Appendix A

Introduction to Finite Difference Methods

How to Read This Chapter

This chapter concerns the problem of solving numerically the partial differential equations that we have encountered in this book. Although several kinds of approaches can be considered (like finite elements or spectral methods), the success of finite differences in image analysis is due to the structure of digital images for which we can associate a natural regular grid. This chapter is an introduction to the main notions that are commonly used when one wants to solve a partial differential equation. From Section A.1 to Section A.2 we will consider only the one-dimensional case and focus on the main ideas of finite differences. Section A.3 will be more applied: It will concern the discretization of certain approaches detailed in this book. More precisely:

- Section A.1 introduces the main definitions and theoretical considerations about finite-difference schemes (convergence, consistency, and stability, Lax theorem). Every notion is illustrated by developing explicit calculations for the case of the one-dimensional heat equation. Besides the precise definitions, this will help the reader to understand them in a simple situation.

- Section A.2 concerns hyperbolic equations. We start with the linear case and show that if we do not choose an upwind scheme, then the scheme is always unstable. We then investigate the nonlinear case by focusing on the Burgers equation.

- The purpose of Section A.3 is to show how finite-difference schemes can be used in image analysis. We first introduce in Section A.3.1 the main notation and consider the 2-D heat equation. The remainder of Section A.3 is concerned with the discretization of certain PDEs studied in this book:
  - Restoration by energy minimization (Section A.3.2): We detail the discretization of the divergence term, which can also be found for the Perona and Malik equation.
  - Enhancement by Osher and Rudin's shock filters (Section A.3.3): The main interest is to use a flux limiter called minmod.
  - Curve evolution with level sets and especially segmentation with geodesic active contours (Section A.3.4). For the sake of simplicity, we examine separately each term of the model (mean curvature motion, constant speed motion, advection equation). We essentially write their discretization and give some experimental results.

For general presentations of finite difference methods: [316, 112, 182],

To know more about hyperbolic equations: [166, 212].

A.1 Definitions and Theoretical Considerations Illustrated by the 1-D Parabolic Heat Equation

A.1.1 Getting Started

There are many approaches that are used for discretizing a partial differential equation. Among the most important ones we can mention finite differences, finite elements, and spectral methods.

We focus here on finite differences, which are widely used in image processing. This is due to the structure of a digital image as a set of uniformly distributed pixels (see Section A.3).

To present the main ideas we will consider the following well-posed initial-value problem, written in the one dimensional case:[1]

$$\begin{cases} \mathcal{L}v = F, & t > 0, \quad x \in R, \\ v(0,x) = f(x), & x \in R, \end{cases} \tag{A.1}$$

[1] This is an initial-value problem, which means that there is no boundary condition. For initial-boundary-value problems, the discussion that follows needs to be slightly adapted, and we refer to [316] for more details.

where v and F are defined on R, and $\mathcal{L}$ is a differential *linear* operator. The function v denotes the exact solution of (A.1).

Example: One of the easiest equations that we may consider is the 1-D heat equation:

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial x^2}, \quad t > 0, \quad x \in R, \tag{A.2}$$

where $\nu > 0$ is a constant, which is equivalent to

$$\mathcal{L}v = 0 \quad \text{with} \quad \mathcal{L}v = \frac{\partial v}{\partial t} - \nu \frac{\partial^2 v}{\partial x^2}.$$

The initial condition is $v(0,x) = f(x)$. From now on, we shall use this equation to illustrate the different notions to be defined. ■

Our aim is to solve the PDE (A.1) numerically. We begin by discretizing the spatial domain by placing a grid over the domain. For convenience, we will use a uniform grid, with grid spacing $\triangle x$. Likewise, the temporal domain can be discretized, and we denote by $\triangle t$ the temporal grid spacing. The resulting grid in the time–space domain is illustrated in Figure A.1.

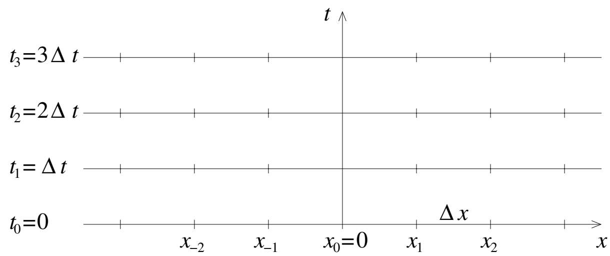


Figure A.1. Grid on the time–space domain.

Solving the problem numerically means finding a discrete function u defined at the points $(n\triangle t, i\triangle x)$ (we will denote by u_i^n the value of u at these points), which is a "good approximation" of v. The function u will be obtained as the solution of a discrete equation, which will be an approximation of (A.1):

$$\begin{cases} L_i^n u_i^n = G_i^n, & i = -\infty, \ldots, +\infty, \\ u_i^0 = f(i\triangle x), \end{cases} \tag{A.3}$$

where L_i^n (respectively G_i^n) corresponds to the discrete approximation of $\mathcal{L}$ (respectively F).[2] Notice that both spatial and temporal derivatives have to be approximated.

Example: Let us show on the 1-D heat equation (A.2) how the discrete equation can

[2]Because of discretization, G_i^n is a priori different from F_i^n, which is simply the value of F in $(n\triangle t, i\triangle x)$

be obtained. In fact, the starting point for writing any finite difference scheme is Taylor expansions. For $\triangle t$ and $\triangle x$ small, we have

$$v((n+1)\triangle t, i\triangle x) = \left(v + \triangle t\frac{\partial v}{\partial t} + \frac{\triangle t^2}{2}\frac{\partial^2 v}{\partial t^2}\right)(n\triangle t, i\triangle x) + \mathcal{O}(\triangle t^3), \tag{A.4}$$

$$v(n\triangle t, (i+1)\triangle x) = \left(v + \triangle x\frac{\partial v}{\partial x} + \frac{\triangle x^2}{2}\frac{\partial^2 v}{\partial x^2}\right)(n\triangle t, i\triangle x) + \mathcal{O}(\triangle x^3), \tag{A.5}$$

$$v(n\triangle t, (i-1)\triangle x) = \left(v - \triangle x\frac{\partial v}{\partial x} + \frac{\triangle x^2}{2}\frac{\partial^2 v}{\partial x^2}\right)(n\triangle t, i\triangle x) + \mathcal{O}(\triangle x^3). \tag{A.6}$$

We recall that $g = \mathcal{O}(\phi(s))$ for $s \in S$ if there exists a constant C such that $|f(s)| \leq C|\phi(s)|$ for all $s \in S$. We say that $g(s)$ is "big $\mathcal{O}$" of $\phi(s)$ or that $g(s)$ is of order $\phi(s)$. In the previous Taylor expansions, notice that the constant C naturally depends on the high-order derivatives of v.

By Equation (A.4), we have

$$\frac{\partial v}{\partial t}(n\triangle t, i\triangle x) = \frac{v_i^{n+1} - v_i^n}{\triangle t} + \mathcal{O}(\triangle t),$$

where we have set $v_i^n = v(n\triangle t, i\triangle x)$. Similarly, by using previous Taylor expansions, we may propose an approximation of the second spatial derivative. By adding (A.5) and (A.6), we have

$$\frac{\partial^2 v}{\partial x^2}(n\triangle t, i\triangle x) = \frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{\triangle x^2} + \mathcal{O}(\triangle x^2). \tag{A.7}$$

Consequently, if we consider the differential operator $\mathcal{L}$ from (A.2), we have

$$\frac{\partial v}{\partial t}(n\triangle t, i\triangle x) - \nu\frac{\partial^2 v}{\partial x^2}(n\triangle t, i\triangle x) = \frac{v_i^{n+1} - v_i^n}{\triangle t} - \nu\frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{\triangle x^2} + \mathcal{O}(\triangle t) + \mathcal{O}(\triangle x^2). \tag{A.8}$$

So a reasonable approximation of equation (A.2) is

$$L_i^n u = 0 \quad \text{with} \quad L_i^n = \frac{u_i^{n+1} - u_i^n}{\triangle t} - \nu\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\triangle x^2}. \tag{A.9}$$

This difference equation (A.9) can also be rewritten in the following form:

$$u_i^{n+1} = (1-2r)u_i^n + r\left(u_{i+1}^n + u_{i-1}^n\right), \quad \text{where} \quad r = \nu\triangle t/\triangle x^2. \tag{A.10}$$

This shows clearly that this scheme is explicit, which means that the values at time $(n+1)\triangle t$ are obtained only from the values at time $n\triangle t$. We will mention how to write implicit schemes at the end of this section. ■

☛ *It is important to realize that the discretized equation replaces the original equation by a new one, and that an exact solution of the discretized problem will lead to an approximate solution of the original PDE, since we introduce a discretization error (the error of replacing a continuous equation by a discrete one).*

For a given approximation (A.3), we would like to know precisely the relations between the discrete equation with the PDE and their respective solutions. In other words, what does it mean that u is an approximation of v, and can we quantify it? Are there any conditions on the grid size ($\triangle t$, $\triangle x$) to yield a "good" approximation? To answer these questions we define precisely the notions of convergence, consistency, and stability in the next sections.

A.1.2 Convergence

The first notion that is essential is to understand what it means that the discrete solution u of (A.3) is an approximation of, or converges to, the solution v of (A.1). To be more precise, we can define pointwise convergence as follows:

Definition A.1.1 (pointwise convergent scheme) *The scheme* (A.3) *approximating the partial differential equation* (A.1) *is pointwise convergent if for any x and t, as $((n+1)\triangle t, i\triangle x)$ converges to (t, x), then u_i^n converges to $v(t,x)$ as $\triangle x$ and $\triangle t$ converge to* 0.

Example: Let us show that the solution of the difference scheme (A.10),

$$\begin{cases} u_i^{n+1} = (1-2r)u_i^n + r\left(u_{i+1}^n + u_{i-1}^n\right), & x \in R, \\ u_i^0 = f(i\triangle x), \end{cases} \tag{A.11}$$

where $r = \nu\triangle t/\triangle x^2$, converges pointwise to the solution of the initial-value problem (A.2):

$$\begin{cases} \dfrac{\partial v}{\partial t} = \nu \dfrac{\partial^2 v}{\partial x^2}, & x \in R, \\ v(0,x) = f(x). \end{cases} \tag{A.12}$$

We will assume that $0 \leq r \leq \frac{1}{2}$ in order to have all the coefficients positive in the difference equation. We need to estimate

$$z_i^n = u_i^n - v(n\triangle t, i\triangle x),$$

where v is the exact solution of the initial-value problem (A.12). Equation (A.8) becomes

$$v_i^{n+1} = (1-2r)v_i^n + r\left(v_{i+1}^n + v_{i-1}^n\right) + \mathcal{O}(\triangle t^2) + \mathcal{O}(\triangle t\triangle x^2). \tag{A.13}$$

Then by subtracting equation (A.13) from (A.11), we have:

$$z_i^{n+1} = (1-2r)z_i^n + r\left(z_{i+1}^n + z_{i-1}^n\right) + \mathcal{O}(\triangle t^2) + \mathcal{O}(\triangle t\triangle x^2), \tag{A.14}$$

and then (since we assumed $0 \leq r \leq \frac{1}{2}$)

$$|z_i^{n+1}| \leq (1-2r)|z_i^n| + r|z_{i+1}^n| + r|z_{i-1}^n| + C(\triangle t^2 + \triangle t\triangle x^2), \tag{A.15}$$

where C is a constant associated with the "big $\mathcal{O}$" terms and depends on the assumed bounds of the higher-order derivatives of v, in space and time. In fact, we will assume that the derivatives v_{tt} and v_{xxxx} (which would appear in the subsequent terms of the Taylor expansion of (A.8)) are uniformly bounded on $[0,t] \times R$. So, by taking the supremum with respect to i in (A.15) we obtain

$$Z^{n+1} \leq Z^n + C(\triangle t^2 + \triangle t\triangle x^2) \quad \text{with} \quad Z^n = |z^n|_{\ell^\infty} \equiv \sup_{i\in Z} \{|z_i^n|\}. \tag{A.16}$$

Applying (A.16) repeatedly yields

$$\begin{aligned} Z^{n+1} &\leq Z^n + C(\triangle t^2 + \triangle t\triangle x^2) \leq Z^{n-1} + 2C(\triangle t^2 + \triangle t\triangle x^2) \\ &\leq \cdots \leq Z^0 + (n+1)C(\triangle t^2 + \triangle t\triangle x^2). \end{aligned}$$

Since $Z^0 = 0$, the previous inequality implies

$$\left|u_i^{n+1} - v((n+1)\triangle t, i\triangle x)\right| \leq (n+1)\triangle t\ C(\triangle t + \triangle x^2). \tag{A.17}$$

Thus we see that the right-hand side of (A.17) goes to zero as $(n+1)\triangle t \to t$, and $\triangle t$, $\triangle x \to 0$, which means that u converges to v pointwise. Notice that in fact, we have just proven a stronger result than the pointwise convergence:

$$Z^{n+1} = \left|z^{n+1}\right|_{\ell^\infty} \to 0 \tag{A.18}$$

as $(n+1)\triangle t \to t$, and $\triangle t$, $\triangle x \to 0$. ■

The pointwise convergence is in general difficult to prove. So we shall instead use a definition of convergence in terms of an l^p-norm ($p < \infty$) of a difference between the solution of the PDE and the solution of the difference equation. In the following definition, we will use the notation

$$u^{n+1} = (\ldots, u_{-1}^n, u_0^n, u_1^n, \ldots),$$
$$v^{n+1} = (\ldots, v_{-1}^n, v_0^n, v_1^n, \ldots).$$

Definition A.1.2 (convergent scheme) *The scheme* (A.3) *approximating the partial differential equation* (A.1) *is a convergent scheme at time t if as $(n+1)\triangle t \to t$,*

$$\left|u^{n+1} - v^{n+1}\right|_* \to 0 \tag{A.19}$$

as $\triangle x \to 0$ and $\triangle t \to 0$, and where $| \,.\, |_*$ is a norm to be specified.

This definition shows that whenever convergence is being discussed, the norm that is used must be specified. Its choice depends on the problem to be solved. For $z = (\ldots, z_{-1}, z_0, z_1, \ldots)$, typical examples include

$$|z|_{\ell^\infty} = \sup_{i \in Z} \{|z_i|\}, \quad |z|_{\ell^2} = \sqrt{\sum_{i=-\infty}^{i=+\infty} |z_i|^2}, \quad \text{and} \quad |z|_{\ell^{2,\triangle x}} = \sqrt{\sum_{i=-\infty}^{i=+\infty} |z_i|^2 \triangle x}. \tag{A.20}$$

Another important piece of information that we may be interested in is the rate of convergence, i.e., how fast the solution of the difference equation converges to the solution of the PDE. This order of convergence is defined as follows:

Definition A.1.3 (order of convergence) *A difference scheme* (A.3) *approximating the partial differential equation* (A.1) *is a convergent scheme of order (p, q) if for any t, as $(n+1)\triangle t$ converges to t,*

$$\left|u^{n+1} - v^{n+1}\right|_* = \mathcal{O}(\triangle x^p) + \mathcal{O}(\triangle t^q) \tag{A.21}$$

as $\triangle x$ and $\triangle t$ converge to 0.

Example: For the approximation (A.11) of the heat equation, we have in fact proven its convergence for the ℓ^∞ norm (A.18). Moreover, we can verify that this scheme is of order $(2,1)$. ■

The convergence is usually something difficult to prove. Most of the time, its proof is based on the Lax theorem, which we present in the next section.

A.1.3 The Lax Theorem

This theorem gives a sufficient condition for a two-level difference[3] scheme to be convergent:

Theorem A.1.1 (Lax) *A consistent two-level difference scheme for a well-posed linear initial value problem is convergent if and only if it is stable.*

In this theorem we have introduced two new notions:

- *Consistency*: This concerns the error introduced by the discretization of the equation. This error should tend to zero as $\triangle t$ and $\triangle x$ go to zero.
- *Stability*: The intuitive idea is that small errors in the initial condition should cause small errors in the solution. This is similar to the definition of well-posedness of a PDE.

Most of the schemes that are used are consistent. The major problem will be to prove their stability.

The two next sections define precisely these two notions and give the main ideas to ensure that they are satisfied.

A.1.4 Consistency

As in the case of convergence, we can first define the property of a scheme to be *pointwise* consistent with the PDE:

Definition A.1.4 (pointwise consistent) *The scheme* (A.3) *approximating the partial differential equation* (A.1) *is pointwise consistent at point* (t,x) *if for any smooth function* $\phi = \phi(t,x)$,

$$(\mathcal{L}\phi - F)|_i^n - [L_i^n \phi(n\triangle t, i\triangle x) - G_i^n] \rightarrow 0 \qquad \text{(A.22)}$$

as $\triangle x$, $\triangle t \rightarrow 0$ *and* $((n+1)\triangle t, i\triangle x) \rightarrow (t,x)$.

Example: Notice that from equality (A.8), we have in fact just proven that the scheme (A.10) is pointwise consistent with the PDE (A.2). ■

[3] A two-level difference scheme is a scheme where only two different levels of time are present in the difference equation, typically $H(u^{n+1}, u^n) = 0$.

As in the case of convergence, it is usually more interesting to have a definition in terms of norms and not only pointwise. If we write the two-level scheme as

$$u^{n+1} = Qu^n + \triangle t\ G^n, \tag{A.23}$$

where $u^n = (\ldots, u^n_{-1}, u^n_0, u^n_1, \ldots)$, $G^{n+1} = (\ldots, G^n_{-1}, G^n_0, G^n_1, \ldots)$, and Q is an operator acting on the appropriate space, then a stronger definition of consistency can be given as follows:

Definition A.1.5 (consistent) *The scheme* (A.3) *is consistent with the partial differential equation* (A.1) *in a norm* $|\,.\,|_*$ *if the solution* v *of the partial differential equation satisfies:*

$$v^{n+1} = Qv^n + \triangle t G^n + \triangle t \tau^n,$$

where τ^n *is such that*

$$|\tau^n|_* \to 0$$

as $\triangle x$, $\triangle t \to 0$.

The term τ^n is called the truncature term. We may be more precise and define also the order in which τ^n goes to 0.

Definition A.1.6 (truncature error, order of accuracy) *The difference scheme* (A.3) *is said to be accurate of order* (p, q) *if*

$$|\tau^n|_* = \mathcal{O}(\triangle x^p) + \mathcal{O}(\triangle t^q).$$

Remark It is easy to see that if a scheme is of order (p, q), $p, q \geq 1$, then it is a consistent scheme. Also, it can be verified that if a scheme is either consistent or accurate of order (p, q), the scheme is pointwise consistent. ■

Example: Let us discuss the consistency of the scheme

$$\frac{u_i^{n+1} - u_i^n}{\triangle t} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\triangle x}$$

with the PDE $\dfrac{\partial v}{\partial t} = \nu \dfrac{\partial^2 v}{\partial x^2}$, $x \in R$, $t > 0$. If we denote by v the solution of the PDE, then equation (A.8) becomes

$$\frac{v_i^{n+1} - v_i^n}{\triangle t} - \nu \frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{\triangle x^2} = \mathcal{O}(\triangle t) + \mathcal{O}(\triangle x^2).$$

As we can see, we need to be more precise to apply Definitions A.1.6 and A.1.5. In particular, we need to know exactly the terms in $\mathcal{O}(\triangle t) + \mathcal{O}(\triangle x^2)$. In fact, similar calculations have to be done but using Taylor expansions with remainder instead of standard Taylor expansions.

After rewriting the difference scheme in the form of (A.23),

$$u_i^{n+1} = (1 - 2r)u_i^n + r\left(u_{i+1}^n + u_{i-1}^n\right), \quad \text{where} \quad r = \nu \triangle t / \triangle x^2,$$

we can define the truncature error by

$$\triangle t \tau_i^n = v_i^{n+1} - \left\{(1-2r)v_i^n + r\left(v_{i+1}^n + v_{i-1}^n\right)\right\}, \tag{A.24}$$

where v is a solution of the PDE. Then we need to develop the right-hand term of (A.24) by using Taylor expansions with remainder. After some calculation, there exist $t_1 \in]n\triangle t, (n+1)\triangle t[$, $x_1 \in](i-1)\triangle x, i\triangle x[$ and $x_2 \in]i\triangle x, (i+1)\triangle x[$ such that

$$\triangle t \tau_i^n = \frac{\partial^2 v}{\partial t^2}(t_1, i\triangle x)\,\frac{\triangle t}{2} - \nu\left(\frac{\partial^4 v}{\partial x^4}(n\triangle t, x_1) + \frac{\partial^4 v}{\partial x^4}(n\triangle t, x_2)\right)\frac{\triangle x^2}{24}. \tag{A.25}$$

Notice that as we have mentioned, when we write simply $\mathcal{O}(\triangle t)+\mathcal{O}(\triangle x^2)$, we have to be aware that the coefficients involved are not constants but depend on certain derivatives of the solution. This also means that as soon as we will talk about consistency, we will need to make some smoothness assumptions.

To apply Definition A.1.5, we need to choose a norm. If we assume that

$$\frac{\partial^2 v}{\partial t^2} \text{ and } \frac{\partial^4 v}{\partial x^4} \quad \text{are uniformly bounded on } [0,T]\times R \text{ for some } T,$$

then we can then choose the sup-norm to get that this scheme is accurate of order $(2,1)$ with respect to this norm. Otherwise, if we assume

$$\sum_{i=-\infty}^{i=+\infty}\left|\left(\frac{\partial^2 v}{\partial t^2}\right)_i^n\right|^2 \triangle x < A < \infty \quad \text{and} \quad \sum_{i=-\infty}^{i=+\infty}\left|\left(\frac{\partial^4 v}{\partial x^4}\right)_i^n\right|^2 \triangle x < B < \infty,$$

for any $\triangle x$ and $\triangle t$, then the difference scheme is accurate of order $(2,1)$ with respect to the $\ell^{2,\triangle x}$ norm. ■

One important remark that comes out of the previous example is that as soon as one considers the problem of consistency, one needs to choose a norm. It is also important to note that this choice is, in fact, related to some smoothness assumptions on the solution.

Finally, we would like to mention that proving consistency can be very difficult, especially for implicit schemes. We refer the interested reader to [316] for more details.

A.1.5 Stability

To conclude this section, we need to discuss the problem of stability, which is necessary for applying the Lax theorem. Though stability is much easier to establish than convergence, it is still often difficult to prove that a given scheme is stable. Many definitions of stability can be found in the literature, and we present below one that is commonly used:

Definition A.1.7 (stable scheme) *The two-level difference scheme*

$$\begin{cases} u^{n+1} = Qu^n, n \geq 0, \\ u^0 \quad \textit{given}, \end{cases} \tag{A.26}$$

where $u^n = (\ldots, u_{-1}^n, u_0^n, u_1^n, \ldots)$, *is said to be stable with respect to the norm* $|\,.\,|_*$ *if there exist positive constants* $\triangle x_0$ *and* $\triangle t_0$, *and nonnegative* *constants K and β such that*

$$\left|u^{n+1}\right|_* \leq K e^{\beta t} \left|u^0\right|_* \tag{A.27}$$

for $0 \leq t = (n+1)\triangle t$, $0 < \triangle x \leq \triangle x_0$ *and* $0 < \triangle t \leq \triangle t_0$.

Remarks From Definition A.1.7 we may observe the following:

- This definition has been established for homogeneous schemes (A.26). If we have a nonhomogeneous scheme, it can be proved that the stability of the associated homogeneous scheme, along with the convergence, is enough to prove its convergence.
- As for convergence and consistency, we will need to define which norm is used.
- This definition of stability does allow the solution to grow with time.

■

As we already mentioned, there are other definitions for stability. In particular, another common definition is one that does not allow for exponential growth. The inequality (A.27) then becomes

$$\left|u^{n+1}\right|_* \leq K \left|u^0\right|_* , \tag{A.28}$$

which clearly implies (A.27). The interest of (A.27) is that it permits us to include more general situations.

Example: Let us show that the scheme

$$u_i^{n+1} = (1-2r)u_i^n + r\left(u_{i+1}^n + u_{i-1}^n\right) \tag{A.29}$$

is stable for the sup-norm. If we assume that $r \leq \frac{1}{2}$, (A.29) yields

$$|u_i^{n+1}| \leq (1-2r)|u_i^n| + r|u_{i+1}^n| + r|u_{i-1}^n| \leq |u^n|_{\ell^\infty} .$$

If we take the supremum over the right-hand side, we get

$$\left|u^{n+1}\right|_{\ell^\infty} \leq |u^n|_{\ell^\infty} .$$

Hence inequality (A.27) is satisfied with $K = 1$ and $\beta = 0$. Notice that in order to prove the stability, we have assumed that $r = \nu\triangle t/\triangle x^2 \leq \frac{1}{2}$. In this case we say that the scheme is conditionally stable. In the case where there is no restriction on $\triangle x$ and $\triangle t$, we say that the scheme is unconditionally stable. ■

The previous example was a simple case where we were able to prove directly stability, i.e., inequality (A.27) or (A.28). In fact, there are several tools that can be used to prove it. The one that is probably the most commonly used is Fourier analysis, which is used for linear difference schemes with constant coefficients. We recall in Table A.1 the definitions of the Fourier transform and the inverse Fourier transform, for the continuous and discrete settings (i.e., for a vector $u^n = (\ldots, u_{-1}^n, u_0^n, u_1^n, \ldots) \in \ell^2$).

We also recall an important property of the Fourier transform, which is Parseval's identity (see [154] for more details).

	Continuous setting	Discrete setting
Fourier transform	$\hat{v}(t,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\underline{i}\omega x} v(t,x)dx$	$\hat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{m=+\infty} \mathrm{e}^{-\underline{i}m\xi} u_m$
inverse Fourier transform	$v(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{\underline{i}\omega x} \hat{v}(t,\omega)d\omega$	$u_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{+\pi} \mathrm{e}^{\underline{i}m\xi} \hat{u}(\xi)d\xi$
Parseval's identity	$\lvert v\rvert_{L^2(R)} = \lvert\hat{v}\rvert_{L^2(R)}$	$\lvert u\rvert_{\ell^2} = \lvert\hat{u}\rvert_{L^2(-\pi,\pi)}$

Table A.1. Some recalls about the Fourier transform ($\underline{i}^2 = 1$).

Interestingly, the discrete transform has similar properties to those of the continuous one. In particular, to prove the stability of a difference scheme, we will use two main ideas:

- The first is that taking the Fourier transform of a PDE turns it into an ODE. Spatial derivatives are turned into products. For example, we can easily verify that

$$\widehat{v_{xx}}(t,\omega) = -\ \omega^2\ \hat{v}(t,\omega).$$

An analogous idea is valid in the discrete case. Let us consider the "standard" approximation of the second-order derivative that we have been using until now (with $\triangle x = 1$ just to simplify notation):

$$u_{xx}|_k = u_{k+1} - 2u_k + u_{k-1}. \tag{A.30}$$

Then the Fourier transform of $\{u_{xx}\}$ in the discrete setting is

$$\widehat{(u_{xx})} = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{k=+\infty} \mathrm{e}^{-\underline{i}k\xi} u_{xx}|_k \tag{A.31}$$

$$\underset{(A.30)}{=} \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{k=+\infty} \mathrm{e}^{-\underline{i}k\xi} u_{k+1} - 2\underbrace{\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{k=+\infty} \mathrm{e}^{-\underline{i}k\xi} u_k}_{\widehat{u}(\xi)} + \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{k=+\infty} \mathrm{e}^{-\underline{i}k\xi} u_{k-1}.$$

By suitable changes of variable in the previous expression for the first sum ($m = k+1$) and the third one ($m = k-1$), we have

$$\widehat{(u_{xx})} = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{m=+\infty} \mathrm{e}^{-\underline{i}(m-1)\xi} u_m - 2\widehat{u}(\xi) + \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{m=+\infty} \mathrm{e}^{-\underline{i}(m+1)\xi} u_m$$

$$= (\mathrm{e}^{-\underline{i}\xi} - 2 + \mathrm{e}^{+\underline{i}\xi})\hat{u}(\xi) = \boxed{-4\sin^2\left(\frac{\xi}{2}\right)} \; \hat{u}(\xi). \tag{A.32}$$

- The second concerns Parseval's identity. The main interest of this identity is that it is equivalent to prove the inequality (A.27) in the transform space or in the solution space. As a matter of fact, in Definition A.1.7 of stability, the inequality that was required in terms of the energy norm was of the form

  $$\left|u^{n+1}\right|_{\ell^2} \leq K\mathrm{e}^{\beta(n+1)\triangle t} \left|u^0\right|_{\ell^2}. \tag{A.33}$$

  But since $|u|_{\ell^2,\triangle x} = \sqrt{\triangle x}\,|u|_{\ell^2} = \sqrt{\triangle x}\,|\hat{u}|_{\ell^2}$, if we can find a K and a β such that

  $$\left|\hat{u}^{n+1}\right|_{\ell^2} \leq K\mathrm{e}^{\beta(n+1)\triangle t} \left|\hat{u}^0\right|_{\ell^2}, \tag{A.34}$$

  then the same K and β will also satisfy (A.33). So the sequence $\{u^n\}$ will be stable if and only if the sequence $\{\hat{u}^n\}$ is stable in $L^2(-\pi,\pi)$.

These ideas are applied in the following example, where we show how to prove the stability of the discrete scheme associated with the 1-D heat equation.

Example: Let us prove the stability of the difference scheme

$$u_k^{n+1} = ru_{k+1}^n + (1-2r)u_k^n + ru_{k-1}^n, \tag{A.35}$$

where $r = \nu\triangle t/\triangle x^2 \leq \frac{1}{2}$. By doing similar computations as in (A.31)–(A.32), taking the Fourier transform of u^{n+1} with (A.35) leads to

$$\hat{u}^{n+1}(\xi) = r\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{k=+\infty} \mathrm{e}^{-\underline{i}k\xi} u_{k+1}^n + (1-2r)\hat{u}(\xi) + r\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{k=+\infty} \mathrm{e}^{-\underline{i}k\xi} u_{k-1}^n$$

$$= \left(1 - 4r\sin^2\left(\frac{\xi}{2}\right)\right)\hat{u}^n(\xi). \tag{A.36}$$

The coefficient of $\hat{u}^n$ on the right-hand side of (A.36) is called the symbol of the difference scheme (A.35). We denote it by $\rho(\xi)$. Then if we apply the result of (A.36) $n+1$ times, we get

$$\hat{u}^{n+1}(\xi) = (\rho(\xi))^{n+1}\hat{u}^0(\xi).$$

So the condition (A.34) will be satisfied with $K = 1$ and $\beta = 0$ as soon as $|\rho(\xi)| \leq 1$ for all $\xi \in [-\pi,\pi]$, i.e.,

$$\left|1 - 4r\sin^2\left(\frac{\xi}{2}\right)\right| \leq 1. \tag{A.37}$$

This condition is satisfied if $r \leq \frac{1}{2}$. Thus $r \leq \frac{1}{2}$ is a sufficient condition for stability (and along with the consistency, for convergence). It is also necessary. If $r > \frac{1}{2}$, then at least for some ξ, $|\rho(\xi)| > 1$, and then $|\rho(\xi)|^{n+1}$ will be greater than $Ke^{\beta(n+1)\triangle t}$ for any K and β.[4] ■

Another approach that is often used is to consider a discrete Fourier mode for the problem

$$u_k^n = \xi^n e^{ijk\pi\triangle x}, \tag{A.38}$$

where $0 \leq j \leq M$ and the superscript on the ξ term is a multiplicative exponent. The idea is then to insert this general Fourier mode into the difference scheme and find the expression for ξ. A necessary condition for stability is obtained by restricting $\triangle x$ and $\triangle t$ so that $|\xi| \leq 1$ (the ξ^n term will not grow without bound). This method is usually referred to as the discrete von Neumann criterion for stability (see [316] for more details).

Example: Let us apply the discrete von Neumann criterion for stability for the difference scheme

$$u_k^{n+1} = ru_{k+1}^n + (1-2r)u_k^n + ru_{k-1}^n. \tag{A.39}$$

By inserting the general Fourier mode

$$u_k^n = \xi^n e^{ijk\pi\triangle x}$$

in the difference scheme, we easily obtain

$$\xi^{n+1}e^{ijk\pi\triangle x} = \xi^n e^{ijk\pi\triangle x}\left(re^{-ij\pi\triangle x} + (1-2r) + re^{+ij\pi\triangle x}\right).$$

Thus if we divide both sides of the above equation by $\xi^n e^{ijk\pi\triangle x}$, we get

$$\xi = re^{-ij\pi\triangle x} + (1-2r) + re^{+ij\pi\triangle x} = 1 - 4r\sin^2\left(\frac{j\pi\triangle x}{2}\right).$$

By saying that $|\xi| \leq 1$, we recover the previous result (A.37). ■

Example: As we already mentioned, the discretization (A.9) initially proposed for the one-dimensional heat equation was explicit. The values of u at time $(n+1)\triangle t$ were fully determined by the values of u at time $n\triangle t$. One may investigate more general schemes such as

$$u_i^{n+1} = u_i^n + \frac{\nu\triangle t}{h^2}\left(\lambda(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) + (1-\lambda)(u_{i+1}^n - 2u_i^n + u_{i-1}^n)\right).$$

If $\lambda \neq 0$, the scheme is now implicit: One needs to solve a linear system in order to know the solution at time $(n+1)\triangle t$. For $\lambda = 1$ the scheme is fully implicit, and for $\lambda = 0.5$ the so-called Crank–Nicholson scheme is obtained. This difference between explicit and implicit schemes can be simply represented as in Figure A.2.
Depending on the value of λ, we have the following:

- For $\lambda = 0$, the scheme is explicit, of order $\mathcal{O}(\triangle t, h^2)$, and stable under the condition $\triangle t \leq \dfrac{h^2}{2\nu}$. This condition implies that the time step has to be chosen small enough, which will naturally slow down the resolution of the equation.

[4] This is true, since for any sequence of $\triangle t$ (chosen so that $(n+1)\triangle t \to t$) and choice of $\triangle x$ (so that r remains constant) the expression $|\rho(\xi)|^{n+1}$ becomes unbounded, while for sufficiently large values of n, $Ke^{\beta(n+1)\triangle t}$ will be bounded by $Ke^{\beta(t_0+1)}$ for some $t_0 > t$, t_0 near t.

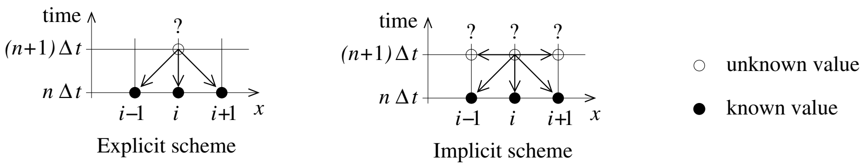


Figure A.2. Relations of dependency: The estimation of u_i^{n+1} depends on the neighbors indicated by an arrow.

- For $\lambda = \frac{1}{2}$, the scheme is implicit, of order $\mathcal{O}(\triangle t^2, h^2)$, and unconditionally stable.
- For $\lambda > \frac{1}{2}$, the scheme is implicit, of order $\mathcal{O}(\triangle t, h^2)$, and unconditionally stable.

We leave it as an exercise to the reader to verify these results, that is, proving consistency.[5] and stability. ■

A.2 Hyperbolic Equations

Let us consider the one-dimensional linear advection equation (also called the transport or wave equation):

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) + a\dfrac{\partial v}{\partial x}(t,x) = 0, \quad t > 0, \quad x \in R \\ v(0,x) = v_0(x), \end{cases} \tag{A.40}$$

where a is a constant. It can be easily verified that the solution is

$$v(t,x) = v_0(x - at). \tag{A.41}$$

Consequently, $v(t,x)$ is constant on lines of slope a, which are called characteristics. This means that the information is propagated in the direction of the sign of a, for example from left to right if a is positive.

In order to solve (A.40) numerically we have to approximate the temporal and spatial derivatives of u. As for the case of the heat equation (see Section A.1), the method is based on the Taylor expansions of v, which we recall

[5] In the example considered throughout this section we have expanded the functions about the index point $(n\triangle t, i\triangle x)$, and it was reasonably obvious that this was the correct point about which to expand. However, in some situations, the consistency cannot be proved if the point about which to expand is not adapted. The decision about which point to expand must be made by carefully considering how we expect the difference scheme to approximate the PDE. Typically, to prove the consistency of the Crank–Nicholson scheme, it is logical to consider the consistency of the scheme at the point $\left(\left(n+\frac{1}{2}\right)\triangle t, i\triangle x\right)$

here:

$$v((n+1)\triangle t, i\triangle x) = \left(v + \triangle t\frac{\partial v}{\partial t} + \frac{\triangle t^2}{2}\frac{\partial^2 v}{\partial t^2}\right)(n\triangle t, i\triangle x) + \mathcal{O}(\triangle t^3), \tag{A.42}$$

$$v(n\triangle t, (i+1)\triangle x) = \left(v + \triangle x\frac{\partial v}{\partial x} + \frac{\triangle x^2}{2}\frac{\partial^2 v}{\partial x^2}\right)(n\triangle t, i\triangle x) + \mathcal{O}(\triangle x^3), \tag{A.43}$$

$$v(n\triangle t, (i-1)\triangle x) = \left(v - \triangle x\frac{\partial v}{\partial x} + \frac{\triangle x^2}{2}\frac{\partial^2 v}{\partial x^2}\right)(n\triangle t, i\triangle x) + \mathcal{O}(\triangle x^3). \tag{A.44}$$

The temporal derivative can be approximated using (A.42) by

$$\frac{\partial v}{\partial t}(n\triangle t, i\triangle x) = \frac{v_i^{n+1} - v_i^n}{\triangle t} + \mathcal{O}(\triangle t).$$

As far as the spatial derivative is concerned, there are several possibilities:

- From (A.43) we have $\frac{\partial v}{\partial x}(n\triangle t, i\triangle x) = \frac{v_{i+1}^n - v_i^n}{\triangle x} + \mathcal{O}(\triangle x)$ (forward difference).
- From (A.44) we have $\frac{\partial v}{\partial x}(n\triangle t, i\triangle x) = \frac{v_i^n - v_{i-1}^n}{\triangle x} + \mathcal{O}(\triangle x)$ (backward difference).
- By subtracting (A.44) from (A.43) we have
$\frac{\partial v}{\partial x}(n\triangle t, i\triangle x) = \frac{v_{i+1}^n - v_{i-1}^n}{2\triangle x} + \mathcal{O}(\triangle x^2)$ (centered difference).

Consequently, there are three different possibilities for the discrete scheme of (A.40):

$$u_i^{n+1} = u_i^n + a\triangle t \begin{cases} \delta_x^+ u_i^n & \left(\equiv \dfrac{u_{i+1}^n - u_i^n}{\triangle x}\right) \quad \text{forward scheme,} \\ \delta_x u_i^n & \left(\equiv \dfrac{u_{i+1}^n - u_{i-1}^n}{2\triangle x}\right) \quad \text{centered scheme,} \\ \delta_x^- u_i^n & \left(\equiv \dfrac{u_i^n - u_{i-1}^n}{\triangle x}\right) \quad \text{backward scheme.} \end{cases}$$

Let us first consider the centered approximation, which is of order $(2, 1)$:

$$u_n^{n+1} = u_i^n - a\triangle t \delta_x u_i^n = u_i^n - \frac{a\triangle t}{2\triangle x}(u_{i+1}^n - u_{i-1}^n). \tag{A.45}$$

Let us examine its stability. As explained in Section A.1.5, a common approach is to use the discrete von Neumann criterion, which consists in inserting the general Fourier mode

$$u_k^n = \xi^n \mathrm{e}^{ijk\pi\triangle x} \quad (0 \leq j \leq M) \tag{A.46}$$

into the difference scheme. We recall that the superscript on the ξ term is a multiplicative exponent. By doing analogous calculations as for the difference scheme (A.39), we find that

$$\xi = 1 - \underline{i}\frac{a\triangle t}{\triangle x}\sin(j\pi\triangle x) \text{ and then } |\xi|^2 = 1 + \left(\frac{a\triangle t}{\triangle x}\right)^2 \sin^2(j\pi\triangle x) \geq 1 \quad \forall j,$$

which means that this scheme is always unstable.

☛ *The reason for this is that we did not take into account the nature of the equation. As there is a propagation, that is, the information is propagated in a certain direction, the numerical scheme should take it into account.*

To take this observation into account, one may propose the following scheme:

$$u_i^{n+1} = u_i^n - \begin{cases} a \ \triangle t \ \delta_x^- u_i^n & \text{if } a > 0, \\ a \ \triangle t \ \delta_x^+ u_i^n & \text{if } a < 0. \end{cases}$$

This can be rewritten as

$$u_i^{n+1} = u_i^n - \triangle t[\max(0,a)\delta_x^- u_i^n + \min(0,a)\delta_x^+ u_i^n]. \tag{A.47}$$

We call (A.47) an upwind scheme, because it uses values in the direction of information propagation. Let us see again the stability.

- Case $a > 0$: By replacing (A.46) in (A.47), we let the reader see for himself that we get

  $$|\xi| = 1 - 2C(1-C)(1-\cos(j\pi\triangle x)) \quad \text{with} \quad C = \frac{a\triangle t}{\triangle x} > 0.$$

  It will be less than or equal to 1 if and only if $C \leq 1$.

- Case $a < 0$: If we set $C = \frac{a\triangle t}{\triangle x} < 0$, similar calculus yield $-C \leq 1$.

To summarize, the stability condition is

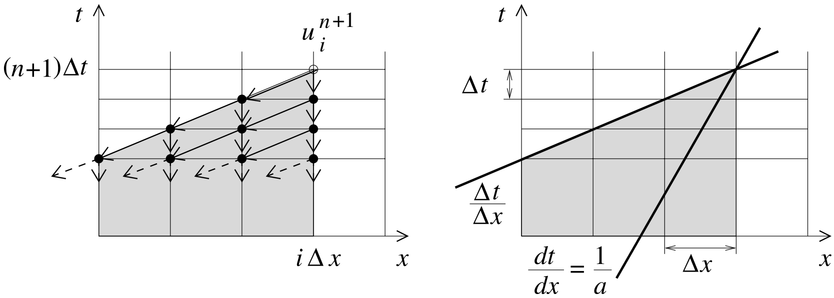
$$|a|\frac{\triangle t}{\triangle x} \leq 1. \tag{A.48}$$

It is usually called CFL, in reference to the authors Courant–Friedrichs–Lewy (in 1928).

This condition may be interpreted in terms of domain of dependence. In the continuous case, as has been mentioned, the information is propagated along the characteristics, and their equation is $\frac{dt}{dx} = \frac{1}{a}$. In the discrete case, if $a > 0$, then (A.47) becomes

$$u_i^{n+1} = \left(1 - \frac{a\triangle t}{\triangle x}\right) u_i^n + \frac{a\triangle t}{\triangle x} u_{i-1}^n.$$

This allows us to define the discrete domain of dependence of u_i^{n+1}: u_i^{n+1} depends on u_i^n and u_{i-1}^n, u_i^n depends on u_i^{n-1} and u_{i-1}^{n-1}, etc. (see Figure



(a) Discrete domain of dependence of u_i^{n+1} for (A.47) with $a > 0$

(b) Interpretation of the CFL condition

Figure A.3. Definition and interpretation of the discrete domain of dependence.

A.3). Moreover, the CFL condition means

$$\frac{\triangle t}{\triangle x} \leq \frac{1}{a} = \frac{dt}{dx},$$

which signifies that the characteristic line has to be included in the discrete domain of dependence.

☛ *The discrete domain of dependence must contain the exact continuous domain of dependence.*

In the nonlinear case the study is of course more complicated. For example, let us examine the nonlinear Burgers equation

$$\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial x} = 0 \tag{A.49}$$

with the initial condition

$$v(0,x) = v_0(x) = \begin{cases} 1 & \text{if } \ x \leq 0, \\ 1 - x & \text{if } \ 0 < x < 1, \\ 0 & \text{if } \ x \geq 1. \end{cases} \tag{A.50}$$

Here the propagation speed depends on the value of u itself. We may try as in the linear case to get an explicit solution of (A.49)–(A.50). As it is classical for hyperbolic equations, the method of characteristics can be used. Let us suppose that u is a smooth solution of (A.49)–(A.50) and let $x(t)$ be an integral curve of the differential equation

$$\frac{dx}{dt}(t) = v(t, x(t)), \quad x(0) = x_0.$$

We claim that u is constant along the characteristic curve $x(t)$. Indeed, since u is a solution of (A.49), we have

$$\begin{aligned}\frac{d}{dt}(v(t,x(t)) &= \frac{\partial v}{\partial t}(t,x(t)) + \frac{dx}{dt}(t)\,\frac{\partial v}{\partial x}(t,x(t)) \\ &= \frac{\partial v}{\partial t}(t,x(t)) + v(t,x(t))\,\frac{\partial v}{\partial x}(t,x(t)) = 0.\end{aligned}$$

Therefore

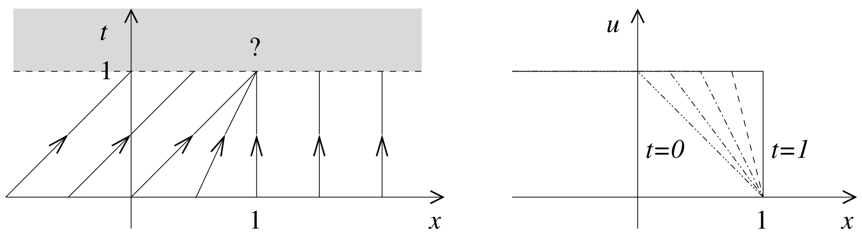
$$\frac{dx}{dt}(t) = v(0,x(0)) = v_0(x_0)$$

and

$$x(t) = x_0 + t\,v_0(x_0).$$

According to the definition of u_0 we deduce that

$$x(t) = \begin{cases} x_0 + t & \text{if } x_0 \le 0, \\ x_0 + (1-x_0)\,t & \text{if } 0 \le x_0 \le 1, \\ x_0 & \text{if } x_0 \ge 1. \end{cases}$$



(a) Characteristics (b) Solution at different times

Figure A.4. Behavior of the Burgers equation (A.49). The left-hand figure represents the characteristics while the right-hand one shows some solution at different times. As we can observe, characteristics intersect at $t = 1$, and after this time, it is not clear how to define the solution.

For $t < 1$ the characteristics do not intersect. Hence, given a point (t, x), $t < 1$, we draw the characteristic passing through this point, and we determine the corresponding point x_0:

$$x_0 = \begin{cases} x - t & \text{if } x \le t, \\ \dfrac{x-t}{1-t} & \text{if } t \le x \le 1, \\ x & \text{if } x \ge 1, \end{cases}$$

and we get the following continuous solution for $t < 1$:

$$u(t,x) = \begin{cases} 1 & \text{if } \ x \le t, \\ \dfrac{1-x}{1-t} & \text{if } \ t \le x \le 1, \\ 0 & \text{if } \ x \ge 1. \end{cases}$$

It consists of a front moving from left to right for $t < 1$. At $t = 1$, the characteristics collide, and beyond this collision time it is not clear how to define the solution uniquely. This discontinuity phenomenon is known as shock. What about for $t \ge 1$? Is it possible to define a unique solution? We must devise some way to interpret a less regular notion of solution. Let $\varphi : [0,+\infty[\times R \to R$ be smooth with compact support. We call φ a test function. We first observe that (A.49) can be written in a conservative form:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(v^2) = 0.$$

Then multiplying the above equality by φ, we deduce

$$\int_0^\infty \int_{-\infty}^\infty \left(\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(v^2) \right) \varphi \, dx\, dt = 0,$$

and by integrating by parts this last equation

$$\begin{aligned} &\int_0^\infty \int_{-\infty}^\infty v(t,x)\frac{\partial \varphi}{\partial t}(t,x)\, dx\, dt \ + \int_{-\infty}^\infty v_0(x)\,\varphi(0,x)\, dx \\ &\qquad + \int_0^\infty \int_{-\infty}^\infty \frac{1}{2} v^2(t,x) \frac{\partial \varphi}{\partial x}(t,x)\, dx\, dt = 0. \end{aligned} \tag{A.51}$$

We derive (A.51) supposing v to be smooth, but it is still valid if v is only bounded.

Definition A.2.1 *We say that* $v \in L^\infty((0,\infty) \times R)$ *is a weak solution of* (A.49) *if equality* (A.51) *holds for each test function* φ*.*

So according to this definition we are now going to search for discontinuous solutions. But before doing that, what can be deduced from (A.51)? Let us suppose in some open domain $\Omega \subset (0,\infty) \times R$ that v is smooth on either side of a smooth curve $x = \xi(t)$. Let us denote by Ω_L (respectively Ω_R) the part of Ω on the left (respectively on the right) of $x = \xi(t)$. We assume that v has limits v_- and v_+ on each side of $x = \xi(t)$: $v_\pm = \lim_{\varepsilon \to 0} u((t,\xi(t)) \pm \varepsilon N)$, where N is the normal vector to $x = \xi(t)$ given by $N = (-\xi'(t), 1)^T$.

Now by choosing a test function φ with compact support in Ω but that does not vanish along $x = \xi(t)$ we get

$$\iint\limits_{\Omega_L} \left(v(t,x)\frac{\partial \varphi}{\partial t}(t,x) + \frac{1}{2}v^2(t,x)\frac{\partial \varphi}{\partial x}(t,x) \right)\, dx\, dt + \iint\limits_{\Omega_R} \left(v(t,x)\frac{\partial \varphi}{\partial t}(t,x) + \frac{1}{2}v^2(t,x)\frac{\partial \varphi}{\partial x}(t,x) \right)\, dx\, dt = 0.$$

But since φ has compact support within Ω, we have

$$\begin{aligned}
&\iint\limits_{\Omega_L} \left(v(t,x)\frac{\partial \varphi}{\partial t}(t,x) + \frac{1}{2}v^2(t,x)\frac{\partial \varphi}{\partial x}(t,x) \right)\, dx\, dt \\
&= -\iint\limits_{\Omega_L} \left(\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(v^2) \right) \varphi(t,x)\, dx\, dt + \int\limits_{x=\xi(t)} \left(-\xi'(t)v_- + \frac{v_-^2}{2} \right) \varphi(t,x) dl \\
&= \int\limits_{x=\xi(t)} \left(-\xi'(t)v_- + \frac{v_-^2}{2} \right) \varphi\, dl,
\end{aligned}$$

since v is a smooth solution satisfying (A.49) in Ω_L. Similarly, we have

$$\iint\limits_{\Omega_R} \left(v(t,x)\frac{\partial \varphi}{\partial t}(t,x) + \frac{1}{2}v^2(t,x)\frac{\partial \varphi}{\partial x}(t,x) \right)\, dx\, dt = -\int\limits_{x=\xi(t)} \left(-\xi'(t)v_+ + \frac{v_+^2}{2} \right) \varphi(t,x) dl.$$

Adding these two last identities, we obtain

$$\int\limits_{x=\xi(t)} \left(-\xi'(t)v_- + \frac{v_-^2}{2} \right) \varphi(t,x) dl - \int\limits_{x=\xi(t)} \left(-\xi'(t)v_+ + \frac{v_+^2}{2} \right) \varphi(t,x) dl = 0. \tag{A.52}$$

Since φ is arbitrary, we easily deduce from (A.52)

$$\xi'(t)\,(v_+ - v_-) = \frac{1}{2}\left(v_+^2 - v_-^2\right). \tag{A.53}$$

For a general equation of the form $\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(f(v)) = 0$, we should have obtained

$$\xi'(t)\,(v_+ - v_-) = (f(v_+) - f(v_-)). \tag{A.54}$$

Identity (A.54) is known as the *Rankine–Hugoniot* condition. It may be read as

$$\text{speed of discontinuity} \times \text{jump of } v = \text{jump of } f(v).$$

Unfortunately, (A.53) and (A.54) are necessary conditions for the existence of discontinuous solutions but they are not sufficient to ensure uniqueness. For example, if we consider again (A.49) with the initial condition

$$v_0(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geq 0, \end{cases}$$

then it is easy to check that

$$v(t,x) = \begin{cases} 0 & \text{if } x < t/2, \\ 1 & \text{if } x > t/2, \end{cases}$$

is a weak solution of (A.49) satisfying the Rankine–Hugoniot condition. However, we can find another such solution

$$\hat{u}(t,x) = \begin{cases} 1 & \text{if } x > t, \\ x/t & \text{if } 0 < x < t, \\ 0 & \text{if } x < 0. \end{cases}$$

☛ *Thus, in general, weak solutions are not unique, and we have to find a further criterion that ensures uniqueness. Such a condition exists; it is called entropy.*

We do not continue the investigation of the theoretical difficulties of hyperbolic equations of conservation laws, since the general theory is complex, and it is far beyond the scope of this Appendix to review it. We refer the interested reader to [166, 212] for the complete theory. Of course, these difficulties are still present when we try to discretize these equations. For example, let us consider again Burgers equation

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{1}{2}\dfrac{\partial}{\partial x}(v^2) = 0, \\ v(0,x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{otherwise.} \end{cases} \end{cases} \tag{A.55}$$

If we rewrite (A.55) in the quasilinear form

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0, \tag{A.56}$$

then a natural finite-difference scheme inspired from the upwind method for (A.40) and assuming that $v \geq 0$ is:

$$\begin{cases} u_i^{n+1} = u_i^n - \dfrac{\triangle t}{\triangle x} u_i^n (u_i^n - u_{i-1}^n), \\ u_i^0 = \begin{cases} 1 & \text{if } j < 0, \\ 0 & \text{otherwise.} \end{cases} \end{cases} \tag{A.57}$$

Then it is easy to verify that $u_i^n = u_i^0$ for all i and n regardless of the step sizes $\triangle t$ and $\triangle x$. Therefore, as $\triangle t$ and $\triangle x$ tend to zero, the numerical solution converges to the function $\overline{v}(t,x) = v_0(x)$. Unfortunately, $\overline{v}(t,x)$ is not a weak solution. The reason is that discretizing the Burgers equation written in the form (A.56) is not equivalent for nonsmooth solutions. For nonsmooth solutions, the product vv_x does not necessarily have a meaning (even weakly). So, studying the Burgers equation written in a conservative form is the right approach. But in this case we have to define the numerical schemes that agree with this form. These schemes exist and are called *conservative schemes.*

Let us consider a general hyperbolic equation of conservation laws:

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{1}{2}\dfrac{\partial}{\partial x}(f(v)) = 0, \\ v(0,x) = v_0(x). \end{cases} \tag{A.58}$$

We say that the numerical scheme is in conservation form if it can be written as

$$u_i^{n+1} = u_i^n - \frac{\triangle t}{\triangle x}\left[F(u_{i-p}^n, u_{i-p+1}^n, \ldots, u_{i+q}^n) - F(u_{i-p-1}^n, u_{i-p}^n, \ldots, u_{i+q-1}^n)\right] \tag{A.59}$$

for some function F of $(p+q+1)$ arguments. F is called the numerical flux function. Of course, some consistency relations between F and f have to be satisfied. For example, if $p = 0$ and $q = 1$, then (A.59) becomes

$$u_i^{n+1} = u_i^n - \frac{\triangle t}{\triangle x}\left[F(u_i^n, u_{i+1}^n) - F(u_{i-1}^n, u_i^n)\right]. \tag{A.60}$$

In fact, for hyperbolic equations it is often preferable to view u_i^n as an approximation of an average of $v(n\triangle t, x)$ defined by

$$u_i^n = \frac{1}{\triangle x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} v(n\triangle t, x)\, dx, \tag{A.61}$$

where $x_{i\pm\frac{1}{2}} = \left(i \pm \frac{1}{2}\right)\triangle x$. From the definition of a weak solution of (A.58) and by choosing a particular test function φ, we can show that if u is a weak solution, then

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} v((n+1)\triangle t, x)\, dx$$
$$= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} v(n\triangle t, x)\, dx - \int_{n\triangle t}^{(n+1)\triangle t} [f(v(t, x_{i+\frac{1}{2}})) - f(v(t, x_{i-\frac{1}{2}})).]\, dt$$

Then dividing by $\triangle x$, we get from (A.61)

$$u_i^{n+1} = u_i^n - \frac{1}{\triangle x}\int_{n\triangle t}^{(n+1)\triangle t} [f(v(t, x_{i+\frac{1}{2}})) - f(v(t, x_{i-\frac{1}{2}}))]\, dt. \tag{A.62}$$

So, comparing (A.60) and (A.62) it is natural to choose

$$F(u_i^n, u_{i+1}^n) = \frac{1}{\triangle t} \int_{n\triangle t}^{(n+1)\triangle t} f(v(t, x_{i+\frac{1}{2}}))\, dt, \tag{A.63}$$

and then the scheme defined by (A.60) will be consistent with the original conservation law if F reduces to f for the case of a constant solution; i.e., if $v(t,x) \equiv c$, then necessarily

$$F(c, c) = f(c) \quad \forall c \in R. \tag{A.64}$$

This is the definition of a consistent scheme. This notion is very important, since according to the Lax–Wendroff theorem [166, 212], if the numerical scheme is consistent and in a conservative form, and if the resulting sequence of approximated solutions converges, then necessarily the limiting function is a weak solution of the conservation law.

Unfortunately, consistency and conservative form are not sufficient in general to capture the correct discontinuous solution. For example, schemes might develop undesirable oscillations. These conditions are related to the entropy condition mentioned above. Since we have not developed at all this notion, we will say no more about the numerical approximation of hyperbolic equations, and we refer to [166, 212] for more development.

We summarize all the numerical concerns by saying that a monotone (which means that the numerical flux function F is a monotone increasing function of each of its arguments), consistent and conservative scheme always captures the solution we would like to get (the unique entropic weak solution).

A.3 Difference Schemes in Image Analysis

A.3.1 Getting Started

In this section we would like to show how certain PDEs studied in this book can be discretized. The generic form of these PDEs is

$$\begin{cases} \mathcal{L}v = F \quad (t,x,y) \in R^+ \times \Omega, \\ \dfrac{\partial v}{\partial N}(t,x,y) = 0 \;\text{ on }\; R^+ \times \partial\Omega \quad \text{(Neumann boundary condition)}, \\ v(0,x,y) = f(x,y) \qquad \text{(initial condition)}, \end{cases} \tag{A.65}$$

where Ω is the image domain and N is the normal to the boundary of Ω, denoted by $\partial\Omega$. $\mathcal{L}$ is generally a second-order differential operator such as (see, for example, Section 3.3)

$$\frac{\partial v}{\partial t}(t,x,y) + H(x, y, v(t,x,y), \nabla v(t,x,y), \nabla^2 v(t,x,y)) = 0.$$

Example: One of the simplest PDEs presented in image analysis is the heat equation (see Section 3.3.1, where it is analyzed):

$$\begin{cases} \dfrac{\partial v}{\partial t} = \nu \Delta v = \nu \left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} \right) & (t,x,y) \in R^+ \times \Omega \\ \dfrac{\partial v}{\partial N}(t,x,y) = 0 \quad \text{on} \quad R^+ \times \partial\Omega, \\ v(0,x,y) = f(x,y), \end{cases} \tag{A.66}$$

where ν is a positive constant. ■

As already mentioned, finite differences are widely used in image processing, which is due to the digital structure of an image as a set of pixels uniformly distributed (see Section 1.2). It is then very easy and natural to associate with an image a uniform grid, as presented in Figure A.5. Since there is no reason to choose it differently, the grid spacing in the x and y directions is usually equal:

$$\triangle x = \triangle y = h.$$

Notice that in many articles from the computer vision literature, it is even chosen as $h = 1$, which means that the pixel size is chosen as the unit of reference. We will call the positions (ih, jh) vertices, nodes, or pixels equivalently. We will denote by $v_{i,j}^n$ (respectively $u_{i,j}^n$) the value of the exact solution (respectively the discrete solution) at location (ih, jh) and time $n\triangle t$.

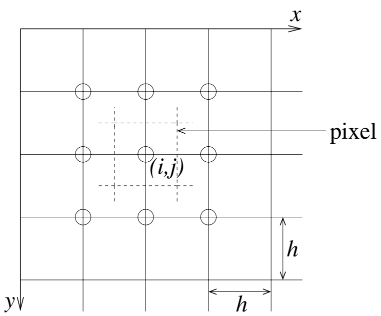


Figure A.5. Grid on the space domain. The circles indicate the vertices that belong to the 3×3 neighborhood of the vertex (i, j).

Example: The PDE (A.66) is an initial-boundary value problem. To discretize it, we need to consider the following:

- The equation. To find the difference scheme associated with the heat equation (A.66), we can proceed as in the one-dimensional case (see Section A.1.1), that is, by using Taylor expansions expanded about the index point $(n\triangle t, ih, jh)$. Naturally, the simplest method is to consider separately the discretization of each second-order derivative in x and y, which is equivalent to using the one-dimensional approximation. By doing so, we obtain

$$\left.\frac{\partial v}{\partial t} - \nu \triangle v\right|_{i,j}^{n} = \frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{\triangle t} - \nu \frac{v_{i+1,j}^{n} + v_{i-1,j}^{n} + v_{i,j+1}^{n} + v_{i,j-1}^{n} - 4v_{i,j}^{n}}{h^2} + \mathcal{O}(\triangle t) + \mathcal{O}(h^2).$$

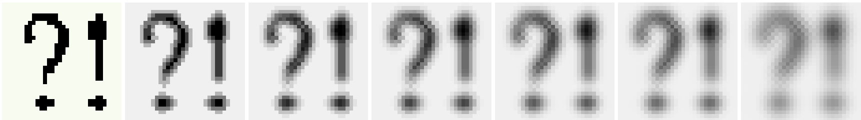
Then the difference scheme that we can propose is

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \frac{\nu \triangle t}{h^2}\left(u_{i+1,j}^{n} + u_{i-1,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n} - 4u_{i,j}^{n}\right). \tag{A.67}$$

It is of order $(2, 1)$.

- The boundary condition. The Neumann boundary condition can be taken into account by a symmetry procedure. If the value of a pixel (vertex) that is outside the domain is needed, we use the value of the pixel that is symmetric with respect to the boundaries.
- The initial condition is simply: $u_{i,j}^{0} = g_{i,j}$, where g is the discretization of f.

To illustrate this algorithm, we show in Figure A.6 some iterations as applied to a very simple image.



$n = 0$ $n = 1$ $n = 2$ $n = 3$ $n = 4$ $n = 5$ $n = 10$

Figure A.6. Example of results with the scheme (A.67) at different times (iterations), as applied to a simple and small image (32×32).

Notice that this example shows clearly the propagation of the information as the number of iterations increases. ■

Remark The scheme (A.67) was obtained by discretizing the Laplacian as a sum of the second-order derivatives in the x and y directions (see Figure A.8):

$$\triangle v|_{i,j} \approx \frac{v_{i+1,j}^{n} + v_{i-1,j}^{n} + v_{i,j+1}^{n} + v_{i,j-1}^{n} - 4v_{i,j}^{n}}{h^2}. \tag{A.68}$$

Clearly, this discretization does not take into account the 2-D nature and properties of this operator. To illustrate what we mean by "2-D nature and properties," we can remark that the Laplacian operator is rotationally invariant. If we apply a rotation of center (x, y) to the image v (with any angle $\theta \in [0, 2\pi[$), then $\triangle v(x, y)$ remains constant for all θ, as it should be for the discretization. Naturally, as we consider a discrete domain, we may ask only that $\triangle v|_{i,j}$ keep constant under rotations of $\pi/4$, as depicted in Figure A.7. This is not the case for discretization (A.68), since we obtain 1 or 2 (for $h = 1$) depending on the situation.

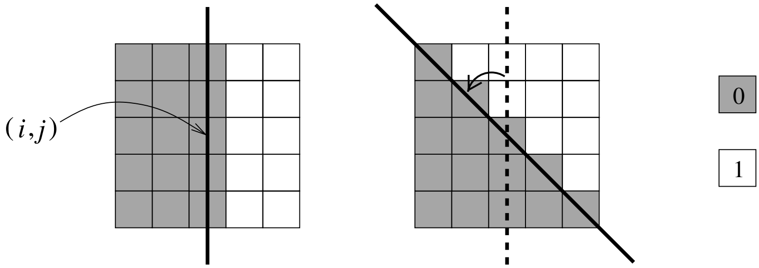


Figure A.7. Example of a binary image representing a vertical edge and the same image after a rotation of $\pi/4$ radians. Rotationally invariant operators estimated at the point in the middle should yield the same value in both situations.

To overcome this difficulty, we need to use the complete 3×3 neighborhood. We may propose the following approximation:

$$\begin{aligned} \triangle v|_{i,j} \approx{}& \lambda \frac{v_{i+1,j} + v_{i-1,j} + v_{i,j+1} + v_{i,j-1} - 4v_{i,j}}{h^2} \\ &+ (1-\lambda)\frac{v_{i+1,j+1} + v_{i-1,j+1} + v_{i+1,j-1} + v_{i-1,j-1} - 4v_{i,j}}{2h^2}, \end{aligned} \tag{A.69}$$

where $\lambda \in [0,1]$ is a constant to be chosen. We can verify that this approximation is consistent. Applying this operator (A.69) in the two situations from Figure A.7 and saying that both results should be equal yields $\lambda = \frac{1}{3}$, hence the approximation.

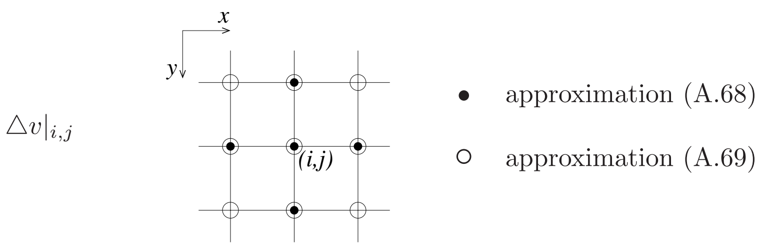


Figure A.8. Representation of the vertices involved in the finite-difference schemes.

Similarly, we can propose a discretization for the first-order derivatives in x and y that is consistent with the fact that the norm of the gradient is invariant under rotation. As we will see further, a second-order centered approximation of the first derivative in x is

$$\left.\frac{\partial v}{\partial x}\right|_{i,j} \approx \delta_x v_{i,j} = \frac{v_{i+1,j} - v_{i-1,j}}{2h}, \tag{A.70}$$

which can also be written in the y direction. The vertices involved in the estimation (A.70) are represented in Figure A.9. As for the case of the Laplacian, these approximations are in fact "one-dimensional" and do not really take advantage of the fact that the data is of dimension 2. This is visible if we consider the value of the norm of the gradient of u in the two situations described in Figure A.7: We obtain either $\frac{1}{2}$ or $\frac{1}{\sqrt{2}}$. The solution is to use more pixels in the estimation of the derivatives. In particular, we may suggest the following approximation:

$$\left.\frac{\partial v}{\partial x}\right|_{i,j} \approx \lambda \frac{v_{i+1,j} - v_{i-1,j}}{2h} + \frac{(1-\lambda)}{2}\left(\frac{v_{i+1,j+1} - v_{i-1,j+1}}{2h} + \frac{v_{i+1,j-1} - v_{i-1,j-1}}{2h}\right), \tag{A.71}$$

where λ is a parameter to be chosen. Applying the operator (A.71) in the two situations of Figure A.9 and by saying that both results should be equal yields $\lambda = \sqrt{2} - 1$, hence the approximation.

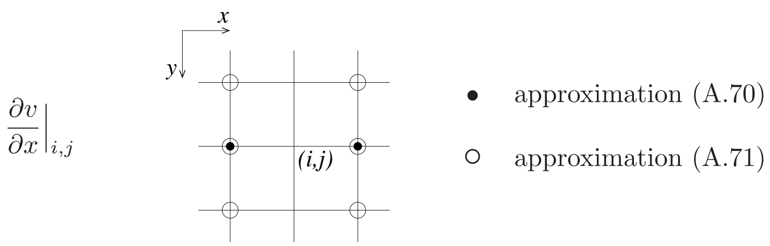


Figure A.9. Representation of the vertices involved in the finite-difference schemes.

Finally, we would like to mention that not only is using more points in the approximations good for rotation-invariance properties, but, practically, the result is also less sensitive to noise. The reason is that it is equivalent to perform a smoothing of the data before the estimation. ■

A.3.2 Image Restoration by Energy Minimization

We first consider the image restoration problem as presented in Section 3.2. By introducing a dual variable b, the problem becomes to minimize with respect to v and b the functional

$$J_\varepsilon(v,\ b) = \frac{1}{2}\int_\Omega |Rv - v_0|^2\, dx + \lambda \int_\Omega (b\,|\nabla v|^2 + \psi_\varepsilon(b))dx.$$

The so-called half-quadratic minimization algorithm consists in minimizing successively J_ε with respect to each variable. The algorithm is (see Section 3.2.4 for more detail) as follows: C++

`For` (v^0, b^0) `given`

- $v^{n+1} = \underset{v}{argmin}\, J_\varepsilon(v, b^n)$ `i.e.,` $\begin{cases} R^*Rv^{n+1} - \operatorname{div}(b^n \nabla v^{n+1}) = 0 \,\texttt{in}\, \Omega, \\ b^n \dfrac{\partial v^{n+1}}{\partial N} = 0 \text{ on } \partial\Omega. \end{cases}$
- $b^{n+1} = \underset{b}{argmin}\, J_\varepsilon(v^{n+1}, b)$ `i.e.,` $b^{n+1} = \dfrac{\phi'\left(\left|\nabla v^{n+1}\right|\right)}{2\left|\nabla v^{n+1}\right|}.$
- `Go back to the first step until convergence.`

`The limit` (v^∞, b^∞) `is the solution`

As far as discretization is concerned, the only term that may be difficult to approximate is the divergence operator. So for $b \geq 0$ and v given at nodes (i,j) the problem is to find an approximation at the node (i,j) for $\operatorname{div}(b\nabla v)$. This kind of term is naturally present as soon as there is a regularization with a ϕ function (see, for instance, optical flow, Section 5.3.2, or sequence segmentation, Section 5.3.3). The diffusion operator used in the Perona and Malik model is also of the same kind (see Sections 3.3.1 and 3.3.2).

Since this divergence operator may be rewritten as:

$$\operatorname{div}(b\nabla v) = \frac{\partial}{\partial x}\left(b\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(b\frac{\partial v}{\partial y}\right),$$

we can use the previous one-dimensional approximation and combine them. For example, if we use the central finite-difference approximation (A.70), we have

$$\begin{aligned} \operatorname{div}(b\nabla v)|_{i,j} &\approx \delta_x(b_{i,j}\delta_x v_{i,j}) + \delta_y(b_{i,j}\delta_y v_{i,j}) \\ &= \frac{1}{4h^2}\Big(b_{i+1,j}v_{i+2,j} + b_{i-1,j}v_{i-2,j} + b_{i,j+1}v_{i,j+2} + b_{i,j-1}v_{i,j-2} \\ &\qquad - (b_{i+1,j} + b_{i-1,j} + b_{i,j+1} + b_{i,j-1})v_{i,j}\Big). \end{aligned} \tag{A.72}$$

The main drawback of this representation is that it involves only the points $((i \pm 2)h, (j \pm 2)h)$, and none of the 3×3 neighborhood (see also Figure A.10). This may be nonrobust for noisy data or when there is considerable variation in this region. Another possibility is to combine forward and backward differences (see Section A.2):

$$\begin{aligned} \operatorname{div}(b\nabla v)|_{i,j} &\approx \delta_x^+(b_{i,j}\delta_x^- v_{i,j}) + \delta_y^+(b_{i,j}\delta_y^- v_{i,j}) \\ &= \frac{1}{h^2}\Big(b_{i+1,j}v_{i+1,j} + b_{i,j}v_{i-1,j} + b_{i,j+1}v_{i,j+1} + b_{i,j}v_{i,j-1} \\ &\qquad - (b_{i+1,j} + b_{i,j+1} + 2b_{i,j})v_{i,j}\Big). \end{aligned}$$

This approximation now involves the 3×3 neighborhood, but it introduces an asymmetry: The values of b at $((i-1)h, jh)$ and $(ih, (j-1)h)$ are not used. A solution is to use the following approximation for the derivatives:

$$\delta_x^* v_{i,j} = \frac{v_{i+\frac{1}{2},j} - v_{i-\frac{1}{2},j}}{h} \quad \text{and} \quad \delta_y^* v_{i,j} = \frac{v_{i,j+\frac{1}{2}} - v_{i,j-\frac{1}{2}}}{h},$$

where $v_{i\pm\frac{1}{2},j\pm\frac{1}{2}}$ is the value of v at location $((i\pm\frac{1}{2})h, (j\pm\frac{1}{2})h)$, which can be obtained by interpolation. As for (A.70), it is a second-order approximation. Then we have

$$\begin{aligned} \operatorname{div}(b\nabla v)|_{i,j} &\approx \delta_x^*(b_{i,j}\delta_x^* v_{i,j}) + \delta_y^*(b_{i,j}\delta_y^* v_{i,j}) \\ &= \frac{1}{h^2}\Big(b_{+0}v_{i+1,j} + b_{-0}v_{i-1,j} + b_{0+}v_{i,j+1} + b_{0-}v_{i,j-1} \\ &\qquad - (b_{+0} + b_{-0} + b_{0+} + b_{0-})v_{i,j}\Big), \end{aligned} \tag{A.73}$$

where $b_{\pm 0} = b_{i\pm\frac{1}{2},j}$ and $b_{0\pm} = b_{i,j\pm\frac{1}{2}}$. Notice that since we applied the operators δ_x^* and δ_y^* twice, this approximation uses the values of v only at $((i \pm 1)h, (j \pm 1)h)$ (see Figure A.10). However, interpolation is needed for b.

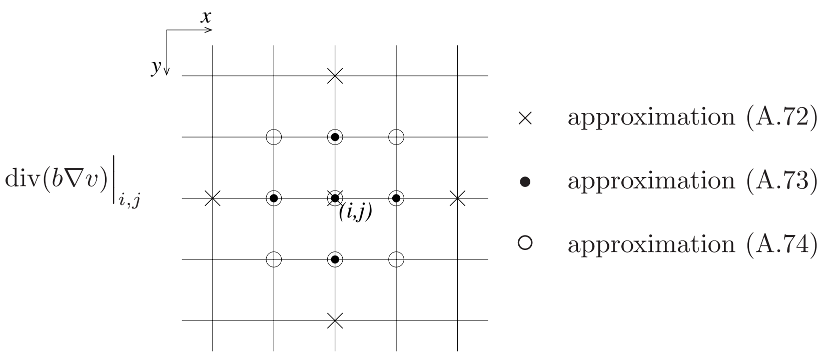


Figure A.10. Vertices involved in the approximation of the divergence term for the different schemes.

As mentioned previously for the estimation of the Laplacian, it would also be interesting to take into account the diagonal values. Then, we can look for an approximation such that

$$\operatorname{div}(b\nabla v)|_{i,j} \approx \tag{A.74}$$

$$\begin{aligned} &\frac{\lambda_p}{h^2}\left(b_{+0}v_{i+1,j} + b_{-0}v_{i-1,j} + b_{0+}v_{i,j+1} + b_{0-}v_{i,j-1} - \beta_p v_{i,j}\right) \\ &+ \frac{\lambda_d}{h^2}\left(b_{++}v_{i+1,j+1} + b_{--}v_{i-1,j-1} + b_{-+}v_{i-1,j+1} + b_{+-}v_{i+1,j-1} - \beta_d v_{i,j}\right) \end{aligned} \tag{A.75}$$

with $b_{\pm\pm} = b_{i\pm\frac{1}{2},j\pm\frac{1}{2}}$,

$$\begin{cases} \beta_p = b_{0+} + b_{0-} + b_{+0} + b_{-0}, \\ \beta_d = b_{++} + b_{--} + b_{+-} + b_{-+}, \end{cases}$$

and where λ_p and λ_d are two weights to be chosen. The first condition is that the scheme must be consistent, and it can be verified that this implies

$$\lambda_p + 2\lambda_d = 1. \tag{A.76}$$

Now there remains one degree of freedom. Two possibilities can been considered:

- The first is to choose (λ_p, λ_d) constant, and for instance equal to $(\frac{1}{2}, \frac{1}{4})$, giving a privilege to the principal directions.
- The second is to choose (λ_p, λ_d) by taking into account the orientation of the gradient of v, as described in Figure A.11.

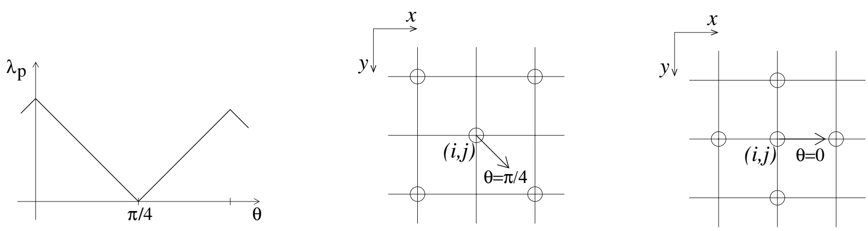


Figure A.11. Adaptive choice of the coefficients (λ_p, λ_d) as a function of θ, the orientation of ∇v. The two right-hand figures show which points will be used in the discretization of the divergence term in two specific situations.

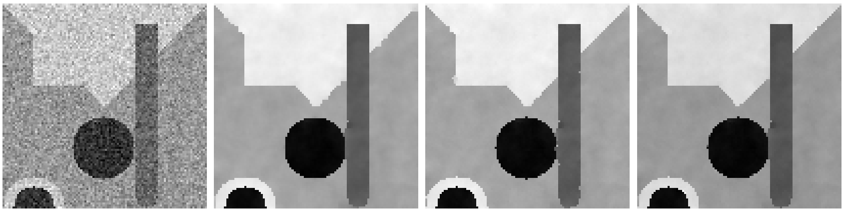
We tested these different discretizations as applied to a simple image with geometric structures (see Figure A.12). From left to right, an improvement in the results can be perceived (by observing the restoration of the horizontal and vertical edges). It is the adaptive choice that gives the best result.

A.3.3 Image Enhancement by the Osher and Rudin Shock Filters

This section concerns the shock-filter equation discussed in Section 3.3.3 and proposed by Osher and Rudin [261]:

$$\frac{\partial v}{\partial t} = -\left|\nabla v\right| F(L(v)), \tag{A.77}$$

where:



Initial image	$(1.0, 0.0)$	$(0.5, 0.25)$	Adaptive
	SNR=19.2	SNR=20.9	SNR=21.5

Figure A.12. Numerical tests for the different discretizations of the divergence term (the choice of (λ_p, λ_d) is indicated below the images).

- F is a Lipschitz function satisfying $F(0) = 0$, $\text{sign}(s)F(s) > 0$ $(s \neq 0)$, for example $F(s) = \text{sign}(s)$.
- L is a second-order edge detector, for example

  $$L(v) = \Delta v = v_{xx} + v_{yy} \quad \text{or} \quad L(v) = \frac{1}{|\nabla v|^2}\,(v_x^2\, v_{xx} + 2v_x v_y v_{xy} + v_y^2 v_{yy}),$$

  which corresponds to the second derivative of v in the direction of the normal to the isophotes.

Equation (A.77) involves two kinds of terms: a first-order term $|\nabla v|$ and a second-order term $F(L(v))$:

- L is discretized with central finite differences.
- $|\nabla v|$ has to be approximated with more care. v_x and v_y are approximated using the minmod operator $m(\alpha, \beta)$. For instance,

  $$v_x|_i = m(\delta_x^- v_i, \delta_x^+ v_i),$$

  where

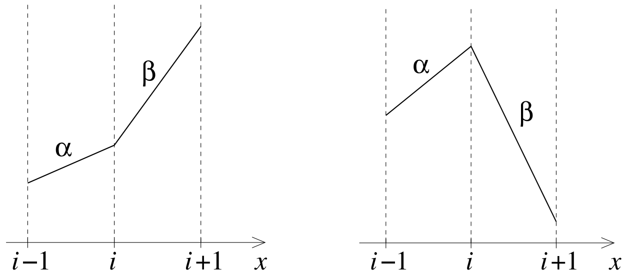
  $$m(\alpha, \beta) = \begin{cases} \text{sign}(\alpha)\ \min(|\alpha|, |\beta|) & \text{if } \ \alpha\beta > 0, \\ 0 & \text{if } \ \alpha\beta \leq 0. \end{cases}$$

  This function is usually called a flux limiter. As shown in Figure A.13, it permits us to choose the lowest slope, or zero in case of a local extremum (this prevents instabilities due to noise).

To summarize, the approximation of (A.77) is then given by

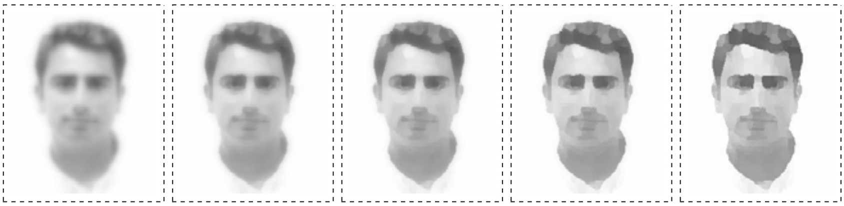
$$u_{i,j}^{n+1} = u_{i,j}^n - \frac{\Delta t}{h} \sqrt{(m(\delta_x^+ u_{i,j}^n,\ \delta_x^- u_{i,j}^n)^2 + (m(\delta_y^+ u_{i,j}^n,\ \delta_y^- u_{i,j}^n))^2}\, F_{i,j}(L(u^n)),$$

where $F_{i,j}(L(u)) = F(L_{i,j}(u))$. We show an example in Figure A.14 and refer to Section 3.3.3 for more detail.



(a) $v_x \approx \min(\alpha, \beta)$ (b) $v_x \approx 0$

Figure A.13. Approximation of the first derivative using the minmod function.



$t = 0$ $t = 1$ $t = 2$ $t = 4$ $t = 10$

Figure A.14. Example of the shock filter on a blurred image of a face. It shows clearly that this filter reconstructs a piecewise constant image that is not satisfying from a perceptual point of view (the result does not look like a real image).

A.3.4 Curve Evolution with the Level-Set Method

In this section we briefly discuss the discretization of the PDEs governing curve evolution. We examine only the case where these curves are identified as level sets of the same function $u(t, x)$ (Eulerian formulation). Of course, we have in mind the geodesic active contours model given by (see Section 4.3.3)

$$\frac{\partial v}{\partial t} = g(\,|\nabla I|\,)\ |\nabla v|\ \mathrm{div}\left(\frac{\nabla v}{|\nabla v|}\right) + \alpha\, g(\,|\nabla I|\,)\ |\nabla v| + \nabla g \cdot \nabla v. \quad \text{(A.78)}$$

As mentioned before, (A.78) involves two kinds of terms: a parabolic term (the first one) and hyperbolic terms (the last two). One can easily imagine that the discretization of each term needs an appropriate treatment, according to its nature (parabolic or hyperbolic). The main idea is that parabolic terms can be discretized by central finite differences, while hyperbolic terms need to be approximated by nonoscillatory upwind schemes. For the sake of clarity we start by examining the evolution driven by each of these terms.

For a detailed description of the above schemes and other numerical questions not developed here, we refer the reader to [300, 262].

Mean Curvature Motion

Let us consider

$$\begin{cases} \dfrac{\partial v}{\partial t} = |\nabla v| \operatorname{div}\left(\dfrac{\nabla v}{|\nabla v|}\right), \\ v(0,x,y) = v_0(x,y). \end{cases} \tag{A.79}$$

Equation (A.79) is a parabolic equation and has diffusive effects (like the heat equation). So, the use of upwind schemes is inappropriate, and classical central differences are used, C++

$$u_{i,j}^{n+1} = u_{i,j}^n + \triangle t \sqrt{(\delta_x u_{i,j}^n)^2 + (\delta_y u_{i,j}^n)^2}\, K_{i,j}^n,$$

where $K_{i,j}^n$ is the central finite-difference approximation of the curvature:

$$K = \operatorname{div}\left(\frac{\nabla v}{|\nabla v|}\right) = \frac{v_{xx}v_y^2 + v_{yy}v_x^2 - 2v_xv_yv_{xy}}{(v_x^2+v_y^2)^{3/2}}.$$

We let the reader write the expression of $K_{i,j}^n$. Unfortunately, the discretization of (A.79) is not as easy as it may appear. At some points (t,x), ∇v can be undefined, or $|\nabla v| = 0$, or $|\nabla v| = +\infty$. This situation can occur even in very simple cases [273]. For example, let us consider the shrinking of a unit circle in two dimensions which corresponds to

$$v(0,x,y) = \sqrt{x^2+y^2} - 1, \tag{A.80}$$

i.e., v_0 is the signed distance to the unit circle. Equation (A.79) is rotationally invariant, and if we search for the solution of the form

$$v(t,x,y) = \phi(t, \sqrt{x^2+y^2}),$$

we easily get $v(t,x,y) = \sqrt{x^2+y^2+2t} - 1$, from which we deduce

$$\nabla v = \frac{1}{\sqrt{x^2+y^2+2t}}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad |\nabla v| = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+2t}},$$
$$\frac{\nabla v}{|\nabla v|} = \frac{1}{\sqrt{x^2+y^2}}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \operatorname{div}\left(\frac{\nabla v}{|\nabla v|}\right) = \frac{1}{\sqrt{x^2+y^2}},$$

so the two last quantities are not defined at the origin, and effectively a spike occurs at the origin (see [273]). Moreover, the interface $\Gamma(t) = \{(x,y);\ v(t,x,y) = 0\}$ is the circle $x^2+y^2 = 1-2t$, and on $\Gamma(t)$ we have $|\nabla v|(t) = \sqrt{1-2t}$. Therefore $v(t,x,y)$ becomes more and more flat as the interface evolves and disappears at $t = \frac{1}{2}$. To circumvent this type of problem, we have to find a numerical trick that prevents the gradient norm from vanishing (or blowing up). This can be realized by reinitializing the function v from time to time to a signed distance function.

More precisely, we run (A.79) until some step n; then we solve the auxiliary PDE C++

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + \text{sign}(v)(|\nabla \phi| - 1) = 0, \\ \phi(0, x, y) = v(n\triangle t, x, y). \end{cases} \tag{A.81}$$

The resulting solution (as t tends to infinity), denoted by ϕ^∞, is a signed distance function whose zero-level set is the same as that of the function $v(n\triangle t, x, y)$. We refer to Section 4.3.4 for more details about this equation. Then we can run again (A.79) with the initial data $v(0, x, y) = \phi^\infty(x, y)$. Practically, this reinitialization has to be done every $n = 20$ iterations of the curve evolution equation, and it is usually performed about every 5 to 10 iterations of (A.81).[6]

Remark There exists another way to avoid doing the reinitialization step. It consists in considering a modified equation (A.79) that has the property of maintaining the norm of the gradient of the solution equal to one. For further details see [168, 273]. ■

An example of mean curvature motion is shown in Figure A.15. Notice that if we let the evolution run until convergence, any curve transforms into a circle and then collapses.

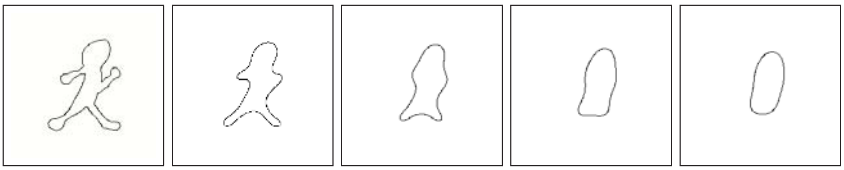


Figure A.15. Example of mean curvature motion.

Constant Speed Evolution

The second example is given by

$$\begin{cases} \dfrac{\partial v}{\partial t} = c\ |\nabla v|, \\ v(0, x, y) = v_0(x, y), \end{cases} \tag{A.82}$$

where c is a constant. This equation describes a motion in the direction normal to the front (the corresponding Lagrangian formulation of (A.82) is $\frac{\partial \Gamma}{\partial t}(t, p) = c\ N(t, p)$, where N is the normal to $\Gamma(t)$). For $c = 1$, it is also referred to as *grass fire*, since it simulates a grass fire wave-front propagation.

[6]These numbers are just an indication and naturally depend on the kind of equation to be solved and on the time steps.

Equation (A.82) is approximated by a nonoscillatory upwind scheme:

$$u_{i,j}^{n+1} = u_{i,j}^n + \triangle t \nabla^+ u_{i,j}^n,$$

where

$$\nabla^+ u_{i,j}^n = \Big[\max(\delta_x^- u_{i,j}^n, 0)^2 + \min(\delta_x^+ u_{i,j}^n, 0)^2 + \max(\delta_y^- u_{i,j}^n, 0)^2 + \min(\delta_y^+ u_{i,j}^n, 0)^2 \Big]^{1/2}.$$

We show in Figures A.16 and A.17 two examples of constant speed motions.

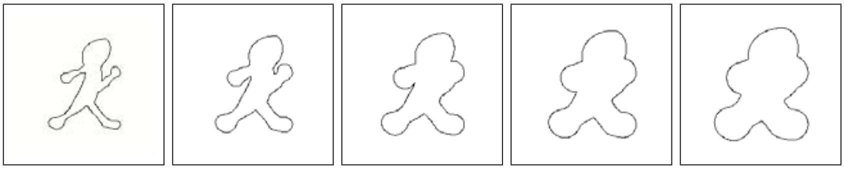


Figure A.16. Example of constant speed motion ($c = 1$, grass fire).

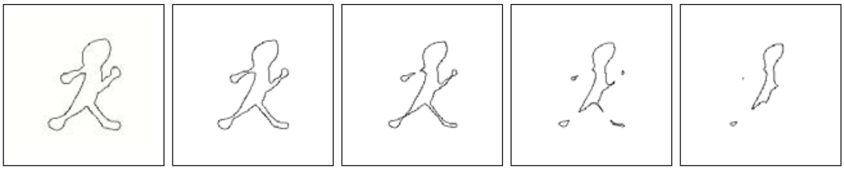


Figure A.17. Example of constant speed motion ($c = -1$).

Remark Motions like equation (A.82) and more generally with a monotone speed have the following property: Every point is crossed once and only once by the curve during its evolution. Notice that this is not the case for mean curvature motion. This property can be used to derive an efficient numerical approach called the *fast marching algorithm* [325, 299]. It is beyond the scope of this Appendix to explain this method, and we refer to the original articles and to [300] for more detail. ■

The Pure Advection Equation

We consider here the equation

$$\begin{cases} \dfrac{\partial v}{\partial t} = A(x,y) \cdot \nabla v, \\ v(0,x,y) = v_0(x,y), \end{cases} \tag{A.83}$$

where $A(x,y) = (A_1(x,y), A_2(x,y))$. For (A.83) we use a simple upwind scheme, i.e., we check the sign of each component of A and construct a one-side upwind difference in the appropriate direction:

$$u_{i,j}^{n+1} = u_{i,j}^n + \triangle t \Big[\max((A_1)_{i,j}^n, 0)\delta_x^- u_{i,j}^n + \min((A_1)_{i,j}^n, 0)\delta_x^+ u_{i,j}^n \\ \max((A_2)_{i,j}^n, 0)\delta_y^- u_{i,j}^n + \min((A_2)_{i,j}^n, 0)\delta_y^+ u_{i,j}^n \Big].$$

Image Segmentation by the Geodesic Active Contour Model

C++

Now we can consider the geodesic active contour model (A.78), which can be seen as the sum of the previous discretization. So the discrete scheme is

$$\begin{aligned} u_{i,j}^{n+1} = u_{i,j}^n + \triangle t \Big[& g_{i,j}\ K_{i,j}^n [(\delta_x u_{i,j}^n)^2 + (\delta_y u_{i,j}^n)^2]^{\frac{1}{2}} \\ & + \alpha [\max(g_{i,j}, 0)\nabla^+ + \min(g_{i,j}, 0)\nabla^-] u_{i,j}^n \\ & + \max(g_{x\,i,j}, 0)\delta_x^- u_{i,j}^n + \min(g_{x\,i,j}, 0)\ \delta_x^+ u_{i,j}^n \\ & + \max(g_{y\,i,j}, 0)\ \delta_y^- u_{i,j}^n + \min(g_{y\,i,j}, 0)\ \delta_y^+ u_{i,j}^n \Big]. \end{aligned}$$

where $\nabla^- u_{i,j}^n$ is obtained from $\nabla^+ u_{i,j}^n$ by inverting the signs plus and minus.

Figure A.18 shows a typical example (see Figure 4.13 for the complete evolution).

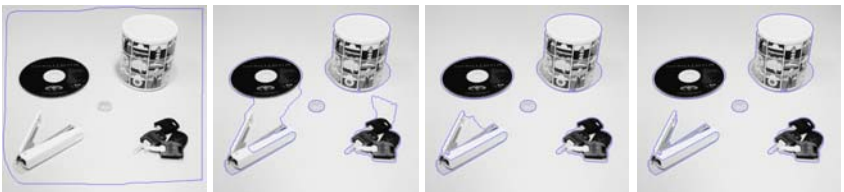


Figure A.18. Example of segmentation. Different iterations are displayed.

Remark From a numerical point of view, all the equations presented in this section involve local operations. Since we are interested only in the curve, it is enough to update the values in a band around the current position of the curve, also called a *narrow band*. Naturally, this region (band) has to be updated as the curve evolves. See, for instance, [300] for more details. ■

Appendix B

Experiment Yourself!

How to Read This Chapter

The objective of this book was to explain the underlying mathematics of the PDE-based and variational approaches used in image processing. In Appendix A we explain the main ideas of finite difference methods, which allow one to discretize continuous equations.

As a further step, we wish to provide programming tools so that readers can implement the approaches and test them on their own data.

A web site is associated with this second edition. This web site contains some related links, complementary of information, and also source code that allow the reader to test easily some variational and PDE-based approaches.

`http://www-sop.inria.fr/books/imath`

Section B.1 justifies the technical choices for the software developments, which is essentially the chosen image processing library. Section B.2 gives a nonexhaustive list of the C++ codes available online and shows an example of CImg code.

B.1 The CImg Library

The chosen programming language is the object-oriented language C++, which is freeware and a very efficient language. Bjarne Stroustrup [311] is the designer and original implementor of C++. The interested reader will also find a wide variety of books and online tutorials on this language.

Many image processing libraries are proposed online. We chose the CImg library, which stands for "Cool Image," developed by David Tschumperlé in 2000. The CImg library is simple to use and efficient:

- The CImg Library is an open source C++ toolkit for image processing. It provides simple classes and functions to load, save, process, and display images in your own C++ code.
- It is highly portable and is fully functional on Unix/X11, Windows, MacOS X, and FreeBSD operating systems. It should compile on other systems as well (possibly without display capabilities).
- It consists of only a single header file CImg.h, which must be included in your C++ program source.
- It contains useful image processing algorithms for image loading/saving, displaying, resizing/rotating, filtering, object drawing (text, lines, faces, curves, ellipses,...), etc.
- Images are instanced by a class able to represent images up to 4 dimensions wide (x, y, z, v) (from 1-D scalar signals to 3-D volumes of vector-valued pixels), with template pixel types.
- It depends on a minimal number of libraries: you can compile it with only standard C libraries. No need for exotic libraries and complex dependencies.
- Additional features appear with the use of ImageMagick, libpng, or libjpeg: install the ImageMagick package or link your code with libpng and libjpeg to be able to load and save compressed image formats (GIF, BMP, TIF, JPG, PNG, ...). Available for any platform.

The CImg package includes full documentation and many examples to help the developer in his first steps.

`http://cimg.sourceforge.net`

B.2 What Is Available Online?

Table B.1 gives a sample of approaches whose CImg codes are available from the book web site. The proposed codes correspond in general to approaches explained in the book. The symbol C++ written in the margin will indicate to the reader that some code is available. The list in Table B.1 is not exhaustive, and we invite the reader to consult regularly the book web site, which will contain updated information.

Tables B.2 and B.3 give an idea of the CImg code necessary to run a heat equation. Table B.2 is the main code, which calls the function `get_isotropic2d` defined in Table B.3. We refer the reader to the book web site for more information.

External contributors are encouraged to submit their own C++ source codes. The procedure is described at the book web site. We hope that such an initiative will enable readers to experiment and compare different approaches without too much effort. We thank in advance new contributors, who will help us to develop this free source code database for PDE-based approaches.

Image restoration		
Description	Ref.	Sections
Half Quadratic Minimization	[108]	3.2.4 and A.3.2
Heat equation	[43, 5, 198]	3.3.1
Perona-Malik equation	[275]	3.3.1 and 3.3.2
Coherence enhancing diffusion	[336]	3.3.1
Shock filters	[261]	3.3.3 and A.3.3

Level-sets and segmentation		
Description	Ref.	Sections
Mean curvature motion	[147]	4.3.2 and A.3.4
Geodesic active contours	[85, 223, 222]	4.3.5 and A.3.4
Distance function initialization	[223, 312]	4.3.4 and A.3.4
Vector–valued regularization	[324]	5.5

Some applications		
Description	Ref.	Sections
Inpainting	[320, 321]	5.1
Optical flow	[184]	5.3.2

Table B.1. Example of approaches proposed as CImg source code. Please consult the book web site for an updated list. Contributors are welcome.

```
#define cimg_plugin "pde_plugin.h"
#include "CImg.h"
using namespace cimg_library;

int main(int argc,char **argv) {

  CImg<float> img("my_image.gif");
  CImgDisplay disp(img,"heat flow");

  img.get_isotropic2d(1000,−5,&disp);

  while (!disp.closed) disp.wait();

  return 0;
}
```

Table B.2. CImg main code for the heat equation.

```
//! 2D isotropic smoothing with the classical heat flow PDE
/**
   Return an image that has been smoothed by the classical isotropic heat flow PDE
   \param nb_iter = Number of PDE iterations
   \param dt = PDE time step. If dt<0, -dt represents the maximum PDE velocity
                    that will be applied on image pixels (adaptative time step)
   \param disp = Display used to show the PDE evolution. If disp==NULL,
                         no display is performed.
**/
CImg<T> get_isotropic2d(const int nb_iter=100, const float dt=0,
                                            CImgDisplay *disp=NULL) const {

    CImg<float> img(*this), veloc(*this,false);

    // Iterative process
    for (unsigned int iter=0; iter<nb_iter; iter++) {

        // Estimation of the Laplacian
        CImg_3x3(I,float);
        cimg_mapV(img,k) cimg_map3x3(img,x,y,0,k,I) {
            const float
                ixx = Inc + Ipc − 2*Icc,
                iyy = Icn + Icp − 2*Icc;
            veloc(x,y,k) = ixx + iyy;
        }

        // Estimation of the optimal time step
        float xdt = dt;
        if (dt<0) {
            CImgStats stats(veloc,false);
            xdt = −dt/cimg::abs(cimg::max(stats.min,stats.max));
        }

        // Update the image
        img += xdt*veloc;

        // Display resulting image
        if (disp) {
            if (disp−>resized) disp−>resize();
            img.display(*disp);
        }
    }

    // Returns final result
    return img;
}
```

Table B.3. CImg code for heat equation.

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