A. The $P = NP$ Hypothesis

Roughly, there are two kinds of solvable problems that are not too difficult–those that can be solved easily and those that can be solved with difficulty. In particular, when we say that a problem is difficult, we mean that (i) if there is an algorithmic solution to this problem we can quickly check it and (ii) an algorithmic solution will require an impossibly long time to yield an output. Let us denote by E the class of problems that can be solved easily and by D the class of problems that can be solved with difficulty. Then it is interesting to see whether these two classes are equal (i.e., whether $E = D$). Although it seems obvious to state that $E \neq D$, this is a long-standing open problem of computer science and mathematics. The problem is not directly connected to hypercomputation, but an affirmative answer to the problem (i.e., a proof that $P = NP$, which is the "real" name of the problem) will have a great impact on computer science and consequently on hypercomputation.

Da Costa and Doria have obtained some interesting results concerning the $P = NP$ problem. However, one cannot fully appreciate their importance without understanding the $P = NP$ problem. Thus, in the next two paragraphs I will briefly present the relevant theory. As usual, readers familiar with this theory can safely skip the next two paragraphs.

A Turing machine that has more than one next state for some combinations of the symbol just read and the current state is called a nondeterministic Turing machine. A polynomial-time Turing machine is a Turing machine that produces output in polynomially bounded time t (i.e., a machine that always halts after at most $p(n)$ steps, where n is the length of the input and $p(n)$ is a polynomial; see footnote on page 80). The class of decision problems that can be solved by a polynomially bounded deterministic Turing machine is denoted by P. Also, the class of decision problems that can be solved by a polynomially bounded nondeterministic Turing machine is denoted by NP. The $P = NP$ hypothesis can be precisely specified in terms of the Boolean satisfiability problem, or SAT problem for short; therefore, we need to explain SAT.

Assume that $X = \{x_1, x_2, \ldots, x_n\}$ is a finite set of Boolean variables and $\overline{X} = \{\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}\}$, where $\overline{x_1}$, $\overline{x_2}$,..., $\overline{x_n}$ are new symbols standing for the negations of x_1, x_2,..., x_n. The set $X \cup \overline{X}$ is called the set of (positive and negative) literals. A clause C is a nonempty set of literals, that is, $C \subseteq X \cup \overline{X}$. A Boolean formula in conjunctive normal form is a set of clauses defined on X. A truth assignment for a formula F is a function from X to the set $\{tt, ff\}$. A truth assignment T satisfies F if for all $C \in F$ there is at least one variable x_i such that either $T(x_i) = tt$, where $x_i \in C$, or $T(x_i) = ff$, where $\overline{x_i} \in C$. A formula F is called *satisfiable* if there is a truth assignment that satisfies it. More generally, we denote by SAT the following problem: given a Boolean formula F in conjunctive normal form, is it satisfiable?

According to da Costa and Doria, the $P = NP$ hypothesis can be (informally) stated as follows: There is a polynomial-time Turing machine $\mathscr{M}_m$, where m is its Gödel number, such that it correctly "guesses" a satisfying line of truth values for every input $x \in$ SAT. The authors start from an exotic formulation of the $P = NP$ hypothesis, which is consistent with ZFC, and from it they derive a consistency result for $P = NP$. More specifically, if we assume that f is a strictly increasing total recursive function with one argument, then we make the following hypothesis.

Hypothesis A.1 There is a Turing machine $\mathscr{M}_m$, with Gödel number m, and natural numbers a, b such that for every $x \in$ SAT, the output $\Psi^1_{\mathscr{M}}(x)$ is a satisfying line for x, and the number $t_m(x)$ of steps performed by the machine with input x is at most $|x|^{\mathsf{f}(a)} + \mathsf{f}(b)$, where $|x|$ denotes the length of the input in bits.

This hypothesis will be written $[P = NP]^{\mathsf{f}}$, while the standard formulation of the $P = NP$ hypothesis will be written $[P = NP]$. Using a special function F, da Costa and Doria have proved the following result (see [43] for more details and [42] for a discussion of earlier results).

Proposition A.1

(i) If ZFC *has a model with standard arithmetic, the equivalence*

$$[P = NP]^{\mathsf{F}} \leftrightarrow [P = NP]$$

holds for that model.

(ii) If ZFC *is consistent, then* $[P = NP]^{\mathsf{F}}$ *is consistent with* ZFC.

(iii) If ZFC + $[P = NP]^{\mathsf{F}}$ *is ω-consistent,*[1] *then* $[P = NP]$ *is consistent with* ZFC.

The last result can be rephrased as follows: if there is a model for a consistent theory ZFC + $[P = NP]^{\mathsf{F}}$, where formal polynomial Turing machines act as intuitively expected, then ZFC + $[P = NP]$ is consistent. In addition, we note that these results are in the line of previous results

1. A system F is said to be ω-consistent if there is no formula $\varphi(x)$ such that F proves $\neg\varphi(n)$ for each natural number n, and yet F proves $\exists x\, \varphi(x)$.

by Richard DeMillo and Richard Lipton (consistency of $[P = NP]$ with fragments of arithmetic; see [49]) and more recently by Attila Máté (consistency of $[P = NP]$ with Peano arithmetic, given an involved technical model-theoretic condition; see [120]).

The point is that if $[P = NP]$ is proved by ZFC, then the da Costa-Doria condition holds trivially. Yet it is known that if $[P = NP]$ is independent of ZFC, then $[P = NP]$ will hold only for models for ZFC with nonstandard arithmetic, and it is not immediately apparent why there should be a nonstandard model with the nice, desirable behavior they are asking for. Their condition for the consistency of $[P = NP]$ with a strong theory is simple and intuitive: in plain words, there is a model for the theory where formal objects behave as expected by our intuition.

Although the results presented so far seem to be supportive of the equality of the two classes, still there are results that seem to be supportive of exactly the opposite. In particular, Ralf-Dieter Schindler [171] defined the classes classes P_{ω^ω} and NP_{ω^ω}, which are supposed to be the corresponding P and NP classes for infinite-time Turing machines. Then he showed that $P_{\omega^\omega} \neq NP_{\omega^\omega}$. Also, Vinay Deolalikar, Joel Hamkins, and Ralf Schindler [50] have shown by extending previous results and by defining new classes of problems, just as Schindler did, that $P_\alpha \neq NP_\alpha \cap \text{co-}NP_\alpha$. But all these contradicting results show that the $P = NP$ problem is far from being settled.

B. Intractability and Hypercomputation

In classical computation theory, first of all it is important to know whether a problem is solvable. And when a problem is solvable, it is equally important to see whether it can be solved efficiently. For example, if there is a problem that is solvable in principle, but whose solution can be computed only in millions of years, then this problem is *practically* not solvable. Generally speaking, complexity theory is the branch of the theory of computation that studies whether computable problems are practically computable, that is, whether the resources required during computation to solve a problem are reasonable.

In Appendix A we presented the class of problems that can be solved by a polynomially bounded nondeterministic Turing machine, that is, the class *NP*. When some problem can be solved by an algorithm that can be *reduced* to one that can solve any problem in *NP*, then it is called *NP*-hard. A problem that is both *NP* and *NP*-hard is called *NP*-complete.

Assume that we have a problem Π that is as difficult as the halting problem (formally, its degree of insolvability is at most $\mathbf{0}'$). It follows that if a hypermachine can solve Π, it will be able to solve any *NP*-complete problem, since *NP*-complete problems can be solved by some algorithm. Clearly, when one finds a general solution to an noncomputable problem, then no one will care whether the solution is optimal, provided it is feasible. In other words, in cases like this, efficiency becomes an empty word. However, a number of researchers in complexity theory have questioned the ability of certain hypermachines to efficiently solve *NP*-complete problems, and thus question the feasibility of these machines. In particular, Aaronson [1] presented a summary of such objections. Aaronson examines a number of approaches to hypercomputation and by using a number of supposedly knockout arguments aimed at showing that hypercomputation proposals cannot solve *NP*-complete problems and thus cannot solve noncomputable problems.

First of all, one should not forget that it is one thing to efficiently solve *NP*-complete problems and another to solve noncomputable problems. Compare airplanes and cars: airplanes can cross oceans and continents, something no car can do, but airplanes cannot be used to go to work every morning. For the moment, I will ignore this objection in order to present the arguments against hypercomputation. There are two basic objections to the feasibility of hypermachines. These two objections can be summarized as follows: first, it is not known whether quantum mechanics remains valid in the regime tested by quantum computing (in other words, are quantum computers feasible?), and second, it is not known whether quantum gravity imposes limits that make infeasible various models of hypercomputation that are based on properties of spacetime.

The reader may recall that the second objection has been discussed in Chapter 8. Our response was that there is no experimental evidence that time and space are granular, but on the contrary, there are experimental indications that space and time are continuous. Quantum gravity is based on the hypothesis that space and time are granular, while relativity theory assumes that space and time are continuous. Also, at least one model of hypercomputation does not rely on the space and time granularity hypothesis. Thus, hypercomputation cannot by ruled out based on this hypothesis. And since there is no proof of the space and time granularity hypothesis, we can safely assume that space and time are indeed continuous. The first objection is more serious despite the vast literature on quantum computing. Nowadays, there are no general-purpose quantum computers available. Indeed, there are many obstacles that scientists and engineers have to overcome before the first general-purpose quantum computer is constructed. However, it seems that most problems are being successfully tackled one after the other (for example, see [14, 6, 165, 59, 32]). So it is not unreasonable to expect that general-purpose quantum computers will be constructed in a few years. After all, it took only a few years to go from the Turing machine to the first general-purpose digital computer.

C. Socioeconomic Implications

As mentioned on page 85, cognitive scientists were probably the first scholars to fully adopt the computational metaphor in their research programs. Later, the computational metaphor was adopted by economists, sociologists, and others. In particular, economists who bought into the metaphor were hoping, and still hope, to be able to convincingly *explain* fluctuations in oil and currency prices, *predict* upcoming stock market crashes, *forecast* economic growth, etc. Naturally, one may wonder whether such expectations are reasonable. Certainly, one should not expect to find a full-fledged answer to this question in an appendix of a book on hypercomputation. Nevertheless, this appendix serves as a brief exposition of our ideas concerning the use of the computational metaphor in economics and sociology in the light of hypercomputation and the view that the human mind has capabilities that transcend the Church–Turing barrier.

John Forbes Nash's noncooperative game theory has a central importance in modern economic theory. More specifically, this theory deals with how intelligent individuals interact with one another in an effort to achieve their own goals. Consequently, an economic system is a very complex system, which, nevertheless, can be modeled by computers, as Axel Stig Bengt Leijonhufvud has suggested [110]. In addition, Leijonhufvud went on to give a (new) formulation of what an economy is.

Conjecture C.1 *An economy is best conceived of as a network of interacting processors, each one with less capability to process information than would be required of a central processor set to solve the overall allocation problem for the entire system.*

Consequently, by assuming this conjecture, Leijonhufvud's further claim that "[T]he economy should be looked at as a machine that has to 'compute' the equilibrium" is not an exaggeration at all. Needless to say, these ideas had a profound impact on the development of the field of *computational economics* (i.e., a new field of economics that utilizes the computational metaphor to analyze economic phenomena).

Agent-based computational economics (ACE) "is the computational study of economic processes modeled as dynamic systems of interacting agents."[1] Observe that ACE is based on Leijonhufvud's conjecture regarding the "true" nature of an economy. More specifically, ACE is based on the assumption that agents (i.e., individuals, social groups, institutions, biological entities, and physical entities) are computable entities, which is a hypothesis we contest in this book. On the other hand, it is possible that in certain cases the interaction of noncomputable agents may result in a computable behavior, but this is something no one can really guarantee. Also, if we assume that economic behavior is hypercomputational, then we may use the theory presented in this book as a starting point for a hypercomputational economics. Indeed, the economist Wolfram Latsch proposed something similar in [108], but he argues for a noncomputable economic theory using Penrose's ideas as a starting point.

Latsch examines evolutionary economics and its relationship to complexity. More specifically, he notes that "evolutionary economics is interested in the emergence of order out of complex processes" and then shows how evolutionary economics is related to Wolfram's view of the cosmos. In particular, Wolfram is convinced that cellular automata, which are self-reproducing finite-state machines that may show very complex behavior from very simple rules, are computing devices that can simulate, if not *implement*, everything in this world. Thus, one would expect that the use of cellular automata would be a panacea for economics. However, da Costa and Doria have shown that if there is a mathematical model for some market economy, it is not possible to algorithmically decide whether the economy has reached some equilibrium set of prices (see [44] for a recent, but not so formal, discussion of these results). A direct consequence of this result is that one cannot practically "compute the future."

Kumaraswamy Vela Velupillai discussed in [211] why, in his opinion, mathematical economics is unreasonably ineffective. This has prompted him to look for an alternative formalization of economics. So he has concluded that a "reasonable and effective mathematisation of economics entails Diophantine formalisms." This implies that economics should suffer from all the limits and constraints imposed by the Church-Turing thesis. A broad-minded view should seriously take into consideration hypercomputation as presented in this book, at least as a basis for a foundation of economics.

An economy is part of a society, and it is not an exaggeration to say that economic activities are actually social activities. So it was not surprising to see the computational metaphor find its way into the social sciences. Indeed, *computational sociology* is a recently developed branch of sociology that uses the computational metaphor to analyze social phenomena (see [65] for a thorough presentation of this new branch of sociology).[2]

1. See `http://www.econ.iastate.edu/tesfatsi/ace.htm` for more information.
2. In a way, computational sociologists try to analyze social phenomena by creating virtual worlds much like the virtual worlds presented in movies like "The Matrix" (written and

The techniques, the methodologies, and the ideas employed in computational sociology are similar, if not identical, to those employed in computational economics. Consequently, computational sociology should and, in our opinion does, suffer from the same problems computational economics does. Apart from this, *culture*[3] is one aspect of any society that has to be taken into account in any serious analysis of social phenomena.

Roy F. Baumeister [10] has examined how culture has affected our mental capabilities. More specifically, he asserts that meaning and language actually prove that human thought cannot be reduced to brain activity. Baumeister argues that thought is more than just neuron firing. For him, "[H]uman thought generally uses language and meaning, which are social realities that are not contained in the brain." Obviously, this idea is akin to Searle's biological naturalism in particular, and hypercomputation in general. Note that this view is actually an attack against reductionism in neurobiology, which assumes that thinking can be reduced to neuron firing.

In order to defend his ideas, Baumeister argues that culture, with its meaning and language, is like the Internet. In particular, he asserts that maintaining that "human thought is contained in the brain, or is nothing more than brain cell activity, is like saying that the Internet is contained inside your computer, or that the Internet is nothing more than electrical activity inside your computer" [10, p. 185]. As was explained in Chapter 5, the Internet cannot be described by the operation of a single computer, while computers connected to the Internet can actually accomplish more than isolated machines. Similarly, brains connected to culture do more and better things than an isolated brain.

In conclusion, one may say that computer simulations in economics and sociology may provide some insight into certain aspects of economic or social phenomena, but they cannot give any definitive answers to crucial problems of computation.

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directed by Andy Wachowski and Larry Wachowski; see `http://whatisthematrix.warnerbros.com/` for more information) and "The Thirteenth Floor" (screenplay by Josef Rusnak and Ravel Centeno-Rodriguez, based on the book "Simulacron 3" by Daniel Francis Galouye, and directed by J. Rusnak; see `http://www.imdb.com/title/tt0139809/` for more information).

3. For example, see `http://www.isanet.org/portlandarchive/bada.html` for a brief discussion of what culture actually is.

D. A Summary of Topology and Differential Geometry

D.1 Frames

The presentation of this section is based on [212].

Definition D.1.1 A poset (or partially ordered set) is a set P equipped with a binary relation $\leq$ called a partial order that satisfies the following laws:

reflexivity $a \leq a$, for all $a \in P$;

transitivity if $a \leq b$ and $b \leq c$, then $a \leq c$, for all $a, b, c \in P$;

antisymmetry if $a \leq b$ and $b \leq a$, then $a = b$ for all $a, b \in P$.

Definition D.1.2 A totally ordered set is a poset $(P, \leq)$ whose binary relation satisfies the following additional law, making it a total order.

comparability (trichotomy law) for any $a, b \in P$ either $a \leq b$ or $b \leq a$.

Two totally ordered sets $(A, \leq)$ and $(B, \leq')$ are order isomorphic if there is a bijection $f : A \to B$ such that for all $a_1, a_2 \in A$, if $a_1 \leq a_2$ then $f(a_1) \leq' f(a_2)$. The order type is the property of a totally ordered set that remains when the set is considered not with respect to the properties of its elements but with respect to their order. The order type of $(A, \leq)$ is denoted by $|A|$. For example, the order type of $(\mathbb{N}, \leq)$ is ω.

Definition D.1.3 Assume that $(P, \leq)$ is a poset, $X \subseteq P$ and $y \in P$. Then y is a *meet* (or *greatest lower bound*) for X if

- y is a *lower bound* for X, that is, if $x \in X$ then $y \leq x$, and
- if z is any other lower bound for X then $z \leq y$.

The meet for X is denoted by $\bigwedge X$. If $X = \{a, b\}$, then the meet for X is denoted by $a \wedge b$.

Definition D.1.4 Suppose that $(P, \leq)$ is a poset, $X \subseteq P$, and $y \in P$. Then y is a *join* (or *least upper bound*) for X if

- y is an *upper bound* for X, that is, if $x \in X$, then $x \leq y$, and
- if z is any other upper bound for X then $y \leq z$.

The join for X is denoted by $\bigvee X$. In addition, if $X = \{a, b\}$, then the join for X is denoted by $a \vee b$.

Definition D.1.5 A poset $(A, \leq)$ is a *frame* if

(i) every subset has a join,

(ii) every finite subset has a meet, and

(iii) binary meets distribute over joins:

$$x \wedge \bigvee Y = \bigvee \{x \wedge y : y \in Y\}.$$

A function between two frames is a frame homomorphism if it preserves all joins and finite meets.

D.2 Vector Spaces and Lie Algebras

In general, we can say that an algebraic structure (or algebra) consists of one or more sets closed under one or more operations satisfying some axioms. A subalgebra consists of subsets of the sets an algebra consists of, while the algebraic operations are now restricted to these subsets. Let us define some common algebraic structures.

Definition D.2.1 A quadruple $(G, \cdot, {}^{-1}, 1)$, where G is a set, $\cdot : G \times G \to G$ a binary operation, ${}^{-1} : G \to G$ a unary operation, and $1 \in G$ a distinguished element called the *unit* element, is an *abelian* group if

(i) $\forall g_1, g_2, g_3 \in G : g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$,

(ii) $\forall g \in G : g \cdot 1 = 1 \cdot g = g$,

(iii) $\forall g \in G : g \cdot g^{-1} = g^{-1} \cdot g = 1$, and

(iv) $\forall g_1, g_2 \in G : g_1 \cdot g_2 = g_2 \cdot g_1$.

Definition D.2.2 A set S together with two binary operators + (the "addition" operator) and * (the "multiplication" operator) is called a ring if it satisfies the following properties:

(i) $(a + b) + c = a + (b + c)$,

(ii) $a + b = b + a$,

(iii) $0 + a = a + 0 = a$,

(iv) $a + (-a) = (-a) + a = 0$,

(v) $(a * b) * c = a * (b * c)$, and

(vi) $a * (b + c) = (a * b) + (a * c)$ and $(b + c) * a = (b * a) + (c * a)$.

Definition D.2.3 A commutative ring (i.e., $a * b = b * a$) with a "multiplication" unit 1 (i.e., $1 * a = a * 1 = a$) with the property that for all $a \neq 0$ there exists an element a^{-1} such that $a^{-1} * a = a * a^{-1} = 1$, is called a field.

The set $\mathbb{R}$ ($\mathbb{C}$) and the operations of real (complex) number addition and multiplication define a field.

Definition D.2.4 A *vector* or *linear* space over the field F is an abelian group V, whose group operation is usually written as $+$, that is equipped with *scalar multiplication*, which is a mapping $F \times V \to V$ that is usually denoted by $(c, \mathbf{x}) \mapsto c\mathbf{x}$. In addition, the following axioms must be fulfilled for all $c, c_1, c_2 \in F$ and $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in V$:

(i) $c_1(c_2\mathbf{x}) = (c_1c_2)\mathbf{x}$,

(ii) $(c_1 + c_2)\mathbf{x} = c_1\mathbf{x} + c_2\mathbf{x}$,

(iii) $c(\mathbf{x}_1 + \mathbf{x}_2) = c\mathbf{x}_1 + c\mathbf{x}_2$, and

(iv) $1\mathbf{x} = \mathbf{x}$.

Note that the unit element of V is denoted by $\mathbf{0}$. Also, the elements of V are called vectors. A finite subset $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ of a linear space V is called *linearly independent* if

$$k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_n\mathbf{x}_n = \mathbf{0} \Rightarrow k_1 = k_2 = \cdots = k_n = 0.$$

Every expression of the form $k_1\mathbf{x}_1 + \cdots + k_n\mathbf{x}_n$ is called a *linear combination* of $\mathbf{x}_1, \ldots, \mathbf{x}_n$. A subset $B \subset V$ is a basis of the linear space V if it is linearly independent and every element of V is a linear combination of B. The dimension of a linear finite-dimensional space X is equal to the cardinality of a basis of X. A linear space that is not finite-dimensional is an infinite-dimensional space.

Lie algebras are special cases of vector spaces.

Definition D.2.5 A Lie algebra L is a vector space over some field together with a bilinear multiplication $[,] : L \times L \to L$, called the bracket, that satisfies two simple properties:

(i) $[a, b] = -[b, a]$ (anticommutativity) and

(ii) $[a[b, c]] = [[a, b], c] + [a, [b, c]]$ (Jacobi identity).

An associative algebra L, that is, an algebraic structure with a product ab that is associative (i.e., $a(bc) = (ab)c$), can be transformed into a Lie algebra $\bar{L}$ by the bilinear multiplication

$$[a, b] = ab - ba.$$

D.3 Topological Spaces: Definitions

The presentation that follows is based on [113].

Definition D.3.1 Let X be a nonempty set. A class $\mathscr{T}$ of subsets of X is a *topology* on X if $\mathscr{T}$ satisfies the following axioms:

(i) the (trivial) subsets X and the empty set $\emptyset$ are in $\mathscr{T}$;

(ii) the intersection of any *finite number* of members of $\mathscr{T}$ is a member of $\mathscr{T}$; and

(iii) the union of any (even infinite) number of members of $\mathscr{T}$ is a member of $\mathscr{T}$.

The members of $\mathscr{T}$ are called *open sets* and the pair $(X, \mathscr{T})$ is called a *topological space*. If $\mathcal{O}$ is an open set containing a point $x \in X$, then $\mathcal{O}$ is called an *open neighborhood* of x.

Let Σ be an alphabet; we are going to define some of the standard topologies on Σ^* and Σ^ω. The class $\tau_* = 2^{\Sigma^*} = \{A \mid A \subseteq \Sigma^*\}$ is the discrete topology on Σ^*. The Cantor topology on Σ^ω is $\tau_C = \{A\Sigma^\omega \mid A \subseteq \Sigma^*\}$, and (Σ^ω, τ_C) is called the Cantor space over Σ. Note that if $A, B \subseteq \Sigma^*$, then $AB = \{ab \mid a \in A, b \in B\}$.

Assume that $(P, \leq)$ is a poset; a nonempty subset $S \subseteq P$ is *directed* if for all $x, y \in S$ there is a $z \in S$ such that $x, y \leq z$.

Definition D.3.2 A poset $(P, \leq)$ is a *directed complete partial order* (or *dcpo*, for short) when every directed subset has a join.

A subset U of a dcpo D is *Scott open* if

(i) U is an upper set, that is if $x \in U$ and $x \leq y$ this implies that $y \in U$, and

(ii) U is inaccessible by directed joins, that is for every directed $S \subseteq D$,

$$\bigvee S \in U \Rightarrow S \cap U \neq \emptyset.$$

The collection of all Scott open sets on D is called the *Scott topology* and is denoted by σ_D.

Definition D.3.3 Let $(X, \mathscr{T})$ be a topological space. A point $p \in X$ is an *accumulation point* of a subset A of X if every open set $\mathcal{O}$ containing p also contains a point of A different from p, that is,

$$\mathcal{O} \text{ open}, p \in \mathcal{O} \Rightarrow (\mathcal{O} \setminus \{p\}) \cap A \neq \emptyset.$$

Definition D.3.4 Assume that $(X, \mathscr{T})$ is a topological space. Then a subset $\mathcal{A}$ of X is a *closed set* if its complement $\mathcal{A}^{\complement}$ is an open set.

Definition D.3.5 Suppose that $\mathcal{A}$ is a subset of X, where $(X, \mathscr{T})$ is a topological space. Then the *closure* of $\mathcal{A}$, denoted by $\mathrm{Cl}(\mathcal{A})$, is the intersection of all closed supersets of $\mathcal{A}$.

Definition D.3.6 Assume that $\mathcal{A}$ is a subset of X, where $(X, \mathscr{T})$ is a topological space. Then a point $p \in \mathcal{A}$ is called an *interior point* of $\mathcal{A}$ if p belongs to an open set $\mathcal{O}$ that is a subset of $\mathcal{A}$. The set of interior points of $\mathcal{A}$ is denoted by $\mathrm{Int}(\mathcal{A})$.

Definition D.3.7 A subset $\mathcal{A}$ of X, where $(X, \mathscr{T})$ is a topological space, is *nowhere dense in* X if $\mathrm{Int}(\mathrm{Cl}(\mathcal{A})) = \emptyset$.

Definition D.3.8 A subset $\mathcal{A}$ of X, where $(X, \mathscr{T})$ is a topological space, is called *dense* if $\mathrm{Cl}(\mathcal{A}) = X$.

Definition D.3.9 Suppose that p is a point in X, where $(X, \mathscr{T})$ is a topological. Then a subset N of X is a *neighborhood* of p if and only if N is a superset of an open set $\mathcal{O}$ containing p.

Definition D.3.10 Let $(X, \mathscr{T})$ be a topological space. A class $\mathscr{B}$ of open subsets of X is a *base* for the topology $\mathscr{T}$ if every open set $\mathcal{O} \in \mathscr{T}$ is the union of members of $\mathscr{B}$.

Definition D.3.11 Let $(X, \mathscr{T})$ be a topological space. A class $\mathscr{S}$ of open subsets of X is a *subbase* for the topology $\mathscr{T}$ if finite intersections of members of $\mathscr{S}$ form a base for $\mathscr{T}$.

Now we need to define the notion of a map between topological spaces.

Definition D.3.12 Assume that $(X, \mathscr{T}_1)$ and $(Y, \mathscr{T}_2)$ are topological spaces. Then a function $f : X \to Y$ is *continuous* if the inverse image $f^{-1}(\mathcal{O})$ of every open subset $\mathcal{O}$ of Y is an open subset of X.

Metric spaces are special cases of topological spaces. In addition, a metric space induces a topological space. Let us now define a metric space.

Definition D.3.13 Suppose that X is a nonempty set. Then a function $f : X \times X \to \mathbb{R}$ is called a *metric* or *distance function* on X if it satisfies, for every $a, b, c \in X$, the following axioms:

(i) $d(a, a) = 0$;

(ii) $d(a, b) > 0$ if $a \neq b$;

(iii) $d(a, b) = d(b, a)$; and

(iv) $d(a, c) \leq d(a, b) + d(b, c)$.

The real number $d(a, b)$ is called the *distance* from a to b.

Given a metric space (X, d), the distance between a point $p \in X$ and a nonempty subset A of X is defined by

$$d(p, A) = \inf\left\{d(p, a) | a \in A\right\}.$$

In addition, the distance between two nonempty subsets A and B of X is defined by

$$d(A, B) = \inf\left\{d(a, b) | a \in A, b \in B\right\}.$$

The *diameter* of a nonempty subset A of X is defined by

$$d(A) = \sup\left\{d(a, b) | a, b \in A\right\}.$$

The *open sphere* with center p and radius δ is defined by

$$S(p, \delta) = \left\{x | d(p, x) < \delta\right\}.$$

Equally important is the notion of a *normed space.*

Definition D.3.14 Assume that V is a vector space over the field F. Then a function that assigns to each vector $\mathbf{v} \in V$ the quantity $\|\mathbf{v}\| \in F$ is a *norm* on V if it satisfies, for all $\mathbf{v}, \mathbf{u} \in V$ and $k \in F$, the following axions:

(i) $\|\mathbf{v}\| \geq 0$ and $\|\mathbf{v}\| = 0$ if $\mathbf{v} = \mathbf{0}$;

(ii) $\|\mathbf{v} + \mathbf{u}\| \leq \|\mathbf{v}\| + \|\mathbf{u}\|$;

(iii) $\|k\mathbf{v}\| = k\|\mathbf{v}\|$.

A vector space V together with a norm is called a *normed vector space*.

A *cover* of a subset $\mathcal{A}$ of a topological space is a collection $\{\mathcal{C}_i\}$ of subsets of X such that $\mathcal{A} \subset \cup_i \mathcal{C}_i$. In addition, if each $\mathcal{C}_i$ is open, then it is called an *open cover*. Furthermore, if a finite subcollection of $\{\mathcal{C}_i\}$ is also a cover of $\mathcal{A}$, then $\{\mathcal{C}_i\}$ includes a *finite subcover*.

Definition D.3.15 A subset $\mathcal{A}$ of a topological space X is *compact* if every open cover of $\mathcal{A}$ contains a finite subcover.

Two subsets $\mathcal{A}$ and $\mathcal{B}$ of a topological space X are said to be *separated* if

(i) $\mathcal{A}$ and $\mathcal{B}$ are disjoint and

(ii) neither contains an accumulation point of the other.

Definition D.3.16 A subset $\mathcal{A}$ of a topological space X is *disconnected* if there exist open subsets $\mathcal{O}$ and $\mathcal{O}'$ of X such that $\mathcal{A} \cap \mathcal{O}$ and $\mathcal{A} \cap \mathcal{O}'$ are disjoint nonempty sets whose union is $\mathcal{A}$. In this case, $\mathcal{A} \cup \mathcal{O}$ is called a *disconnection* of $\mathcal{A}$. A set is *connected* if it is not disconnected.

An easy way to construct a new topological space from existing ones is by *multiplying* them.

Definition D.3.17 Assume that $\{(X_i, \mathscr{T}_i)\}$ is a collection of topological spaces and that $X = \prod_i X_i$ (i.e., X is the product of the sets X_i). The smallest topology $\mathscr{T}$ on X with respect to which all the projections $\pi_i : X \to X_i$ are continuous is called the *product topology*. The product set X with the product topology $\mathscr{T}$ (i.e., $(X, \mathscr{T})$) is called the *product (topological) space*.

Suppose that X is a metric space. A sequence $a_1, a_2, \ldots$ in X is a *Cauchy sequence* if for every $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$ it holds that $d(a_n, a_m) < \varepsilon$. Similarly, if X is a normed vector space, then $\mathbf{a}_1, \mathbf{a}_2, \ldots$ is a Cauchy sequence if for every $\varepsilon > 0$, there is an $n_o \in \mathbb{N}$ such that for all $n, m > n_0$ it holds that $\|\mathbf{a}_n - \mathbf{a}_m\| < \varepsilon$.

A metric space (X, d) is complete if every Cauchy sequence $a_1, a_2, \ldots$ in X converges to some point $p \in X$.

Definition D.3.18 A topological space X is a *Hausdorff space* if each pair of distinct points $a, b \in X$ belong to disjoint open sets.

D.4 Banach and Hilbert Spaces

Definition D.4.1 A Banach space is a normed vector space over $\mathbb{R}$ or $\mathbb{C}$ that is complete in the metric $\|\mathbf{x} - \mathbf{y}\|$.

We denote by $C([a, b])$ the Banach space of all continuous functions f on $[a, b]$, endowed with the uniform norm $\| f \|_\infty = \sup_x\{|f(x)|\}$. By obvious modifications the interval $[a, b]$ can be replaced by cubes (i.e., subsets of $\mathbb{R}^3$), squares (i.e., subsets of $\mathbb{R}^2$), etc.

Definition D.4.2 A Hilbert space H is a Banach space in which the norm is given by an inner product $H \times H \to F \mapsto \langle \mathbf{x}|\mathbf{y}\rangle$. The inner product of a Hilbert space induces a vector norm in a natural way:

$$\|\mathbf{x}\|^2 = \langle \mathbf{x}|\mathbf{x}\rangle.$$

A function $T : X \to Y$, where X and Y are linear spaces over the same field F, is called an operator.

Definition D.4.3 An operator $T : X \to Y$ is called *linear* if

(i) for all $\mathbf{x}_1, \mathbf{x}_2 \in X$, $T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_2) + T(\mathbf{x}_2)$, and

(ii) for all $k \in F$ and all $\mathbf{x} \in X$, $T(k\mathbf{x}) = kT(\mathbf{x})$.

It customary to write $T\mathbf{x}$ instead of $T(\mathbf{x})$. Let us now define some special linear operators.

Definition D.4.4 A linear operator $T : X \to Y$ is called bounded if there is a constant $m \geq 0$ such that

$$\|T\mathbf{x}\| \leq m \cdot \|\mathbf{x}\|, \quad \forall \mathbf{x} \in X.$$

The smallest m satisfying the inequality above is called the norm of the operator T. In addition, the domain of the operator $T : X \to Y$ is usually not X but a subspace $\mathscr{D}(T)$ that is dense in X.

Definition D.4.5 A linear operator $T : \mathscr{D}(T) \to Y$ is called closed if for $(\mathbf{x}_n) \in \mathscr{D}(T)$,

$$\Big(\|\mathbf{x}_n - \mathbf{x}\| \to \mathbf{0}\Big) \wedge \Big(\|T\mathbf{x}_n - y\| \to 0\Big) \Rightarrow \Big((\mathbf{x} \in \mathscr{D}(T)) \wedge (T\mathbf{x} = \mathbf{y}\Big).$$

Let $T : H \to H$ be a bounded linear operator. Then the adjoint operator T^* is defined by

$$\langle T\mathbf{x} \mid \mathbf{y}\rangle = \langle \mathbf{x} \mid T^*\mathbf{y}\rangle, \quad \forall \mathbf{x}, \mathbf{y} \in H.$$

A closed operator $T : H \to H$ is called self-adjoint if $T = T^*$.

Suppose that T is a closed operator. Then a number λ belongs to the spectrum of T if the operator $(T - \lambda)$ does not have a bounded inverse. A number λ is called an eigenvalue of an operator T if there exists a nonzero vector $\mathbf{x}$, which is the corresponding eigenvector, such that $T(\mathbf{x}) = \lambda\mathbf{x}$.

A closed operator $T : H \to H$ is *effectively determined* if there is a computable sequence $\{e_n\}$ in H such that the pairs $\{(e_n, Te_n)\}$ form an effective generating set for the graph of T. Note that *effective generating set* means that $\{(e_n, Te_n)\}$ is computable in $H \times H$ and that the linear span of $\{(e_n, Te_n)\}$ is dense in the graph of T.

D.5 Manifolds and Spacetime

The short exposition of manifolds that follows is based on the introduction to differential geometry provided in [79].

Let us denote by $\mathbb{R}^n$ the set of n-tuples $(x_1, \ldots, x_n)$, where $-\infty < x_i < +\infty$. This set forms an n-dimensional vector space over $\mathbb{R}$ called *Euclidean space of dimension n*. Since a Euclidean space is a metric space, it is also a topological space with the natural topology induced by the metric. The metric topology on $\mathbb{R}^n$ is called the Euclidean topology. We say that $(x_1, \ldots, x_n) \in \frac{1}{2}\mathbb{R}^n$ if $x_i \leq 0$, $i = 1, \ldots, n$. A map φ of an open set $\mathcal{O} \subset \mathbb{R}^n$ to an open set $\mathcal{O}' \subset \mathbb{R}^m$ is of class C^r if the coordinates $(x'_1, \ldots, x'_m) = \varphi(p)$ in $\mathcal{O}'$ are r-times continuously differentiable functions (i.e., the rth derivatives exist and are continuous) of the coordinates $(x_1, \ldots, x_n)$ of $p \in \mathcal{O}$. When a map is C^r for all $r \geq 0$, then it is called a C^∞ map.

A function $f : \mathcal{O} \to \mathcal{O}$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^n$, is locally Lipschitz if for each open set $\mathcal{U} \subset \mathcal{O}$ with compact closure, there is a constant C such that for all $p, q \in \mathcal{U}$, $|f(p) - f(q)| \leq C|p - q|$, where $|p|$ means

$$\sqrt{\Big(x_1(p)\Big)^2 + \cdots + \Big(x_n(p)\Big)^2}.$$

A map φ is locally Lipschitz, denoted by C^{1-}, if the coordinates of $\varphi(p)$ are locally Lipschitz functions of the coordinates of p. In addition, a map φ is C^{r-} if it is C^{r-1} and if the $(r-1)$th derivatives of the coordinates of $\varphi(p)$ are locally Lipschitz functions of the coordinates of p.

Suppose that $P \subset \mathbb{R}^n$ and $P' \subset \mathbb{R}^m$. Then a map $\varphi : P \to P'$ is a C^r map if it is the restriction of a map $\psi : \mathcal{O} \to \mathcal{O}'$, where $\mathcal{O}$ and $\mathcal{O}'$ are open sets and include P and P', respectively. We are now ready to give a general definition of the notion of C^r manifold.

Definition D.5.1 A C^r n-dimensional manifold $\mathscr{M}$ is a set $\mathscr{M}$ together with a C^r *atlas* $\{\mathscr{U}_i, \varphi_i\}$, $i = 1, \ldots, n$, which is a collection of *charts* $(\mathscr{U}_i, \varphi_i)$, where each $\mathscr{U}_i$ is a subset of $\mathscr{M}$ and φ_i injectively maps $\mathscr{U}_i$ to open sets in $\mathbb{R}^n$ such that

(i) the sets $\mathscr{U}_i$ cover $\mathscr{M}$, that is,

$$\mathscr{M} = \bigcup_i \mathscr{U}_i,$$

(ii) if $\mathscr{U}_k \cap \mathscr{U}_l \neq \emptyset$, the map $\varphi_k \circ \varphi_l^{-1}$, which maps $\varphi_l(\mathscr{U}_k \cap \mathscr{U}_l)$ to $\varphi_k(\mathscr{U}_k \cap \mathscr{U}_l)$, is a C^r map of an open subset of $\mathbb{R}^n$ to an open subset of $\mathbb{R}^n$.

Each $\mathscr{U}_i$ is a *local coordinate neighborhood*, and the local coordinates x^i are defined by the map φ_i, that is, if $p \in \mathscr{U}_i$, then the coordinates of p are the coordinates of $\varphi_i(p)$ in $\mathbb{R}^n$.

An atlas $\{\mathscr{U}_i, \varphi_i\}$ is *locally finite* if every $p \in \mathscr{M}$ has an open neighborhood that intersects only a finite number of the sets $\mathscr{U}_i$. A manifold $\mathscr{M}$ is called *paracompact* if for every atlas $\{\mathscr{U}_i, \varphi_i\}$ there is a locally finite atlas $\{\mathscr{V}_j, \psi_j\}$ such that each $\mathscr{V}_j$ is contained in some $\mathscr{U}_i$. A connected Hausdorff manifold is paracompact if there is a countable collection of open sets such that any open set is the union of members of this collection.

A C^k *curve* $\lambda(t)$ in $\mathscr{M}$ is a C^k map of an interval of the real line $\mathbb{R}^1$ into $\mathscr{M}$. The *vector* (contravariant vector) $(\partial/\partial t)_\lambda|_{t_0}$ tangent to the C^1 curve $\lambda(t)$ at the point $\lambda(t_0)$ is the operator that maps each C^1 function f at $\lambda(t_0)$ into the number $(\partial/\partial t)_\lambda|_{t_0}$. This means that $\left(\frac{\partial f}{\partial t}\right)_\lambda$ is the derivative of f in the direction of $\lambda(t)$ with respect to the parameter t, or

$$\left(\frac{\partial f}{\partial t}\right)_\lambda \Big|_t = \lim_{s\to 0} \frac{1}{s}\Big\{f(\lambda(t+s) - f(\lambda(t))\Big\}.$$

The curve parameter t obeys the relation $\left(\frac{\partial}{\partial t}\right)_\lambda t = 1$.

If $(x^1, \ldots, x^n)$ are local coordinates in a neighborhood of p, then

$$\left(\frac{\partial f}{\partial t}\right)_\lambda \Big|_{t_0} = \sum_{j=1}^{n} \frac{dx^j(\lambda(t))}{dt}\Big|_{t=t_0} \cdot \frac{\partial f}{\partial x^j}\Big|_{\lambda(t_0)} = \frac{dx^j}{dt}\frac{\partial f}{\partial x^j}\Big|_{\lambda(t_0)}.$$

This means that every tangent vector at a point p can be expressed as a linear combination of the coordinate derivatives

$$\left(\frac{\partial}{\partial x^1}\right)\Big|_p, \ldots, \left(\frac{\partial}{\partial x^n}\right)\Big|_p.$$

The space of all tangent vectors to $\mathscr{M}$ at p, denoted by T_p, is an n-dimensional vector space. A *one-form* ω at p is a real-valued linear function on the space T_p of vectors at p. The space of all one-forms at p is denoted by T_p^*. From the spaces T_p and T_p^* we can form the Cartesian product

$$\Pi_r^s = \underbrace{T_p^* \times \cdots \times T_p^*}_{r \text{ factors}} \times \underbrace{T_p \times \cdots \times T_p}_{s \text{ factors}}.$$

Clearly, if $(\omega_1, \ldots, \omega_r, a_1, \ldots, a_s) \in \Pi_r^s$, then the a_i's are vectors and the ω_is are one-forms. A *tensor of type* (r, s) *at* p is a function on Π_r^s that is linear in each argument.

A *metric tensor* $\mathbf{g}$ at a point $p \in \mathscr{M}$ is a symmetric tensor of type $(0, 2)$ at p. A tensor $\mathbf{T}$ of type $(0, 2)$ is symmetric if $T_{ab} = \frac{1}{2}(T_{ab} + T_{ba})$. The *signature* of $\mathbf{g}$ at p is the number of positive eigenvalues of the matrix (g_{ab}) at p, minus the negative ones. A metric whose signature is $(n-2)$ is called a *Lorentz metric*. We are now ready to give the definition of the mathematical model of a spacetime.

Definition D.5.2 A spacetime, that is, the collection of all events, is a pair $(\mathscr{M}, \mathbf{g})$, where $\mathscr{M}$ is a connected four-dimensional Hausdorff C^∞ manifold and $\mathbf{g}$ is a Lorentz metric.

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