A

The Classical Mittag-Leffler Theorem Derived from Bourbaki's

This is the Mittag-Leffler theorem from complex variables.

Theorem A.1 (Mittag-Leffler theorem). *Let $\varnothing \neq \Omega \subset \mathbb{C}$ be open, let $\{c_1, c_2, c_3, \ldots\}$ be a discrete subset of Ω, and let $(r_n)_{n=1}^\infty$ be a sequence of rational functions of the form*

$$r_n(z) = \sum_{j=1}^{m_n} \frac{a_{j,n}}{(z - c_n)^j} \qquad (n, m_n \in \mathbb{N},\ a_{1,n}, \ldots, a_{m_n,n} \in \mathbb{C},\ z \in \Omega \setminus \{c_n\}).$$

Then there is a meromorphic function f on Ω with $\{c_1, c_2, c_3, \ldots\}$ as its set of poles such that, for each $n \in \mathbb{N}$, the singular part of f at c_n is r_n.

This theorem is usually treated in courses on complex variables, and a proof can be found in probably any text on the subject (such as [Conway 78], for example). But what does this theorem have to do with Theorem 2.4.14? Following [Esterle 84], we show in this appendix that the Mittag-Leffler theorem can, in fact, be derived from Theorem 2.4.14. Besides Theorem 2.4.14, the proof also requires (of course) some knowledge of complex variables, as well as further topological background from Sections 3.1 to 3.4.

We first need to bring complete metric spaces into the picture. To this end, we prove a lemma.

Lemma A.2. *Let $\varnothing \neq \Omega \subset \mathbb{R}^m$ be open. Then there is a sequence $(K_n)_{n=1}^\infty$ of compact subsets of Ω with the following properties.*

(i) $\Omega = \bigcup_{n=1}^\infty K_n$;

(ii) $K_n \subset \mathring{K}_{n+1}$ *for all* $n \in \mathbb{N}$;

(iii) *For each $n \in \mathbb{N}$, every component of $\mathbb{R}^m_\infty \setminus K_n$ contains a component of $\mathbb{R}^m_\infty \setminus \Omega$.*

Proof. If $\Omega = \mathbb{R}^m$, letting $K_n := B_n[0]$ for $n \in \mathbb{N}$ will do.

Hence, we may suppose that $\Omega \neq \mathbb{R}^m$. We may then define

$$K_n := \left\{ x \in \Omega : \|x\| \le n \text{ and } \operatorname{dist}(x, \mathbb{R}^m \setminus \Omega) \ge \frac{1}{n} \right\} \qquad (n \in \mathbb{N}).$$

It is easy to see that (i) and (ii) are satisfied.

Let $n \in \mathbb{N}$, and let C be a component of $\mathbb{R}^m_\infty \setminus K_n$, where $\mathbb{R}^m_\infty$ is the one-point compactification of $\mathbb{R}^m$.

Case 1: $\infty \in C$.

Let C_∞ be the component of $\mathbb{R}^m_\infty \setminus \Omega$ containing ∞. Then $C_\infty \subset \mathbb{R}^m_\infty \setminus K_n$ is connected and contains ∞. Consequently, $C_\infty \subset C$ must hold.

Case 2: $\infty \notin C$.

The subset $C_0 := \{x \in \mathbb{R}^m : \|x\| > n\} \cup \{\infty\}$ of $\mathbb{R}^m_\infty \setminus K_n$ is connected. Since C is a component of $\mathbb{R}^m_\infty \setminus K_n$, it follows that either $C_0 \subset C$ or $C_0 \cap C = \varnothing$. Since $\infty \in C_0$ whereas $\infty \notin C$, the first alternative cannot occur, so that $C_0 \cap C = \varnothing$; that is, $\|x\| \le n$ for all $x \in C$ and therefore, by the definition of K_n, $\operatorname{dist}(x, \mathbb{R}^m \setminus \Omega) < \frac{1}{n}$ for all $x \in C$. Consequently, there is $x_0 \in \mathbb{R}^m \setminus \Omega$ such that $B_{\frac{1}{n}}(x_0) \cap C \neq \varnothing$. Note that $B_{\frac{1}{n}}(x_0) \subset \mathbb{R}^m \setminus K_n$ by the definition of K_n. Since $B_{\frac{1}{n}}(x_0)$ is connected, Proposition 3.4.16 yields that $B_{\frac{1}{n}}(x_0) \subset C$. As in the first case, we see that C contains the component of $\mathbb{R}^m_\infty \setminus \Omega$ containing x_0.
□

With the help of Lemma A.2, we can introduce a metric on the space of continuous functions on an open subset of $\mathbb{R}^m$.

Proposition A.3. *Let $\varnothing \neq \Omega \subset \mathbb{R}^m$ be open, and let $(K_n)_{n=1}^\infty$ be a sequence as in Lemma* A.2. *For $f, g \in C(\Omega, \mathbb{F})$ define*

$$d_n(f,g) := \sup\{|f(x) - g(x)| : x \in K_n\} \qquad (n \in \mathbb{N})$$

and

$$d(f,g) := \sum_{n=1}^\infty \frac{1}{2^n} \frac{d_n(f,g)}{1 + d_n(f,g)}.$$

Then:

(i) *d is a metric on $C(\Omega, \mathbb{F})$ such that $d(f+h, g+h) = d(f,g)$ for all $f, g, h \in C(\Omega, \mathbb{F})$;*
(ii) *The topology on $C(\Omega, \mathbb{F})$ induced by d is $\mathcal{T}_\mathcal{K}|_{C(\Omega,\mathbb{F})}$, where $\mathcal{K}$ is the collection of all compact subsets of Ω;*
(iii) *The metric space $(C(\Omega, \mathbb{F}), d)$ is complete.*

Proof. (i) is clear.

For (ii), let $\mathcal{C} := \{K_n : n \in \mathbb{N}\}$. It is routine to check that d induces the topology $\mathcal{T}_\mathcal{C}|_{C(\Omega,\mathbb{F})}$. Since $\mathcal{C} \subset \mathcal{K}$, it is clear that $\mathcal{T}_\mathcal{K}|_{C(\Omega,\mathbb{F})}$ is finer than $\mathcal{T}_\mathcal{C}|_{C(\Omega,\mathbb{F})}$. On the other hand, $\left\{ \overset{\circ}{K}_n : n \in \mathbb{N} \right\}$ is an open cover for Ω. Hence, for any $K \in \mathcal{K}$, there is $n \in \mathbb{N}$ such that $K \subset \overset{\circ}{K}_n \subset K_n$. It follows that $\mathcal{T}_\mathcal{K}|_{C(\Omega,\mathbb{F})}$ and $\mathcal{T}_\mathcal{C}|_{C(\Omega,\mathbb{F})}$ coincide.

Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in $(C(\Omega,\mathbb{F}),d)$. Then, for each $x \in \Omega$, the sequence $(f_n(x))_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{F}$, so that $f(x) := \lim_{n\to\infty} f_n(x)$ exists. It is routine to check that $(f_n)_{n=1}^{\infty}$ converges to $f\colon \Omega \to \mathbb{F}$ uniformly on each $K \in \mathcal{K}$. Let $x_0 \in \Omega$, and let $\epsilon > 0$ be such that $\overline{B_\epsilon(x_0)} \subset \Omega$. Since $(f_n)_{n=1}^{\infty}$ converges to f uniformly on $\overline{B_\epsilon(x_0)}$, it follows that $f|_{\overline{B_\epsilon(x_0)}}$ is continuous, and since $\overline{B_\epsilon(x_0)}$ is a neighborhood of x_0, the function f is continuous at x_0. This proves (iii). □

To prove Theorem A.1, we apply Theorem 2.4.14 not to all of $C(\Omega,\mathbb{C})$, but to a subspace.

Definition A.4. *Let $\varnothing \neq \Omega \subset \mathbb{C}$ be open. Then $H(\Omega)$ denotes the space of all holomorphic functions on Ω.*

For the following corollary of Proposition A.3, we identify $\mathbb{C}$ with $\mathbb{R}^2$.

Corollary A.5. *Let $\varnothing \neq \Omega \subset \mathbb{C}$ be open, and let d be as in Proposition* A.3. *Then $H(\Omega)$ is a closed subspace of* $(C(\Omega,\mathbb{C}),d)$ *(and therefore complete).*

Proof. Let $(f_n)_{n=1}^{\infty}$ be a sequence in $H(\Omega)$ that converges to $f \in C(\Omega,\mathbb{C})$ with respect to d and thus, by Proposition A.3(ii), uniformly on all compact subsets of Ω. It is well known that this forces f to be holomorphic, too (see, for example, [Conway 78, 2.1 Theorem]). □

We can now prove Theorem A.1 with the help of Theorem 2.4.14.

Proof (of Theorem A.1). Let $(K_n)_{n=1}^{\infty}$ be a sequence as specified by Lemma A.2, and let $\Omega_{n-1} := \mathring{K}_n$ for $n \in \mathbb{N}$. For each $n \in \mathbb{N}_0$, let $\tilde{d}_n$ be a metric on $H(\Omega_n) \subset C(\Omega_n,\mathbb{C})$ as specified by Proposition A.3.

Let $n \in \mathbb{N}_0$ be fixed, and let $S_n := \{m \in \mathbb{N} : c_m \in \Omega_n\}$. Since K_{n+1} is compact, and since $\{c_1, c_2, \ldots\}$ is discrete, each S_n is finite (and possibly empty). Hence, the rational function $R_n := \sum_{m\in S_n} r_m$ is well defined (the sum is finite). Let X_n be the set of those meromorphic functions f on Ω_n such that $f - R_n$ has a holomorphic extension to all of Ω_n, and define a metric on it via

$$d_n(f,g) := \tilde{d}_n(f - R_n, g - R_n) \qquad (f,g \in X_n).$$

It follows from Corollary A.5 that (X_n, d_n) is a complete metric space. For $n \in \mathbb{N}$, let $\phi_n\colon X_n \to X_{n-1}$ denote the restriction map. In view of Proposition A.3(ii), it is clear that ϕ_n is continuous.

We claim that ϕ_n has dense range. Let $g \in X_{n-1}$, so that $g - R_{n-1} \in H(\Omega_{n-1})$. Since the rational functions r_m for $m \in S_n \setminus S_{n-1}$ have their poles off Ω_{n-1}, it follows that $g - R_n$ is in $H(\Omega_{n-1})$ as well. Due to Lemma A.2 and Runge's approximation theorem [Conway 78, 1.14 Corollary], we can find a sequence $(q_m)_{m=1}^{\infty}$ of rational functions with poles off Ω_{n-1} (which therefore belong to $H(\Omega_{n-1})$) such that $\tilde{d}_{n-1}(g - R_n, q_m) \to 0$. It follows that

$$
\begin{aligned}
d_{n-1}(g, \phi_n(q_m + R_n)) &= \tilde{d}_{n-1}(g - R_{n-1}, q_m + R_n - R_{n-1}) \\
&= \tilde{d}_{n-1}(g - R_n + (\underbrace{R_n - R_{n-1}}_{\in H(\Omega_{n-1})}), q_m + (\underbrace{R_n - R_{n-1}}_{\in H(\Omega_{n-1})})) \\
&= \tilde{d}_{n-1}(g - R_n, q_m) \\
&\to 0
\end{aligned}
$$

as $m \to \infty$, and consequently, $\phi_n(X_n)$ is dense in X_{n-1}.

From Theorem 2.4.14, we conclude that $\bigcap_{n=1}^{\infty}(\phi_1 \circ \cdots \circ \phi_n)(X_n)$ is dense in X_0 and thus, in particular, is not empty. Let $(g_n)_{n=0}^{\infty}$ be a sequence such that $g_n \in X_n$ for $n \in \mathbb{N}_0$ and $\phi_n(g_n) = g_{n-1}$ for $n \in \mathbb{N}$. Define $f\colon \Omega \setminus \{c_1, c_2, \ldots\} \to \mathbb{C}$ by letting $f(z) := g_n(z)$ if $z \in \Omega_n \setminus \{c_m : m \in S_n\}$. Since $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$, this defines a meromorphic function on Ω with the required properties. $\square$

B

Failure of the Heine–Borel Theorem in Infinite-Dimensional Spaces

We first show that the Heine–Borel theorem holds in all finite-dimensional, normed spaces.

The following is the crucial assertion for this.

Proposition B.1. *Let E be a finite-dimensional, linear space (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$), and let $\|\cdot\|$ and $|||\cdot|||$ be norms on E. Then there is a constant $C \geq 0$ such that*

$$\|x\| \leq C|||x||| \quad \textit{and} \quad |||x||| \leq C\|x\| \qquad (x \in E).$$

Proof. Let $e_1, \ldots, e_n \in E$ be a basis for E. For $x = \lambda_1 e_1 + \cdots + \lambda_n e_n$, let

$$|x| := \max\{|\lambda_1|, \ldots, |\lambda_n|\}.$$

Clearly, $|\cdot|$ is a norm on E.

Set $C_1 := \|e_1\| + \cdots + \|e_n\|$, and note that

$$\|x\| \leq |\lambda_1|\|e_1\| + \cdots + |\lambda_n|\|e_n\| \leq C_1|x| \qquad (x \in E).$$

Next, we show that there is $C_2 \geq 0$ with $|x| \leq C_2\|x\|$ for all $x \in E$.

Assume otherwise. Then there is a sequence $(x_m)_{m=1}^{\infty}$ in E with $|x_m| > m\|x_m\|$ for $m \in \mathbb{N}$. Let

$$y_m := \frac{x_m}{|x_m|} \qquad (m \in \mathbb{N}).$$

For each $m \in \mathbb{N}$, there are unique $\lambda_{1,m}, \ldots, \lambda_{n,m} \in \mathbb{F}$ with $y_m = \sum_{j=1}^{n} \lambda_{j,m} e_j$. It follows that

$$1 = |y_m| = \max\{|\lambda_{1,m}|, \ldots, |\lambda_{n,m}|\} \qquad (m \in \mathbb{N}).$$

In particular, the sequence $((\lambda_{1,m}, \ldots, \lambda_{n,m}))_{m=1}^{\infty}$ is bounded in $\mathbb{F}^n$ and thus has—by the Bolzano–Weierstraß theorem (for $\mathbb{R}^n$ if $\mathbb{F} = \mathbb{R}$ and for $\mathbb{R}^{2n}$ if $\mathbb{F} = \mathbb{C}$)—a convergent subsequence, say $((\lambda_{1,m_k}, \ldots, \lambda_{n,m_k}))_{k=1}^{\infty}$ with limit

$(\lambda_1, \ldots, \lambda_n)$. It follows that $(y_{m_k})_{k=1}^{\infty}$ converges, with respect to $|\cdot|$, to $y := \lambda_1 e_1 + \cdots + \lambda_n e_n$, so that necessarily $|y| = 1$ and thus $y \neq 0$. Since $\|\cdot\| \leq C_1 |\cdot|$, we see that $y = \lim_{k\to\infty} y_{m_k}$ as well with respect to $\|\cdot\|$. However,

$$\|y_m\| = \left\| \frac{x_m}{|x_m|} \right\| = \frac{\|x_m\|}{|x_m|} < \frac{1}{m} \to 0,$$

so that $y = 0$. This is impossible.

For $C' := \max\{C_1, C_2\}$, we have

$$\|x\| \leq C'|x| \quad \text{and} \quad |x| \leq C'\|x\| \qquad (x \in E),$$

and in a similar vein, we obtain $C'' \geq 0$ such that

$$|||x||| \leq C''|x| \quad \text{and} \quad |x| \leq C''|||x||| \qquad (x \in E).$$

Consequently, with $C := C'C''$,

$$\|x\| \leq C|||x||| \quad \text{and} \quad |||x||| \leq C\|x\| \qquad (x \in E)$$

holds. □

As an immediate consequence, any two norms on a finite-dimensional vector space E yield equivalent metrics, and if E is a Banach space with respect to one norm, it is a Banach space with respect to *every* norm. Hence, if $\dim E = n$ and if $e_1, \ldots, e_n$ is a basis of E, the map

$$\mathbb{F}^n \to E, \quad (\lambda_1, \ldots, \lambda_n) \mapsto \lambda_1 e_1 + \cdots + \lambda_n e_n$$

is continuous with continuous inverse and carries Cauchy sequences to Cauchy sequences (as does its inverse).

We therefore obtain the following.

Corollary B.2. *Let E be a finite-dimensional, normed space. Then E is a Banach space, and a subset of E is compact if and only if it is closed and bounded.*

Combining this with Proposition 2.4.5(ii) yields the following.

Corollary B.3. *Let E be a normed space, and let F be a finite-dimensional subspace of E. Then F is closed in E.*

By Corollary B.2, the Heine–Borel theorem holds true in any finite-dimensional normed space. For the converse, we require the following.

Lemma B.4 (Riesz' lemma). *Let E be a normed space, and let F be a closed, proper (i.e., $F \neq E$), subspace of E. Then, for each $\theta \in (0, 1)$, there is $x_\theta \in E$ with $\|x_\theta\| = 1$, and $\|x - x_\theta\| \geq \theta$ for all $x \in F$.*

Proof. Let $x_0 \in E \setminus F$, and let $\delta := \operatorname{dist}(x_0, F)$. If $\delta = 0$, the closedness of F implies $x_0 \in F$, which is a contradiction. Hence, $\delta > 0$ must hold. Since $\theta \in (0,1)$, we have $\delta < \frac{\delta}{\theta}$. Choose $y_\theta \in F$ with $0 < \|x_0 - y_\theta\| < \frac{\delta}{\theta}$, and let

$$x_\theta := \frac{y_\theta - x_0}{\|y_\theta - x_0\|},$$

so that trivially $\|x_\theta\| = 1$. Let $x \in F$, and note that

$$\|x - x_\theta\| = \left\| x - \frac{y_\theta - x_0}{\|y_\theta - x_0\|} \right\| = \frac{1}{\|y_\theta - x_0\|} \| \|y_\theta - x_0\| x - y_\theta + x_0 \|.$$

Since $x, y_\theta \in F$, we have $\|y_\theta - x_0\| x - y_\theta \in F$ as well, so that

$$\| \|y_\theta - x_0\| x - y_\theta + x_0 \| \geq \operatorname{dist}(x_0, F) = \delta.$$

Eventually, we obtain

$$\|x - x_\theta\| = \frac{1}{\|y_\theta - x_0\|} \| \|y_\theta - x_0\| x - y_\theta + x_0 \| > \frac{\theta}{\delta}\delta = \delta.$$

Since $x \in F$ was arbitrary, this completes the proof. □

We can now prove the following.

Theorem B.5. *For a normed space E, the following are equivalent.*

(i) *Every closed and bounded subset of E is compact.*
(ii) *The closed unit sphere of E is compact.*
(iii) $\dim E < \infty$.

Proof. (i) $\Longrightarrow$ (ii) is trivial.

(ii) $\Longrightarrow$ (iii): Suppose that $\dim E = \infty$. We construct a sequence in $S_1[0]$ that has no convergent subsequence, so that $S_1[0]$ cannot be compact by Theorem 2.5.10

Choose $x_1 \in E$ with $\|x_1\| = 1$. Since $\dim E = \infty$, the one-dimensional space F_1 spanned by x_1 is not all of E. By Riesz' lemma, there is thus $x_2 \in E$ such that $\|x_2 - x\| \geq \frac{1}{2}$ for $x \in F_1$, so that, in particular, $\|x_2 - x_1\| \geq \frac{1}{2}$. Since $\dim E = \infty$, the two-dimensional space F_2 spanned by $\{x_1, x_2\}$ is also not all of E. Again by Riesz' lemma, there is thus $x_3 \in E$ such that $\|x_3 - x\| \geq \frac{1}{2}$ for $x \in F_2$, and thus, in particular, $\|x_3 - x_j\| \geq \frac{1}{2}$ for $j = 1, 2$. Let F_3 be the linear span of $\{x_1, x_2, x_3\}$, so that $F_3 \neq E$. Appealing again to Riesz' lemma, we obtain $x_4 \in E$, and so on.

Inductively, we thus obtain a sequence $(x_n)_{n=1}^\infty$ in $S_1[0]$ such that

$$\|x_n - x_m\| \geq \frac{1}{2} \qquad (n \neq m).$$

It is clear that no subsequence of $(x_n)_{n=1}^\infty$ can be a Cauchy sequence.

Finally, (iii) $\Longrightarrow$ (i) is Corollary B.2. □

C

The Arzelà–Ascoli Theorem

As we have seen in Example 2.5.13, the Heine–Borel theorem is false for $C([0,1],\mathbb{F})$ (and, more generally, for *every* infinite-dimensional normed space; see Appendix B).

The Arzelà–Ascoli theorem can be thought of as the right substitute for the Heine–Borel theorem in spaces of continuous functions. In this appendix, we derive it from Tychonoff's theorem.

For the statement of the Arzelà–Ascoli theorem, we require two notions: that of relative compactness, which was introduced in Exercise 2.5.7, and that of equicontinuity.

Definition C.1. *Let $(X,\mathcal{T})$ be a topological space, and let (Y,d) be a metric space. Then a family $\mathfrak{F}$ of functions from X to Y is said to be* equicontinuous at $x_0 \in X$ *if, for each $\epsilon > 0$, there is $N \in \mathcal{N}_{x_0}$ such that $d(f(x_0), f(x)) < \epsilon$ for all $f \in \mathfrak{F}$ and $x \in N$. If $\mathfrak{F}$ is equicontinuous at every point of X, we call* $\mathfrak{F}$ equicontinuous.

If $\mathfrak{F}$ consists only of one function, say f, then $\mathfrak{F}$ is equicontinuous if and only if f is continuous.

Let $(K,\mathcal{T})$ be a compact topological space, let (Y,d) be a metric space, and let $f : K \to Y$ be continuous. Then $f(K)$ is compact and therefore has finite diameter, which means that f is actually in $C_b(K,Y)$. In the following result, we have $C(K,Y) = C_b(K,Y)$ equipped with the metric D introduced in Example 2.1.2(d).

Theorem C.2 (Arzelà–Ascoli theorem). *Let $(K,\mathcal{T})$ be a compact topological space, and let (Y,d) be a complete metric space. Then the following are equivalent for $\mathfrak{F} \subset C(K,Y)$.*

(i) *$\mathfrak{F}$ is relatively compact in $C(K,Y)$.*
(ii) (a) *$\{f(x) : f \in \mathfrak{F}\}$ is relatively compact in Y for each $x \in X$, and*
 (b) *$\mathfrak{F}$ is equicontinuous.*

Proof. (i) $\Longrightarrow$ (ii): For $x \in K$, let

$$\pi_x : C(K,Y) \to Y, \quad f \mapsto f(x).$$

Then π_x is continuous, so that $\pi_x\left(\overline{\mathfrak{F}}\right)$ is compact in Y and contains $\{f(x) : f \in \mathfrak{F}\}$. Consequently, $\{f(x) : f \in \mathfrak{F}\}$ is relatively compact in Y. This proves (ii)(a).

Assume towards a contradiction that (ii)(b) is false; that is, there are $x_0 \in X$ and $\epsilon_0 > 0$ such that, for each $N \in \mathcal{N}_{x_0}$, there are $f_N \in \mathfrak{F}$ and $x_N \in N$ such that $d(f_N(x_0), f_N(x_N)) \geq \epsilon_0$. Since $\overline{\mathfrak{F}}$ is compact, the net $(f_N)_{N \in \mathcal{N}_{x_0}}$, where $\mathcal{N}_{x_0}$ is ordered by reversed set inclusion, has a subnet $(f_\alpha)_{\alpha \in \mathbb{A}}$ converging (with respect to D) to some $f \in \overline{\mathfrak{F}}$. Let $N_0 \in \mathcal{N}_{x_0}$ be such that $d(f(x_0), f(x)) < \frac{\epsilon_0}{3}$ for $x \in N_0$ (this is possible because f is continuous), let $\phi : \mathbb{A} \to \mathcal{N}_{x_0}$ be the cofinal map associated with the subnet $(f_\alpha)_{\alpha \in \mathbb{A}}$, and let $\alpha \in \mathbb{A}$ be such that $D(f_\alpha, f) < \frac{\epsilon_0}{3}$ and $\phi(\alpha) \subset N_0$. We then have:

$$\begin{aligned}
&d(f_\alpha(x_0), f_\alpha(x_{\phi(\alpha)})) \\
&\leq d(f_\alpha(x_0), f(x_0)) + d(f(x_0), f(x_{\phi(\alpha)})) + d(f(x_{\phi(\alpha)}), f_\alpha(x_{\phi(\alpha)})) \\
&\leq D(f_\alpha, f) + d(f(x_0), f(x_{\phi(\alpha)})) + D(f_\alpha, f) \\
&< \frac{2\epsilon_0}{3} + d(f(x_0), f(x_{\phi(\alpha)})) \\
&< \frac{2\epsilon_0}{3} + \frac{\epsilon_0}{3}, \qquad \text{because } \phi(\alpha) \subset N_0, \\
&= \epsilon_0.
\end{aligned}$$

This contradicts the choices of f_N and x_N for $N \in \mathcal{N}_{x_0}$. (This part of the proof has not made any reference to the completeness of Y or to the compactness of K.)

(ii) $\Longrightarrow$ (i): Since (a) and (b) are not affected if we replace $\mathfrak{F}$ by its closure, we can suppose without loss of generality that $\mathfrak{F}$ is closed.

Let $(f_\alpha)_\alpha$ be a net in $\mathfrak{F}$. We show that it has a convergent subnet.

For $x \in K$, let $K_x := \overline{\{f(x) : f \in \mathfrak{F}\}}$, so that K_x is compact by (a). Tychonoff's theorem then yields the compactness of the topological product $\prod_{x \in K} K_x$. Hence, $(f_\alpha)_\alpha$ has a subnet $(f_\beta)_{\beta \in \mathbb{B}}$ such that $(f_\beta(x))_{\beta \in \mathbb{B}}$ converges for each $x \in K$. By Exercise 3.2.12(a), this means in particular that, for each $\epsilon > 0$ and $x \in K$, there is $\beta_{x,\epsilon} \in \mathbb{B}$ such that $d(f_\beta(x), f_\gamma(x)) < \epsilon$ for all $\beta, \gamma \in \mathbb{B}$ with $\beta_{x,\epsilon} \preceq \beta, \gamma$.

Fix $\epsilon > 0$. For each $x \in X$, choose an open neighborhood U_x of x such that $d(f(x), f(x')) < \frac{\epsilon}{3}$ for $x' \in U_x$. Clearly, $\{U_x : x \in K\}$ is an open cover for K. Since K is compact, there are $x_1, \ldots, x_n \in K$ such that

$$K = U_{x_1} \cup \cdots \cup U_{x_n}.$$

Choose $\beta_\epsilon \in \mathbb{B}$ such that $d(f_\beta(x_j), f_\gamma(x_j)) < \frac{\epsilon}{3}$ for all $j = 1, \ldots, n$ and $\beta, \gamma \in \mathbb{B}$ with $\beta_\epsilon \preceq \beta, \gamma$. Let $x \in K$, and choose $j \in \{1, \ldots, n\}$ such that $x \in U_{x_j}$. Then we have for $\beta, \gamma \in \mathbb{B}$ with $\beta_\epsilon \preceq \beta, \gamma$:

$$
\begin{aligned}
d(f_\beta(x), f_\gamma(x)) &\le d(f_\beta(x), f_\beta(x_j)) + d(f_\beta(x_j), f_\gamma(x_j)) + d(f_\gamma(x_j), f_\gamma(x)) \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}
$$

It follows that $D(f_\beta, f_\gamma) \le \epsilon$ for $\beta, \gamma \in \mathbb{B}$ with $\beta_\epsilon \preceq \beta, \gamma$, so that $(f_\beta)_{\beta\in\mathbb{B}}$ is a Cauchy net in $C(K, Y)$. Since $B(K, Y)$ is complete by Example 2.4.4(c), it follows from Exercise 3.2.12(b), that $(f_\beta)_{\beta\in\mathbb{B}}$ converges to some $f \in B(K, Y)$. As in Example 2.4.6, where the case of the domain being a metric space was treated, one sees that $f \in C(K, Y)$. □

Let $(K, \mathcal{T})$ be a compact topological space. Then $C(K, \mathbb{F})$ is a normed space, so that it makes sense to speak of bounded sets. As an immediate consequence of Theorem C.2, we obtain what may be construed as an infinite-dimensional Heine–Borel theorem.

Corollary C.3. *Let $(K, \mathcal{T})$ be a compact topological space. Then a subset of $C(K, \mathbb{F})$ is compact if and only if it is closed, bounded, and equicontinuous.*

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