatorname{tr}(C)$ and $\operatorname{tr}_{\mathbb{C}}(A) = \operatorname{tr}(C) + i\operatorname{tr}(D)$.

**Exercise 13.2.5:** Given a representation $\pi\colon \mathfrak{g} \to \operatorname{End}(V)$ of a semisimple Lie algebra $\mathfrak{g}$, extend it to $\mathfrak{g}_{\mathbb{C}}$, then restrict to a compact real form $\mathfrak{u}$, and finally lift the resulting representation of $\mathfrak{u}$ to the simply connected group $U$ with Lie algebra $\mathfrak{u}$.

**Exercise 13.3.1:** $x^2 = -(\det x)\mathbf{1} = k(x)\mathbf{1}$ for all $x \in \mathfrak{sl}_2(\mathbb{R})$ (Cayley–Hamilton).

**Exercise 13.3.3(e):** Exercise 13.3.1 and (b).

**Exercise 13.3.4(b):** Choose $a_0 := h \in \mathfrak{a}$ to define $\Delta^+$.

**Exercise 14.2.7:** Use Exercise 14.2.6 for one direction and argue for the converse direction that all compositions factors $D^k(\Gamma)/D^{k+1}(\Gamma)$ of the derived series are finitely generated abelian groups, hence polycyclic.

**Exercise 14.4.3:**

(c) Consider the simply connected group $G$ with $\mathbf{L}(G) = \mathfrak{g}$ and use that the exponential function of $G/\langle \exp_G \mathfrak{a}\rangle$ is bijective.
(d) Compute the exponential function as in Lemma 14.4.7.
(e) Show that $\exp_G$ is injective if $G$ is simply connected. Assuming the contrary, there exists an $x \in \mathfrak{g}$ such that $\exp_G$ is not injective on the subalgebra $\mathfrak{b} := \mathfrak{a} \rtimes \mathbb{R}x$. By (c), $\mathfrak{b}$ is exponential, and one thus obtains a contradiction to (d).

**Exercise 14.4.4:** Exercise 14.4.1.

**Exercise 14.4.5:** Exercise 14.4.1.

**Exercise 14.5.1:** Corollary 14.5.5 and the proof of Theorem 14.2.7.

**Exercise 15.1.1:** Exercise A.1.3.

**Exercise 15.1.2(a):** Consider the antiholomorphic involution $\sigma$ on $\widetilde{G}_{\mathbb{C}}$ to see that $\eta_G(D)$ is closed.

**Exercise 15.1.4:** Consider the group $G := (\widetilde{\mathrm{SL}_2(\mathbb{R})} \times \mathbb{R})/D$, where

$$D = \big\langle (\exp \pi u, 1), \big(\exp(-\pi u), \sqrt{2}\big) \big\rangle \quad \text{and} \quad u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R}).$$

In this case, $\widetilde{G} = \mathrm{SL}_2(\mathbb{R})\widetilde{\ } \times \mathbb{R}$, $\widetilde{G}_{\mathbb{C}} \cong \mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}$, and we have

$$\eta_G(D) = \big\langle (-\mathbf{1}, 1), (-\mathbf{1}, \sqrt{2}) \big\rangle \supseteq \{0\} \times (\mathbb{Z} + \sqrt{2}\mathbb{Z}) \quad \text{in} \quad Z(\widetilde{G}_{\mathbb{C}}) = \{\pm \mathbf{1}\} \times \mathbb{C}.$$

**Exercise 15.1.5:** $\mathrm{Ad}_G(D) = \{\mathrm{id}_{\mathbf{L}(G)}\}$.

**Exercise 15.1.8:** Note that $h \in Z(H)$ implies that $\mathrm{Ad}(h)|_{\mathbf{L}(H)} = \mathrm{id}_{\mathbf{L}(H)}$, which implies $h \in Z_H(G_0)$.

**Exercise 16.2.1:**

(a) Lemma 14.3.3 and the construction of $\mathfrak{b}$ in the proof of Proposition 16.2.4.
(c) Corollary 5.6.9, Remark 7.4.7, and Lemma 14.3.5.

**Exercise 16.2.2:** $\mathfrak{t}$ contains the center and is closed by Lemma 14.2.6.

**Exercise 16.2.3:** Without loss of generality, $G$ may assumed to be connected and abelian. Then use Exercise 9.3.5.

**Exercise 16.2.4:** Exercise 16.2.3.

**Exercise 16.2.5:** Corollary 14.5.6(e) and Exercise 16.2.4.

**Exercise 16.2.6:** Use Corollary 16.2.8 to find a faithful representation of $G/G'$ with closed range. To this end, it is useful to show that

$$G/G' \cong B/(B \cap G') \times H/H',$$

where $B/(B \cap G')$ is a vector group and $H/H'$ is a torus. This follows from $G' \cong (B \cap G') \rtimes H'$.

**Exercise 16.2.7:** If this is not the case, then, by Exercise 16.2.5, one finds an $x \in \mathfrak{a} \cap \mathfrak{b}$ such that $\varphi(\exp_G \mathbb{R}x)$ is not contained in $\varphi(G)$. By Exercise 16.2.6, $x \in \mathfrak{a} \cap \mathfrak{b} \cap \mathfrak{g}'$. But this element lies in the center of the maximal nilpotent ideal (Exercise 16.2.1(c)).

**Exercise 17.2.1(a):** In $\mathrm{GL}_n(\mathbb{C})$, we have

$$\begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & i\mathbf{1}_q \end{pmatrix} \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix} \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & i\mathbf{1}_q \end{pmatrix}^\top = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & \mathbf{1}_q \end{pmatrix}.$$

**Exercise 17.4.4(ii):** Since $\mathtt{d}q(v) = 2\beta(v, \cdot)$, the singularity of $q$ in $v \in H$ implies that $2\beta(v, \cdot) \in \mathbb{K}^\times \alpha$ because $\beta$ is nondegenerate. Now $\alpha(v) = 1$ leads to $\beta(v, v) \neq 0$, so that all points in the 0-level set of $q$ are regular.

**Exercise 18.3.2(e):** Use (b) and Exercise 18.3.1.

**Exercise A.1.2:** Exercise A.1.1 helps to glue homotopies.

**Exercise A.1.3:**

(c) Consider $H(t, s) := \frac{(1-s)\gamma(t) + s\alpha(t)}{\|(1-s)\gamma(t) + s\alpha(t)\|}$.

(e) Let $p \in \mathbb{S}^n \setminus \operatorname{im} \beta$. Using stereographic projection, where $p$ corresponds to the point at infinity, show that $\mathbb{S}^n \setminus \{p\}$ is homeomorphic to $\mathbb{R}^n$, hence contractible.

**Exercise A.1.5:** Mimic the argument in the proof of Lemma A.1.8.

**Exercise A.2.1:** Consider the map

$$G\colon I^2 \to I^2, \quad G(t,s) := \begin{cases} (2t, s) & \text{for } 0 \leq t \leq \frac{1}{2}, s \leq 1-2t, \\ (1, 2t-1) & \text{for } \frac{1}{2} \leq t \leq 1, s \leq 2t-1, \\ (t + \frac{1-s}{2}, s) & \text{else} \end{cases}$$

and show that it is continuous. Take a look at the boundary values of $F \circ G$.

**Exercise B.1.2(iii):** Use (ii) and collect suitable terms. Conclude that

$$V \otimes W \cong \bigoplus_{i \in I} e_i \otimes W \cong W^{(I)}.$$

**Exercise B.3.1:** Use induction on the dimensional on $V$ and compose $g \in \mathrm{O}(V, \beta)$ with a suitable reflection to set up the induction.

**Exercise B.3.3:** For the third one use the map

$$f\colon \mathbb{H} \times \mathbb{H} \to \mathrm{End}_{\mathbb{R}}(\mathbb{H}), \; f(x,y)(z) := xzy^*.$$

**Exercise B.3.7:** Use (B.9) and the fact that each element of $\mathrm{Spin}(V, \beta)$ is a product of elements of $V^\times$.

**Exercise C.3.3:** It suffices to show that the image of the unit ball is bounded.

# References


[Ad96] Adams, J. F., "Lectures on Exceptional Lie Groups." Edited by Z. Mahmud and M. Mimura, Chicago Lectures in Mathematics, 1996

[ABS64] Atiyah, M. F., R. Bott and A. Shapiro, *Clifford modules*, Topology **3**, suppl. 1 (1964), 3–38

[Bak01] Baker, H. F., *On the exponential theorem for a simply transitive continuous group, and the calculation of the finite equations from the constants of structure*, J. London Math. Soc. **34** (1901), 91–127 contains: BCH formula for the adjoint representation

[Bak05] Baker, H. F., *Alternants and continuous groups*, Lond. M. S. Proc. (2) **3** (1905), 24–47

[BN08] Beltita, D., K.-H. Neeb, *Finite-dimensional Lie subalgebras of algebras with continuous inversion*, Studia Math. **185:3** (2008), 249–262

[Bl73] Blattner, R. J., Quantization and representation theory, in "Harmonic analysis on homogeneous spaces" (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972), pp. 147–165, Amer. Math. Soc., Providence, RI, 1973

[BG81] Borel, A., "Essays in the History of Lie Groups and Algebraic Groups," Amer. Math. Soc., Providence, 2001

[BFM02] Borel, A., R. Friedman, and J. W. Morgan, Almost commuting elements in compact Lie groups, Mem. Amer. Math. Soc. **157** (2002), no. 747, x + 136pp

[Bou70] Bourbaki, N., "Algèbre," Hermann, Paris, 1970

[Bou89] Bourbaki, N., "Lie Groups and Lie Algebras, Chap. 1–3", Springer, Berlin, 1989

[Bou82] Bourbaki, N., *Groupes et algèbres de Lie, Chapitres 9*, Masson, Paris, 1982

[Br36] Brauer, R., *Eine Bedingung für vollständige Reduzibilität von Darstellungen gewöhnlicher und infinitesimaler Gruppen*, Math. Zeitschr. **61** (1936), 330–339

[Cam97] Campbell, J. E., *On a law of combination of operators bearing on the theory of continuous transformation groups*, Proc. of the London Math. Soc. **28** (1897), 381–390

[Cam98] Campbell, J. E., *On a law of combination of operators. (Second paper)*, Proc. of the London Math. Soc. **29** (1898), 14–32



J. Hilgert, K.-H. Neeb, *Structure and Geometry of Lie Groups*,
Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-84794-8,

[Ca94] Cartan, É., "Sur la structure des groupes de transformations finis et continue", thèse, Paris, Nony, 1894

[Ca30] Cartan, É., "La théorie des des groupes finis et continus et l'analysis situs", Mém. Sci. Math. XLII, Gauthier-Villars, Paris, 1930

[CW35] Casimir, H., and B. L. van der Waerden, *Algebraischer Beweis der vollständigen Reduzibilität der Darstellungen halbeinfacher Liescher Gruppen*, Math. Ann. **111** (1935), 1–12

[Ch46] Chevalley, C. "Theory of Lie grooups I", Princeton Univ. Press, Princeton, 1946

[CE48] Chevalley, C. and S. Eilenberg, *Cohomology Theory of Lie Groups and Lie Algebras*, Transactions of the Amer. Math. Soc. **63** (1948) 85–124

[Dix57] Dixmier, J., *L'application exponentielle dans les groupes de Lie résolubles*, Bull. Soc. Math. Fr. 85 (1957), 113–121

[DH97] Djokovic, D., and K. H. Hofmann, *The surjectivity question for the exponential function in real Lie groups: A status report*, J. Lie Theory **7** (1997), 171–199

[Do07] Donaldson, S. K., *Lie algebra theory without algebra*, arXiv: math.DG/0702016v2, 15 Mar 2007

[DK00] Duistermaat, J. J., and J. A. C. Kolk, "Lie groups". Springer, Berlin, 2000

[Dyn47] Dynkin, E. B., *Calculation of the coefficients in the Campbell–Hausdorff formula* (Russian), Doklady Akad. Nauk. SSSR (N.S.) **57** (1947), 323–326

[Dyn53] Dynkin, E. B., "Normed Lie Algebras and Analytic Groups," Amer. Math. Soc. Translation 1953, no. 97, 66 pp

[GW09] Goodman, R., and N. R. Wallach, "Symmetry, Representations, and Invariants," Springer-Verlag, New York, 2009.

[GK96] Goze, M., and Y. Khakimdjanov, "Nilpotent Lie Algebras," Mathematics and Its Applications, Kluwer Acad. Pub., Dordrecht, 1996

[GHV73] Greub, W., Halperin, S., and R. Vanstone, "Connections, Curvature, and Cohomology," Academic Press, New York and London, 1973

[Gr01] Grove, L. C., "Classical Groups and Geometric Algebra," Amer. Math. Soc., Providence, 2001

[dlH00] de la Harpe, P., "Topics in Geometric Group Theory," Chicago Lectures in Math., The Univ. of Chicago Press, Chicago, 2000

[Hau06] Hausdorff, F., *Die symbolische Exponentialformel in der Gruppentheorie*, Leipziger Berichte **58** (1906), 19–48

[Haw00] Hawkins, Th., "Emergence of the Theory of Lie Groups," Springer, Berlin, 2000

[He01] Helgason, S., "Differential Geometry, Lie Groups, and Symmetric Spaces", AMS Graduate Studies in Mathematics **34**, Providence, 2001

[HN91] Hilgert, J., and K.-H. Neeb, "Lie-Gruppen und Lie-Algebren", Vieweg, Braunschweig, 1991

[Hir76] Hirsch, M. W., "Differential Topology," Graduate Texts in Mathematics **33**, Springer-Verlag, Berlin, 1976

[Ho54a] Hochschild, G., *Lie algebra kernels and cohomology*, Amer. J. Math. **76** (1954), 698–716

[Ho54b] Hochschild, G., *Cohomology classes of finite type and finite-dimensional kernels for Lie algebras*, Amer. J. Math. **76** (1954), 763–778

[Ho65] Hochschild, G., "The Structure of Lie Groups, " Holden Day, San Francisco, 1965

[HM06] Hofmann, K. H., and S. A. Morris, "The Structure of Compact Groups," 2nd ed., Studies in Math., de Gruyter, Berlin, 2006

[Kil89] Killing, W., *Die Zusammensetzung der stetigen endlichen Transformationsgruppen II*, Math. Ann. **33** (1889), 1–48

[Kn02] Knapp, A. W., "Lie groups beyond an introduction". Second edition. Progress in mathematics **140**, Birkhäuser, Boston, 2002

[Ko95] Kobayashi, S., "Transformation Groups in Differential Geometry". Springer, Berlin, 1995

[Kos50] J.-L. Koszul, *Homologie et cohomologie des algèbres de Lie*, Bull. Soc. Math. France **78** (1950), 65–127

[KM97] Kriegl, A., and P. W. Michor, "The Convenient Setting of Global Analysis". AMS Surveys and Monographs **53**, AMS, Providence, 1997

[Lai77] Lai, H. L., *Surjectivity of the exponential map on semisimple Lie groups*, J. Math. Soc. Japan **29** (1977), 303–325

[Lai78] Lai, H. L., *Exponential map of a center-free complex simple Lie group*, Osaka J. Math. **27** (1978), 553–560

[La93] Lang, S., "Algebra," Addison Wesley Publ. Comp., London, 1993

[La99] Lang, S., "Fundamentals of Differential Geometry", Grad. Texts Math. **191**, Springer-Verlag, Berlin, 1999

[Le05] Levi, E. E., *Sulla structura dei gruppi finiti e continui*, Atti Accad. Torino **40** (1905), 3–17

[LV94] D. Luminet and A. Valette, *Faithful uniformly continuous representations of Lie groups*, J. London Math. Soc. (2) **49** (1994), 100–108

[ML63] MacLane, S., "Homological Algebra, " Springer-Verlag, Berlin, 1963

[Ma42] Malcev, A. I., *On the representation of an algebra as a direct sum of the radical and a semi-simple subalgebra*, Dokl. Akad. Nauk SSSR, **36:2** (1942), 42–45

[Mi03] Milnor, J., *Towards the Poincaré conjecture and the classification of 3-manifolds*, Notices Amer. Math. Soc. **50:10** (2003), 1226–1233

[Mo66] Morimoto, A., *On the classification of non-compact abelian Lie groups*, Trans. of the Amer. Math. Soc. **123** (1966), 200–228

[MS08] Moskowitz, M., and R. Sacksteder, *On complex exponential groups*, Math. Res. Lett. **15** (2008), 1197–1210.

[Na97] Nahlus, N., *Note on faithful representations and a local property of Lie groups*, Proc. Amer. Math. Soc. **125:9** (1997), 2767–2769

[Ne07] Neeb, K.-H., *Nonabelian extensions of infinite-dimensional Lie groups*, Ann. Inst. Fourier **56** (2007), 209–271

[Ner02] Neretin, Y. A., *A construction of finite-dimensional faithful representation of Lie algebra*, in Proceedings of the 22nd Winter School "Geometry and Physics" (Srni 2002), Rend. Circ. Mat. Palermo (2) Suppl. **71** (2003), 159–161, arXiv:math/0202190 [math.RT]

[Ra72] Raghunathan, M. S., "Discrete Subgroups of Lie Groups," Ergebnisse der Math. **68**, Springer, Berlin, 1972

[RV81] Raymond, F., and A. T. Vasquez, *3-manifolds whose universal coverings are Lie groups*, Topology and its Appl. **12** (1981), 161–179

[Ru86] Rudin, W., "Real and Complex Analysis," McGraw Hill, New York, 1986

[Sai57] Saito, M., *Sur certains groupes de Lie résolubles*, Sci. Pap. Coll. Gen. Educ. Univ. Tokyo **7** (1957), 1–11.

[Sch85] Scharlau, W., "Quadratic and Hermitian Forms," Grundlehren der math. Wiss. **270**, Springer-Verlag, Berlin, 1985

[Sch95] Schottenloher, M., "Geometrie und Symmetrie in der Physik," Vieweg Lehrbuch Mathematische Physik, Vieweg Verlag, Braunschweig, 1995

[Seg83] Segal, D., "Polycyclic Groups, " Cambridge Univ. Press, Cambridge, 1983

[Se88] Serre, J.-P., "Cours d'arithmétique", Le Mathématicien, Presse Universitaire de France, 3[e] edition, 1988

[Wa88] Wallach, N. R., "Real reductive groups I", Academic Press, Boston, 1988

[We95] Weibel, C. A., "An Introduction to Homological Algebra", Cambridge Studies in Advanced Math. **38**, Cambridge Univ. Press, Cambridge, 1995

[We25] Weyl, H., *Theorie der Darstellung kontinuierlicher halbeinfacher Gruppen durch lineare Transformationen.* I, II, III und Nachtrag. Math. Zeitschr. **23** (1925), 271–309; **24** (1926), 328–376, 377–395, 789–791

[Wh36] Whitehead, J. H. C., *On the decomposition of an infinitesimal group*, Proc. Camb. Philos. Soc. **32** (1936), 229–237

[Wie49] Wielandt, H., *Über die Unbeschränktheit der Operatoren der Quantenmechanik*, Math. Ann. **121** (1949), 21

[Wu98] Wüstner, M., *On the surjectivity of the exponential function of solvable Lie groups*, Math. Nachrichten **192** (1998), 255–266

[Wu03] Wüstner, M., *Supplements on the theory of exponential Lie groups*, J. Algebra **265** (2003), 148–170.

[Wu05] Wüstner, M., *The classification of all simple Lie groups with surjective exponential map*. J. Lie Theory **15** (2005), 269–278.

# Index




J. Hilgert, K.-H. Neeb, *Structure and Geometry of Lie Groups*,
Springer Monographs in Mathematics,
DOI 10.1007/978-0-387-84794-8, 