Appendix A
Spherical Harmonics

The goal of this appendix is to prove that the restrictions of harmonic polynomials of degree ℓ to the sphere do in fact correspond to the spherical harmonics of degree ℓ. Recall that in Section 1.6 we used solutions to the Legendre equation (Equation 1.11) to define the spherical harmonics. In this appendix we construct bona fide solutions $\mathbf{P}_{\ell,m}$ to the Legendre equation; then we show that each of the span of the spherical harmonics of degree ℓ is precisely the set of restrictions of harmonic polynomials of degree ℓ to the sphere.

Physicists and chemists know the Legendre functions well. One very useful explicit expression for these functions is given in terms of derivatives of a polynomials.

Definition A.1 *Let ℓ be a nonnegative integer and let m be an integer satisfying $0 \leq m \leq \ell$. Define the ℓ, m* Legendre function *by*

$$\mathbf{P}_{\ell,m}(t) := \frac{(-1)^m}{\ell! 2^\ell}(1-t^2)^{m/2}\partial_t^{\ell+m}(t^2-1)^\ell.$$

For each ℓ, the function $\mathbf{P}_{\ell,0}$ is called the Legendre polynomial of degree ℓ*.*

Note that the so-called Legendre polynomial is in fact a polynomial of degree ℓ, as it is the ℓth derivative of a polynomial of degree 2ℓ. Legendre functions with $m \neq 0$ are often called *associated Legendre functions*.

Recall the Legendre equation (Equation 1.11):

$$(1-t^2)P''(t) - 2tP'(t) + \left(\ell(\ell+1) - \frac{m^2}{(1-t^2)}\right)P(t) = 0. \tag{A.1}$$

Proposition A.1 *The Legendre functions of Definition A.1 satisfy the Legendre equation.*

There are many ways to prove this proposition. Our proof is straightforward, elementary and rather ugly. For a more elegant proof via the "Rodrigues formula," see [WW, Chapter XV] or [DyM, Section 4.12].

Proof. First we will show that the Legendre polynomial of degree ℓ satisfies the Legendre equation with $m = 0$. Then we will deduce that for any $m = 1, \ldots, \ell$, the Legendre function $\mathbf{P}_{\ell,m}$ satisfies the Legendre equation.

When $m = 0$ the Legendre equation reduces to

$$(1-t^2)P''(t) - 2tP'(t) + \ell(\ell+1)P(t) = 0. \tag{A.2}$$

We use the binomial expansion to find the coefficients of the Legendre polynomial of degree ℓ. For convenience, we multiply through by $2^\ell \ell!$:

$$\begin{aligned}(2^\ell \ell!)\mathbf{P}_{\ell,0}(t) &= \partial_t^\ell (t^2-1)^\ell \\ &= \partial_t^\ell \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^{\ell-k} t^{2k} \\ &= \sum_{k=(\ell+\epsilon)/2}^{\ell} \binom{\ell}{k} (-1)^{\ell-k} \frac{(2k)!}{(2k-\ell)!} t^{2k-\ell}.\end{aligned}$$

Differentiating once we find

$$(2^\ell \ell!)\mathbf{P}'_{\ell,0}(t) = \sum_{k=1+(\ell-\epsilon)/2}^{\ell} \binom{\ell}{k} (-1)^{\ell-k} \frac{(2k)!}{(2k-\ell-1)!} t^{2k-\ell-1}$$

and, differentiating again,

$$(2^\ell \ell!)\mathbf{P}''_{\ell,0}(t) = \sum_{k=1+(\ell+\epsilon)/2}^{\ell} \binom{\ell}{k} (-1)^{\ell-k} \frac{(2k)!}{(2k-\ell-2)!} t^{2k-\ell-2},$$

where $\epsilon = 0$ if ℓ is even and $\epsilon = 1$ if ℓ is odd. Hence to show that

$$(1-t^2)\mathbf{P}''_{\ell,0}(t) - 2t\mathbf{P}'_{\ell,0}(t) + \ell(\ell+1)\mathbf{P}_{\ell,0}(t) = 0,$$

it suffices to show the vanishing of the following expression:

$$\sum_{k=1+(\ell+\epsilon)/2}^{\ell} \binom{\ell}{k} \frac{(-1)^{\ell-k}(2k)!}{(2k-\ell-2)!} t^{2k-\ell-2} - \sum_{k=1+(\ell+\epsilon)/2}^{\ell} \binom{\ell}{k} \frac{(-1)^{\ell-k}(2k)!}{(2k-\ell-2)!} t^{2k-\ell} - 2 \sum_{k=1+(\ell-\epsilon)/2}^{\ell} \binom{\ell}{k} \frac{(-1)^{\ell-k}(2k)!}{(2k-\ell-1)!} t^{2k-\ell} + \ell(\ell+1) \sum_{k=(\ell+\epsilon)/2}^{\ell} \binom{\ell}{k} \frac{(-1)^{\ell-k}(2k)!}{(2k-\ell)!} t^{2k-\ell}.$$

We will show that the coefficient of each power of t is zero. The coefficient of t^ℓ is

$$-\frac{(2\ell)!}{(\ell-2)!} - 2\frac{(2\ell)!}{(\ell-1)!} + \ell(\ell+1)\frac{(2\ell)!}{\ell!} = \frac{2\ell}{(\ell-1)!}\left(-(\ell-1) - 2 + (\ell+1)\right) = 0.$$

The coefficients of $t^{\ell-1}$ through t^2 take the form (with an appropriate choice of k, and ignoring an overall factor of $(-1)^{\ell-k}$):

$$-\binom{\ell}{k}\frac{(2k+2)!}{(2k-\ell)!} - \binom{\ell}{k}\frac{(2k)!}{(2k-\ell-2)!} - 2\binom{\ell}{k}\frac{(2k)!}{(2k-\ell-1)!} + \ell(\ell+1)\binom{\ell}{k}\frac{(2k)!}{(2k-\ell)!}$$
$$= \binom{\ell}{k}\frac{(2k)!}{(2k-\ell)!}\Big(-2(\ell-k)(2k+1) - (2k-\ell-1)(2k-\ell) - 2(2k-\ell) + \ell(\ell+1)\Big) = 0.$$

There is one more term: t^1 if ℓ is odd and t^0 if ℓ is even. We will leave the even case to the reader. If ℓ is odd, then the coefficient of t^1 is

$$\binom{\ell}{(\ell+3)/2}(-1)^{\frac{\ell+1}{2}}(\ell+3)! - 2\binom{\ell}{(\ell+1)/2}(-1)^{(\ell-1)/2}(\ell+1)! + \ell(\ell+1)\binom{\ell}{(\ell+1)/2}(-1)^{(\ell-1)/2}(\ell+1)!$$
$$= \binom{\ell}{(\ell+1)/2}(\ell+1)!\left((\ell+2)(\ell+1) + 2 - \ell(\ell+1)\right) = 0.$$

The calculation for the case of even ℓ is similar. So we have shown that the Legendre polynomial $\mathbf{P}_{\ell,0}$ of degree ℓ satisfies the Legendre equation with $m = 0$.

Next we fix an integer m with $1 \leq m \leq \ell$ and show that $\mathbf{P}_{\ell,m}$ satisfies the Legendre equation (Equation A.1). Since the function $\mathbf{P}_{\ell,0}$ satisfies Equation A.2, we have

$$(1-t^2)\mathbf{P}''_{\ell,0}(t) - 2t\mathbf{P}'_{\ell,0}(t) + \ell(\ell+1)\mathbf{P}_{\ell,0}(t) = 0.$$

Differentiating m times with respect to t, we find that

$$\Big((1-t^2)\partial_t^{m+2} - 2(m+1)t\partial_t^{m+1} + \big(\ell(\ell+1) - m(m+1)\big)\partial_t^m\Big)\mathbf{P}_{\ell,0}(t) = 0. \quad \text{(A.3)}$$

Define $c := (-1)^m/(\ell!2^\ell)$. From Definition A.1 we know that $c\,\partial_t^m\mathbf{P}_{\ell,0}(t) = (1-t^2)^{(-\frac{m}{2})}\mathbf{P}_{\ell,m}(t)$. Differentiating this expression twice in a row we obtain

$$\begin{aligned} c\,\partial_t^{m+1}\mathbf{P}_{\ell,0}(t) &= (1-t^2)^{(-\frac{m}{2})}\Big(\frac{mt}{1-t^2}\mathbf{P}_{\ell,m}(t) + \mathbf{P}'_{\ell,m}(t)\Big) \\ c\,\partial_t^{m+2}\mathbf{P}_{\ell,0}(t) &= (1-t^2)^{(-\frac{m}{2})}\Big(\frac{m}{1-t^2} + \frac{(m+2)t^2}{(1-t^2)^2}\Big)\mathbf{P}_{\ell,m}(t) \\ &\quad + (1-t^2)^{(-\frac{m}{2})}\Big(\frac{2mt}{1-t^2}\mathbf{P}'_{\ell,m}(t) + \mathbf{P}''_{\ell,m}(t)\Big). \end{aligned}$$

Here we have used the fact (easily verified by induction) that for any sufficiently differentiable function $f(t)$ we have

$$\partial_t^m(1-t^2)f(t) = (1-t^2)\partial_t^m f + 2mt\partial_t^{m-1}f(t) + m(m-1)\partial_t^{m-2}f(t).$$

Plugging these expressions into Equation A.3, multiplying by $c(1-t^2)^{(m/2)}$ and simplifying we find that

$$0 = (1-t^2)\mathbf{P}''_{\ell,m}(t) - 2t\mathbf{P}'_{\ell,m}(t) + \left(\ell(\ell+1) - \frac{m}{1-t^2}\right)\mathbf{P}_{\ell,m}(t).$$

In other words, the function $\mathbf{P}_{\ell,m}$ satisfies the Legendre equation (Equation A.1). □

It is natural to wonder whether there are any other solutions to Legendre's equation. Since the equation is linear (in P, P' and P''), there should be two solutions for each value of m^2. For $m^2 \neq 0$ there are indeed two solutions: $\mathbf{P}_{\ell,\pm m}$. The case $m^2 = 0$ is discussed in detail in Simmons' undergraduate text on ordinary differential equations [Sim, Sections 28, 29 and 44]. The point is that a solution corresponds to a continuous function on the sphere only if it is bounded near $t = \pm 1$ and only one of the solutions to the Legendre equation is bounded near $t = \pm 1$.

Now we are ready to define the spherical harmonic functions. In Section 1.6 we gave examples for $\ell = 0, 1, 2$; here is the general definition.

Definition A.2 *Let ℓ be a nonnegative integer and let m be an integer satisfying $-\ell \leq m \leq \ell$. Define the ℓ, m* spherical harmonic function $Y_{\ell,m} \colon [0, \pi] \oplus (-\pi, \pi] \to \mathbb{C}$ *by*

$$Y_{\ell,m}(\theta, \phi) := c_{\ell,m} \mathbf{P}_{\ell,|m|}(\cos\theta) e^{im\phi},$$

where the the constant $c_{\ell,m}$ takes the value

$$\sqrt{\frac{(\ell - m)!(2\ell + 1)}{(\ell + m)!4\pi}}.$$

For each ℓ, linear combinations of the vectors

$$\left\{Y_{\ell,m} \colon m = -\ell, \ldots, \ell\right\}$$

are spherical harmonics of degree ℓ.

In fact, every spherical harmonic function is the restriction to the sphere S^2 in $\mathbb{R}^3$ of a harmonic polynomial on $\mathbb{R}^3$. Recall the vector space $\mathcal{Y}^\ell$ of restrictions of harmonic polynomials of degree ℓ in three variables to the sphere S^2 (Definition 2.6).

Proposition A.2 *Suppose ℓ is a nonnegative integer. Then the span of the set $\{Y_{\ell,-\ell}, \ldots, Y_{\ell,\ell}\}$ is $\mathcal{Y}^\ell$.*

Proof. First we will show that the set $\left\{Y_{\ell,m} \colon m = -\ell, \ldots, \ell\right\}$ is linearly independent. Next we will show it is a subset of $\mathcal{Y}^\ell$. The proof ends with a dimension count.

To show linear independence, consider an arbitrary linear combination equalling zero:

$$0 = \sum_{m=-\ell}^{\ell} C_m \mathbf{P}_{\ell,|m|}(\cos\theta) e^{im\phi}.$$

We must show that each $C_m = 0$. By Exercise 2.2, the set

$$\left\{e^{im(\cdot)} \colon m = -\ell, \ldots, \ell\right\},$$

where $e^{im(\cdot)}\colon [0,\pi] \to \mathbb{C}, x \to e^{imx}$, is linearly independent, so we can conclude that for each m we have $C_m \mathbf{P}_{\ell,m} = 0$. Hence we will be done with the proof of linear independence if we can show that for each m, the function $\mathbf{P}_{\ell,m}(\cos\theta)$ is not the zero function. Now $(-1)^m/\ell!2^\ell$ is a nonzero constant, and $(1-\cos^2(\pi/2))^{m/2} \neq 0$, so it suffices to show that $\partial_t^{\ell+m}(t^2-1)^\ell$ is not the zero polynomial. But $(t^2-1)^\ell$ is a polynomial of degree 2ℓ in t, so its first 2ℓ derivatives are nonzero. Since $m \leq \ell$, it follows that $\mathbf{P}_{\ell,m}$ is not the zero function. We have shown the required linear independence.

A longer argument is required to show that $\{Y_{\ell,m}\colon m = -\ell, \dots, \ell\} \subset \mathcal{Y}^\ell$. We begin by showing that for any nonnegative integer k the expression $\partial_t^k(t^2-1)^\ell$ is a polynomial in the variables $\alpha := 1-t^2$ and $\beta := t$. According to the chain rule for partial derivatives we have $\partial_t = 2\beta\partial_\alpha + \partial_\beta$, so applying ∂_t to any polynomial in α and β yields a polynomial in α and β. Consider $(t^2-1)^\ell = \alpha^\ell$, which is a polynomial in α and β. Hence, by induction on k, we can conclude that $\partial_t^k(t^2-1)^\ell$ is a polynomial in α and β.

Another induction on k shows that for any nonnegative integer k the expression $\partial_t^k(t^2-1)^\ell$ is a homogeneous polynomial of degree $2\ell - k$ in the variables $\sqrt{\alpha}$ and β. The key to the inductive step is that $(t^2-1)^\ell = \sqrt{\alpha}^{2\ell}$, a polynomial of degree 2ℓ, while applying $\partial_t = 2\beta\partial_\alpha + \partial_\beta$ lowers the degree (in $\sqrt{\alpha}$ and β) by one.

The point is that if $t = \cos\theta$, then we have

$$
\begin{aligned}
r^2\alpha &= r^2(1-t^2) = r^2\sin^2\theta = x^2 + y^2 \\
r\beta &= r\cos\theta = z.
\end{aligned}
$$

So a polynomial of degree d in $\sqrt{\alpha}$ and β that is also a polynomial in α will be homogeneous of degree d in x, y and z. Setting $k = \ell + m$ and applying the results of our inductions above, we find that

$$r^{\ell-m}\partial_t^{\ell+m}(t^2-1)^\ell$$

is a polynomial of degree $\ell - m$ in x, y, and z. Also,

$$r^m(1-t^2)^{m/2}e^{\pm im\phi} = r^m\sin^m\theta(\cos\phi \pm i\sin\phi)^m = (x \pm iy)^m,$$

which is a homogeneous polynomial in x and y of degree m when $m \geq 0$. Note that $\theta \in [0,\pi]$, so $\sin\theta \geq 0$. Hence the function

$$r^\ell(1-t^2)^{m/2}\partial_t^{\ell+m}(t^2-1)^\ell e^{im\phi}$$

is a polynomial of degree ℓ in x, y and z; by inspection, it is homogeneous. We know from Equation 1.12 and Proposition A.1 that if we evaluate this function at $t = \cos\theta$ we obtain a harmonic function. Restricting this homogeneous polynomial to the sphere we obtain $\mathbf{P}_{\ell,m}$. Hence $\mathbf{P}_{\ell,m} \in \mathcal{Y}^\ell$ for $m = 0, 1, \ldots, \ell$.

Next we show that the harmonic function from Equation 1.12 is a polynomial in x, y and z of degree ℓ even when $m < 0$. To see this, note that (by yet another induction, this time on $-m$ and left to the reader), for any nonnegative integer ℓ and any integer m with $-\ell \leq m < 0$ there is a polynomial q of two variables such that $q(\alpha, \beta)$ has degree $\ell + m$ in $\sqrt{\alpha}$ and β and

$$\partial_t^{\ell+m}(t^2 - 1)^\ell = \alpha^{-m} q(\alpha, \beta).$$

Note that $r^{\ell+m} q(\alpha, \beta)$ is a polynomial of degree $\ell + m$ in x, y and z. Hence, for $m < 0$ we have

$$\begin{aligned} r^\ell (1 - t^2)^{m/2} \partial_t^{\ell+m} (t^2 - 1)^\ell e^{im\phi} &= r^\ell \alpha^{-m/2} (e^{-i\phi})^{-m} q(\alpha, \beta) \\ &= (x - iy)^{-m} r^{\ell+m} q(\alpha, \beta), \end{aligned}$$

which is a polynomial of degree $-m + \ell + m = \ell$ in x, y and z. This polynomial is clearly homogeneous, and by Equation 1.12 it is harmonic. Restricting this homogeneous polynomial to the sphere we obtain $\mathbf{P}_{\ell,m}$. Hence $\mathbf{P}_{\ell,m} \in \mathcal{Y}^\ell$ for $m = -\ell, \ldots, -1$. Thus we have shown that each function $\mathbf{P}_{\ell,m}$ is the restriction to the sphere S^2 of a harmonic polynomial of degree ℓ on $\mathbb{R}^3$. In other words, $\left\{Y_{\ell,m} \colon m = -\ell, \ldots, \ell\right\} \subset \mathcal{Y}^\ell$.

Finally, since the $Y_{\ell,m}$'s are linearly independent, they span a $(2\ell + 1)$-dimensional subset of $\mathcal{Y}^\ell$. But we know by Proposition 7.1 that $\mathcal{Y}^\ell$ has dimension at most $(2\ell + 1)$. Hence $\mathcal{Y}^\ell$ is equal to the span of the $Y_{\ell,m}$'s. □

The following proposition justifies the reliance on spherical harmonics in spherically symmetric problems involving the Laplacian. To state it succinctly, we introduce the vector space $C_2 \subset L^2(\mathbb{R}^3)$ of continuous functions whose first and second partial derivatives are all continuous.

Proposition A.3 *Suppose D is a differential operator of the form*

$$D = \nabla^2 + u(r),$$

where u is a real-valued function of r. Then the vector space

$$K := \left\{f \in L^2(\mathbb{R}^3) \colon f \in C_2 \text{ and } Df = 0\right\}$$

of solutions to the differential equation $Df = 0$ is spanned by solutions of the form $\alpha \otimes Y_{\ell,m}$, where $\alpha \in \mathcal{I}$, ℓ is a nonnegative integer and m is an integer such that $|m| \leq \ell$.

The technical conditions on f are quite reasonable: if a physical situation has a discontinuity, we might look for solutions with discontinuities in the function f and its derivatives. In this case, we might have to consider, e.g., piecewise-defined combinations of smooth solutions to the differential equation. These solutions might not be linear combinations of spherical harmonics.

Proof. Let V denote the set of solutions in $L^2(\mathbb{R}^3)$ obtained by multiplying a spherical harmonic by a spherically symmetric function:

$$V := \left\{\alpha \otimes Y_{\ell,m} \in \mathcal{I} \otimes \mathcal{Y} \colon \alpha \in C_2 \text{ and } D(\alpha \otimes Y_{\ell,m}) = 0\right\}.$$

It suffices to show that $K \cap \left(V^{\perp}\right) = 0$. So suppose that $f \in K \cap \left(V^{\perp}\right)$, i.e., suppose that f and its first and second partial derivatives are continuous, that $Df = 0$ and that f is orthogonal to every solution obtained by separation of variables. We will show that $f = 0$.

By Fubini's Theorem (Theorem 3.1), the function $\|f\|_{S^2}$ defined by

$$\|f\|_{S^2} : r \mapsto \sqrt{\int_{S^2} |f(r,\theta,\phi)|^2 \,\sin\theta \, d\theta d\phi}$$

lies in $\mathcal{I}$ because $f \in L^2(\mathbb{R}^3)$. Now for any nonnegative integer ℓ and any integer m with $|m| \leq \ell$, the function $Y_{\ell,m} f$ is measurable and

$$\int_{\mathbb{R}^3} \left|Y_{\ell,m}(\theta,\phi) f(r,\theta,\phi)\right|^2 r^2 \, dr \, \sin\theta \, d\theta d\phi < \infty$$

because $Y_{\ell,m}$ is bounded and $f \in L^2(\mathbb{R}^3)$. Again by Fubini's Theorem,

$$\alpha_{\ell,m}(r) := \int_{S^2} Y_{\ell,m}(\theta,\phi) f(r,\theta,\phi) \,\sin\theta \, d\theta d\phi = \left\langle Y_{\ell,m}, f(r,\cdot,\cdot)\right\rangle_{S^2}$$

defines a measurable function $\alpha_{\ell,m}$ on $\mathbb{R}^{\geq 0}$. Note that by the Schwarz Inequality (Proposition 3.6) on $L^2(S^2)$ we have

$$\left|\alpha_{\ell,m}\right|^2 = \left|\int_{S^2} Y_{\ell,m}(\theta,\phi) f(\cdot,\theta,\phi) \sin\theta \, d\theta d\phi\right|^2 \leq \left\|Y_{\ell,m}\right\|_{S^2}^2 \|f\|_{S^2}^2 .$$

Since $\left\|Y_{\ell,m}\right\|^2$ does not depend on r and $\|f\|_{S^2} \in \mathcal{I}$, it follows that $\alpha_{\ell,m} \in \mathcal{I}$.

Next we introduce some convenient notation. By Exercise 1.12 we know that $\nabla^2 = \nabla_r^2 + \nabla_{\theta,\phi}^2$, where we set

$$\nabla_r^2 := \partial_r^2 + \frac{2}{r}\partial_r$$
$$\nabla_{\theta,\phi}^2 := \frac{1}{r^2}\partial_\theta^2 + \frac{\cos\theta}{r^2 \sin\theta}\partial_\theta + \frac{1}{r^2 \sin^2\theta}\partial_\phi^2.$$

Note that $\nabla^2_{\theta,\phi}$ is Hermitian-symmetric on $L^2(S^2)$ by Exercise 3.26.

Since $Df = 0$ we have $(\nabla^2_r + u)f = -\nabla^2_{\theta,\phi} f$. Hence for $r \in (0, \infty)$ we have

$$\begin{aligned}
(\nabla^2_r + u)\alpha_{\ell,m}(r) &= \left\langle Y_{\ell,m}, (\nabla^2_r + u) f(r, \cdot, \cdot)\right\rangle_{S^2} \\
&= -\left\langle Y_{\ell,m}, \nabla^2_{\theta,\phi} f(r, \cdot, \cdot)\right\rangle_{S^2} \\
&= -\left\langle \nabla^2_{\theta,\phi} Y_{\ell,m}, f(r, \cdot, \cdot)\right\rangle \\
&= \ell(\ell+1)\left\langle Y_{\ell,m}, f(r, \cdot, \cdot)\right\rangle \\
&= \ell(\ell+1)\alpha_{\ell,m}(r).
\end{aligned}$$

Here the first equality follows from the fact that $f \in C_2$. The technical continuity condition on f and its first and second partial derivatives allows us to exchange the derivative and the integral sign (disguised as a complex scalar product). See, for example, [Bart, Theorem 31.7]. The third equality follows from the Hermitian symmetry of $\nabla^2_{\theta,\phi}$. It follows that $\alpha_{\ell,m} Y_{\ell,m}$ is an element of the kernel of $D = \nabla^2 + u$, as we can verify:

$$\begin{aligned}
(\nabla^2 + u)\alpha_{\ell,m}(r) Y_{\ell,m}(\theta, \phi) &= \left((\nabla^2_r + u)\alpha_{\ell,m}(r)\right) Y_{\ell,m}(\theta, \phi) \\
&\qquad + \alpha_{\ell,m}(r)\nabla^2_{\theta,\phi} Y_{\ell,m}(\theta, \phi) \\
&= \ell(\ell+1)\alpha_{\ell,m}(r) Y_{\ell,m}(\theta, \phi) \\
&\qquad - \alpha_{\ell,m}(r)\ell(\ell+1) Y_{\ell,m}(\theta, \phi) \\
&= 0.
\end{aligned}$$

Hence $\alpha_{\ell,m} \otimes Y_{\ell,m} \in V$. Next we examine the norm of $\alpha_{\ell,m} = 0$ and recall that $f \in V^{\perp}$ by hypothesis:

$$\begin{aligned}
\left\|\alpha_{\ell,m}\right\|^2_{\mathcal{I}} &= \left\langle \alpha_{\ell,m}, \left\langle Y_{\ell,m}, f\right\rangle_{S^2}\right\rangle = \int_0^\infty \int_{S^2} \alpha^*_{\ell,m} Y^*_{\ell,m} f \\
&= \left\langle \alpha_{\ell,m} Y_{\ell,m}, f\right\rangle_{\mathbb{R}^3} \\
&= 0.
\end{aligned}$$

Hence $\alpha_{\ell,m} = 0$. But this implies that for any $h \otimes Y \in (\mathcal{I} \otimes \mathcal{Y})$ we have

$$\langle h \otimes Y, f\rangle_{\mathbb{R}^3} = \int_0^\infty \langle Y, f(r, \cdot, \cdot)\rangle_{S^2}\, r^2\, dr = \left\langle h, \alpha_{\ell,m}\right\rangle_{\mathbb{R}^{\geq 0}} = 0.$$

But, by Proposition 7.5, $\mathcal{I} \otimes \mathcal{Y}$ spans $L^2(\mathbb{R}^3)$. Hence $f = 0$. □

Note that the application of Fubini's Theorem here mirrors the argument in Proposition 7.7. Also note that this proposition could easily be generalized to differential operators of the form

$$\nabla^2 + \mathcal{O},$$

where $\mathcal{O}$ is a differential operator depending only on r. One would need appropriate technical hypotheses on f. Specifically, if we let n denote the minimum of 2 and the order of the differential operator $\mathcal{O}$, then f and all its partial derivatives up to the nth order would have to be continuous.

Appendix B
Proof of the Correspondence between Irreducible Linear Representations of $SU(2)$ and Irreducible Projective Representations of $SO(3)$

In this appendix we prove Proposition 10.6 from Section 10.4, which states that the irreducible projective unitary Lie group representations of $SO(3)$ are in one-to-one correspondence with the irreducible (linear) unitary Lie group representations of $SU(2)$. The proof requires some techniques from topology and differential geometry.

Let us start by stating the definitions and theorems we use from topology. We will use the notion of local homomorphisms.

Definition B.1 *Suppose that M and N are topological spaces, and suppose that $f: M \to N$ is a continuous function. Suppose $m \in M$. Then f is a* local homeomorphism at *m if there is a neighborhood $\tilde{M}$ containing m such that $f|_{\tilde{M}}$ is invertible and its inverse is continuous. If f is a local homeomorphism at each $m \in M$, then f is a* local homeomorphism.

We need a theorem about covering spaces.

Theorem B.1 *Suppose X, Y and Z are topological spaces. Suppose $\pi: Y \to X$ is a finite-to-one local homeomorphism.*[1] *Suppose Z is connected and simply connected. Suppose $f: Z \to X$ is continuous. Then there is a continuous function $\tilde{f}: Z \to Y$ such that $f = \pi \circ \tilde{f}$.*

[1] A function such as π is known as a *covering function*, while a space such as Y is called a *covering space* for X.

For a proof, see [Hat, Proposition 1.30] or [Mas, Theorem 5.1].

Next we introduce the relevant concepts and theorems from differential geometry. First we define local diffeomorphisms.

Definition B.2 *Suppose that M and N are differentiable manifolds*[2] *of the same dimension, and suppose that $f : M \to N$ is a differentiable function. Suppose $m \in M$. Then f is a* local diffeomorphism at m *if there is a neighborhood $\tilde{M}$ containing m such that $f|_{\tilde{M}}$ is invertible and its inverse is differentiable. If f is a local diffeomorphism at each $m \in M$, then f is a* local diffeomorphism.

We will appeal to the Inverse Function Theorem.

Theorem B.2 (Inverse Function Theorem) *Suppose that M and N are manifolds of the same dimension, and suppose that $f : M \to N$ is a differentiable function. Suppose $m \in M$. Suppose the linear transformation $df(m) : T_m M \to T_{f(m)} N$ is invertible. Then f is a local diffeomorphism at m.*

See Boothby [Bo, II.6] or Bamberg and Sternberg [BaS, p. 237] for a proof of the inverse function theorem on $\mathbb{R}^n$. The corresponding theorem for manifolds follows by restricting to coordinate neighborhoods of m and $f(m)$. We will use the following theorem about group actions on differentiable manifolds.

Theorem B.3 *Suppose M is a differentiable manifold, G is a compact Lie group and (G, M, σ) is a group action. Suppose further that*

1. *The action is* free, *i.e., if $g \in G$, $m \in M$ and $(\sigma(g))(m) = m$, then $g = I$.*
2. *The action is* smooth, *i.e., for each $g \in G$, the function*

$$\sigma(g) : M \to M$$

is an infinitely differentiable function.

Then the quotient space M/G (defined in Exercise 4.43) is a differentiable manifold, and the natural projection $\pi : M \to M/G$ is a differentiable function.

A proof of this theorem can be found in [AM, Proposition 4.1.23].

Next, recall the Lie group homomorphism $\Phi : SU(2) \to SO(3)$ defined in Section 4.3.

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[2] Also known as C^∞ *manifolds* or *smooth manifolds*.

Proposition B.1 *The function Φ is a local diffeomorphism. In other words, for any $g \in SU(2)$, there is a neighborhood N containing g such that the restriction $\Phi|_N$ is invertible and its inverse is differentiable.*

Proof. First we show that Φ is a local diffeomorphism at $I \in SU(2)$. We use Equation 4.2 to calculate the derivative of Φ at I: for x, y and z near 0 we have, up to first order in x, y and z,

$$\Phi\left(I + \begin{pmatrix} ix & y+iz \\ -y+iz & ix \end{pmatrix}\right) = \Phi\begin{pmatrix} 1+ix & y+iz \\ -y+iz & 1-ix \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2y & -2z \\ -2y & 1 & 2x \\ 2z & -2x & 1 \end{pmatrix} = I + \begin{pmatrix} 0 & 2y & -2z \\ -2y & 0 & 2x \\ 2z & -2x & 0 \end{pmatrix}.$$

Hence

$$d\Phi(I)\begin{pmatrix} ix & y+iz \\ -y+iz & -ix \end{pmatrix} = \begin{pmatrix} 0 & 2y & -2z \\ -2y & 0 & 2x \\ 2z & -2x & 0 \end{pmatrix}.$$

The kernel of the linear transformation $d\Phi(I)$ from the three-dimensional vector space $T_I SU(2)$ to the three-dimensional vector space $T_I SO(3)$ is trivial, so $d\Phi(I)$ is invertible. Hence by the Inverse Function Theorem (Theorem B.2), Φ is a local diffeomorphism at I.

Next we consider an arbitrary $g_0 \in SU(2)$ and show that Φ is a local diffeomorphism at g_0. Now let N denote a neighborhood of $I \in SU(2)$ on which the restriction $\Phi|_N$ has a differentiable inverse. Since left multiplication by g_0^{-1} is a continuous function on $SU(2)$, the set

$$g_0 N := \{g_0 g : g \in N\}$$

is a neighborhood of g_0. For any $g_0 n \in g_0 N$ we have $\Phi(g_0 n) = \Phi(g)\Phi(n)$. Hence

$$\Phi\Big|_{g_0 N} = L_{\Phi(g_0)} \circ \Phi\Big|_N,$$

where $L_{g_0} : SO(3) \to SO(3)$ denotes left multiplication by $\Phi(g_0)$. Hence the inverse function is

$$\left(\Phi\Big|_{g_0 N}\right)^{-1} = \left(\Phi\Big|_N\right)^{-1} \circ L_{\Phi(g_0^{-1})}.$$

Since $SU(2)$ is a Lie group, the function $L_{\Phi(g_0^{-1})}$ is differentiable; by our choice of N, the function $(\Phi|_N)^{-1}$ is differentiable. Hence $\Phi|_{g_0 N}$ has a differentiable inverse, i.e., Φ is a local diffeomorphism at g_0. But g_0 was arbitrary; hence Φ is a local diffeomorphism. □

Because finite quotients are easier to handle than infinite quotients, it is useful to think of $\mathbb{P}\mathcal{U}(V)$ as a finite quotient of the group $SU(V)$, the set of unitary transformations from V to itself with determinant 1 (Definition 4.2).

Proposition B.2 *Suppose V is a complex scalar product space of finite dimension $n \in \mathbb{N}$. Consider the equivalence relation on the group $SU(V)$ defined by $A \sim B$ if and only if there is a complex number λ such that $\lambda^n = 1$ and $A = \lambda B$. Then $SU(V)/\sim$ is a group and there is a Lie group isomorphism*

$$\mathbb{P}\mathcal{U}(V) \cong SU(V)/\sim .$$

Proof. First we must show that the group operation on $SU(V)$ survives the equivalence. Because we are accustomed to using $[A]$ to denote an element of $\mathbb{P}\mathcal{U}(V)$, we will write elements of $SU(V)/\sim$ as $\{A\}$, where $A \in SU(V)$. Note that if $A_1 = \lambda_A A_2$ and $B_1 = \lambda_B B_2$, with $\lambda_A^n = \lambda_B^n = 1$, then $A_1 B_1 = (\lambda_A \lambda_B) A_2 B_2$, where $(\lambda_A \lambda_B)^n = 1$. So group multiplication survives the equivalence. It follows easily that $SU(V)/\sim$ is a group.

Next we define a function $\Psi\colon (SU(V)/\sim) \to \mathbb{P}\mathcal{U}(V)$ and show it is a group isomorphism. For any $\{A\} \in SU(V)/\sim$, we define

$$\Psi(\{A\}) := [A].$$

Note that $\Psi(\{A\})$ is well defined since any two equivalent elements of $SU(V)$ yield the same element of $\mathbb{P}\mathcal{U}(V)$. The function Ψ is a group homomorphism because, for any $A, B \in SU(V)$ we have

$$\Psi(\{A\}\{B\}) = \Psi(\{AB\}) = [AB] = [A][B] = \Psi(\{A\})\Psi(\{B\}).$$

To see that Ψ is injective, consider $\Psi^{-1}[I]$. If $A \in SU(V)$ and $\Psi(\{A\}) = [I]$, then there must be a complex number λ such that $A = \lambda I$. Notice that

$$\lambda^n = \det(\lambda I) = \det(A) = 1,$$

because $A \in SU(V)$. Hence $A \sim I$, i.e., $\{A\} = \{I\}$. So Ψ is injective. To see that Ψ is surjective, consider $[B]$ for any $B \in \mathcal{U}(V)$. Set $c := \det(B) \neq 0$. Then $\det(c^{-n}B) = 1$, so $c^{-n}B \in SU(V)$. We have

$$\Psi(\{c^{-n}B\}) = [c^{-n}B] = [B].$$

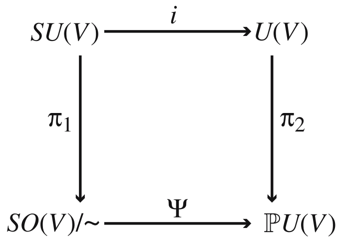

Figure B.1. A commutative diagram for the proof of Proposition B.2. The functions π_1 and π_2 are the natural projection functions. The function i is the inclusion function: any element of $SU(V)$ is automatically an element of $U(V)$.

Hence Ψ is surjective. Since Ψ is an injective and surjective group homomorphism, it is a group isomorphism.

Next we show differentiability. Consider Figure B.1. By construction, the function π_1 is surjective. So given an arbitrary element $c \in SU(V)/\sim$, there is an element $A \in SU(V)$ such that $\pi_1(A) = c$. By Theorem B.3, we know that π_1 is a local diffeomorphism. Hence there is a neighborhood N of A such that $\pi_1|_N$ has a differentiable inverse. The inclusion function is automatically differentiable. Finally, from Theorem B.3 we know that π_2 is a differentiable function. Hence the function

$$\Psi\Big|_N = \pi_2 \circ i \circ \left(\pi_1\Big|_N\right)^{-1}$$

is differentiable. So Ψ is differentiable at c. But c was arbitrary, so Ψ is differentiable. □

Here is the proof of Proposition 10.6.

Proof. (of Proposition 10.6) First we suppose that $(SU(2), V, \rho)$ is a linear irreducible unitary Lie group representation. By Proposition 6.14 we know that ρ is isomorphic to the representation R_n for some n. By Proposition 10.5 we know that R_n can be pushed forward to an irreducible projective representation of $SO(3)$. Hence $[\rho]$ can be pushed forward to an irreducible projective Lie group representation of $SO(3)$.

Conversely, suppose that $(SO(3), \mathbb{P}(V), \sigma)$ is a finite-dimensional projective unitary representation. We want to show that σ is the pushforward of the projectivization of a linear unitary representation ρ of $SU(2)$. In other words, we must show that there is a Lie group representation ρ that makes the diagram in Figure B.2 commutative, and that this ρ is a Lie group representation.

Consider the function $\sigma \circ \Phi\colon SU(2) \to SU(\mathcal{P}^n)/\sim$. This function is continuous, and its domain $SU(2)$ is simply connected, by Exercise 4.27. Let us show that $SU(2)$ is also connected. Since S^3 is path-connected (any two

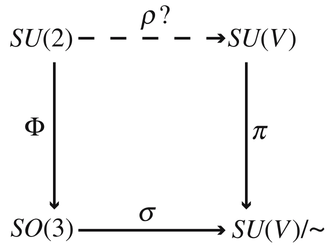

Figure B.2. Commutative diagram for proof that every projective unitary representation of $SO(3)$ comes from a linear representation of $SU(2)$.

points in S^3 lie on a plane through the origin that intersects S^3 in a circle) and $SU(2)$ is topologically equivalent to S^3, we know that $SU(2)$ is connected.

Since the function π is a finite-to-one covering, we can apply Theorem B.1 to conclude that there is a continuous function $\rho\colon SU(2) \to SU(\mathcal{P}^n)$ that makes the diagram in Figure B.2 commutative. Note that $\rho(I) = e^{ik/2\pi(n+1)}$ for some integer k. Without loss of generality, we can assume that $k = 0$: if not, replace ρ by $e^{-ik/2\pi(n+1)}\rho$.

Next we will show that ρ is a group homomorphism. Since Φ and σ are group homomorphisms, we know that $\pi \circ \rho$ is a group homomorphism. Hence, for any $g_1, g_2 \in SU(2)$ we have

$$\rho(g_1 g_2) = e^{\frac{ik}{2\pi n}} \rho(g_1)\rho(g_2),$$

for some $k \in \mathbb{Z}$. In fact, we can use this equation to define k as a function of the pair (g_1, g_2). In other words, consider the function $K\colon SU(2) \times SU(2) \to SU(V)$ defined by

$$K(g_1, g_2) := \rho(g_1 g_2)\rho(g_2)^{-1}\rho(g_1)^{-1}.$$

Since the function ρ is continuous, so is the function K. Since the domain $SU(2) \times SU(2)$ of K is connected, the range of K must be connected. But the range is a subset of the integer multiples of I, and we know that I lies in the range of K because $K(I, I) = I$. So the range must be equal to $\{I\}$. Hence, for any $(g_1, g_2) \in SU(2) \times SU(2)$ we have

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2).$$

Hence ρ is a group homomorphism.

Let us show that ρ is differentiable. Consider Figure B.2. Consider arbitrary $g \in SU(2)$. Because π is a local diffeomorphism, there is a neighborhood N of $\rho(g) \in SU(V)$ such that $\pi|_N$ has a differentiable inverse. By

Proposition B.2, the set $\pi[N]$ must be a neighborhood of the point $\pi \circ \rho(g) = \sigma \circ \Phi(g)$. Let $\tilde{N}$ denote the preimage in $SU(2)$ of the set $\pi[N]$ under the function $\sigma \circ \Phi$. Then $\tilde{N}$ is neighborhood of g. On the neighborhood $\tilde{N}$ we have

$$\rho\Big|_{\tilde{N}} = \left(\pi\Big|_{N}\right)^{-1} \circ \sigma \circ \Phi,$$

where all three functions on the right-hand side are differentiable. Hence $\rho|_{\tilde{N}}$ is differentiable, which implies that ρ is differentiable at g. But g was arbitrary; hence ρ is differentiable on all of $SU(2)$.

We have shown that the projective representation σ is the pushforward of the representation ρ, completing the proof. □

Appendix C
Suggested Paper Topics

- Selection rules and Clebsch–Gordan coefficients.
- Fourier transforms and momentum space.
- Classification of representations of the symmetric group S^n.
- Representations of the Poincaré group and their relation to mass and spin.
- The Peter–Weyl theorem.
- Maximal tori and conjugacy classes of compact groups.
- The Ω^- particle.
- Spin-orbit coupling.
- The hyperfine splitting in hydrogen.
- The crystallographic groups.
- Quarks and representations of $SU(3)$.
- Hilbert spaces (in the mathematical sense).
- The history of the use of hydrogen in modern physics (see Rigden [Ri]).
- Any topic from *Lie Groups and Physics* [St] or *Variations on a Theme by Kepler* [GS].

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Glossary of Symbols and Notation

$:=$	a defining equality, 26
$\hat{K}$	complement of K in $\{1, \ldots, n\}$, 348
$\Im$	the imaginary part of a complex number, 21
$\Re$	the real part of a complex number, 21
$f \circ g$	composition of the functions f and g, 19
$f\|_S$	the restriction of the function f to the set S, 19
$\partial_y f$	the partial derivative of the function f with respect to the variable y, 20
τ	natural isomorphism from a complex scalar product space to its dual, 107, 165
τ	complex conjugation on $\mathbb{C}^n$, 325
$\mathrm{sgn}(\sigma)$	sign of the permutation σ, 75
$[a : b]$	element of the projective space $\mathbb{P}(\mathbb{C}^2)$, 300
$[c_0 : \cdots : c_n]$	element of the projective space $\mathbb{P}(\mathbb{C}^{n+1})$, 303
$\hat{f}$	Fourier transform of f, 26
∇^2	the Laplacian operator, 21
Å	angstrom, i.e., 10^{-10} meters , 9
$\hbar$	Planck's constant, 9

$\mathbf{H}$	the Schrödinger operator , 11
E_n	the n-th energy eigenvalue of the Schrödinger operator for the electron in the hydrogen atom , 12
V_E	eigenspace of the Schrödinger operator corresponding to energy level E , 267
$\mathbf{e}$	charge of the electron, 12
$\mathbf{m}$	mass of the electron, 12
$\mathbf{Z}$	constant factor in Schrödinger operator, 16
$\vert 0\rangle$, $\vert 1\rangle$	basis of kets of a qubit (a.k.a. spin-1/2 particle), 305
$\vert +\mathbf{z}\rangle$, $\vert -\mathbf{z}\rangle$	basis of kets for the state space of a spin-1/2 particle, 305
Π_+, Π_-	spin up and spin down projection operators, 49
$\vert +\mathbf{z}\rangle\langle +\mathbf{z}\vert$	spin up projection operator, 49
ℓ	azimuthal quantum number, 11
m	magnetic quantum number, 11
n	principal quantum number, 10
s	spin quantum number, 11
s, p, d, f	labels for states of the electron, 11
S^2	the unit two-sphere in $\mathbb{R}^3$, 23
S^3	the unit three-sphere in $\mathbb{R}^4$, 25
C_2	complex scalar product space of continuous square-integrable functions on $\mathbb{R}^3$ whose first and second partial derivatives are all continuous, 365
$\mathcal{Y}_4$	complex scalar product space of spherical harmonics on the three-sphere S^3, 285
$\mathcal{Y}_4^n$	complex scalar product space of spherical harmonics of degree n on the three-sphere S^3, 284
$W^\infty(\mathbb{R}^3)$	complex scalar product space of infinitely differentiable functions with all derivatives in $L^2(\mathbb{R}^3)$, 243
$\mathcal{I}$	complex scalar product space of rotation-invariant functions in $L^2(\mathbb{R}^3)$, 158
$C[-1, 1]$	complex scalar product space of continuous complex-valued functions on $[-1, 1]$, 45
$L^2(\mathbb{R}^3)$	complex scalar product space of square-integrable functions on $\mathbb{R}^3$, 80

$L^2(\mathbb{R}^{\geq 0})$ complex scalar product space of square-integrable functions on the nonnegative real axis, 158

$L^2(S^2)$ complex scalar product space of square-integrable functions on the two-sphere, 84

$L^2(S)$ complex scalar product space of square-integrable functions on a set S, 84

$\mathbb{H}$ complex scalar product space of complex-valued harmonic polynomials in three real variables, 52

$\mathbb{H}^\ell$ vector space of homogeneous harmonic polynomials of degree ℓ in three variables, 53

$\mathcal{P}^n$ complex scalar product space of homogeneous polynomials of degree n in two real variables, 47

$\mathbb{H}_4^n$ complex scalar product space of homogeneous harmonic polynomials of degree n in four variables, 284

$\mathcal{P}_3^\ell$ complex scalar product space of homogeneous polynomials of degree ℓ in three real variables, 47

$\mathcal{Y}^\ell$ complex scalar product space of restrictions of harmonic polynomials of degree ℓ on $\mathbb{R}^3$ to the two-sphere S^2, 53

$\mathcal{Y}$ complex scalar product space of restrictions of harmonic polynomials on $\mathbb{R}^3$ to the two-sphere S^2, 54

$\mathbf{Q}$ the algebra of quaternions, 25

$\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ a basis for the quaternions, 25

$\mathbf{P}_{\ell,m}$ Legendre function, 29

R_n representation of $SU(2)$ on homogeneous polynomials of degree n, 137

Q_n representation of $SO(3)$ on homogeneous polynomials of even degree n in two variables, pushforward of R_n, 202

$\Psi_{n\ell m}$ spherical harmonic function on S^3, 290

$Y_{\ell,m}$ spherical harmonic function on S^2, 30

$\langle\cdot,\cdot\rangle$ complex scalar product, 82

$\|\cdot\|$ norm, 94

$\mathbb{T}$ circle group, 112

$SO(2)$ group of rotations of the plane, 112

$SO(3)$ group of rotations in three-dimensional Euclidean space, 117

$SO(4)$ group of rotations of four-dimensional Euclidean space, 120

$\mathbb{T} \times \cdots \times \mathbb{T}$ the n-torus, an n-fold Cartesian product of circles, 206

$\mathcal{T}(S, S)$ group of all invertible functions from a set S to itself, 113

$\mathcal{GL}(V)$ group of invertible linear operators on a vector space V, 113

$\mathcal{U}(V)$ unitary group, i.e., group of unitary operators on a complex scalar product space V, 114

$SU(2)$ special 2×2 unitary group, 118

$SU(V)$ special unitary group, i.e., group of unitary operators of determinant one on a finite-dimensional scalar product space V, 114

(G, V, ρ) a representation ρ of a group G on a vector space V, 127

χ_ρ character of the representation ρ, 141

Φ surjective Lie group homomorphism from $SU(2)$ to $SO(3)$, 123

$\int_{SU(2)} f(g)\, dg$ invariant, volume-one integral on $SU(2)$, 189

$\mathfrak{g}$ Lie algebra, 230

$[\cdot, \cdot]$ Lie bracket, 230

$g\ell(n, \mathbb{C})$ (real) Lie algebra of $n \times n$ matrices with complex entries, 232

$\mathfrak{g}_{\mathbf{Q}}$ Lie algebra of quaternions spanned by $\mathbf{i}, \mathbf{j}, \mathbf{k}$, 231

$\mathcal{H}$ Heisenberg Lie algebra, 239

$g\ell(V)$ Lie algebra of all linear operators on the vector space V, 241

$so(n)$ Lie algebra of $n \times n$ skew-symmetric real matrices, 247

$su(2)$ 2×2 special unitary algebra, 232

$\mathbf{L}$ total angular momentum operator, 243

$\mathbf{U}$ angular momentum operator on polynomials in two real variables, 246

$\mathbf{X}$ raising operator for the representation $\mathbf{U}$, 247

$\mathbf{Y}$ lowering operator for the representation $\mathbf{U}$, 248

$\mathbf{X}_\rho$ raising operator for the representation ρ, 249

$\mathbf{Y}_\rho$ lowering operator for the representation ρ, 249

$\mathbf{R_i}, \mathbf{R_j}, \mathbf{R_k}$ Runge–Lenz operators, 268

$\mathbf{C}$ Casimir operator, 255

$V \cong W$ the representations on the vector spaces V and W are isomorphic, 132

T^* the adjoint of the linear transformation T, 89

$\rho \cong \tilde{\rho}$ the representations ρ and $\tilde{\rho}$ are isomorphic, 132

$\ker T$ kernel of the linear transformation T, 52

$W^{\perp}$ the subspace complementary to W inside another vector space, 86

$\Lambda^n V$ alternate tensor product of n copies of the vector space V, 75

$\mathrm{Sym}^n V$ symmetric tensor product of n copies of V, 75

$\mathbb{P}(V)$ projective space over V, 300

$\Pi_{[W]}$ orthogonal projection onto the subspace $[W]$ of projective space, 344

$[W]$ linear subspace of a projective space $\mathbb{P}(V)$, where W is a subspace of V, 303

$[T]$ projectivization of the linear operator T, 304

$\mathbb{P}\mathcal{U}(V)$ projective unitary group of the vector space V, 318

$S/\sim$ the set of equivalence classes in S modulo the equivalence relation $\sim$, 33

$\langle\cdot,\cdot\rangle_*$ complex scalar product on the dual of a complex scalar product space, 107, 165

V^* dual vector space to V, 72, 164

ρ^* dual to the representation ρ, 166

$\mathrm{Hom}_G(V, W)$ fixed points of the natural representation on $\mathrm{Hom}(V, W)$, 169

$V \oplus W$ Cartesian sum of vector spaces V and W, 62

$V \otimes W$ tensor product of vector spaces V and W, 67

$\rho \oplus \tilde{\rho}$ Cartesian sum of representations ρ and $\tilde{\rho}$, 159

$\rho \otimes \tilde{\rho}$ tensor product of the representations ρ and $\tilde{\rho}$, 160

Π_k projection onto the k-th summand of a Cartesian sum, 63

$\mathrm{Hom}(V, W)$ complex scalar product space of linear transformations from V to W, 73, 169

unirrep unitary irreducible representation, 195

Index

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(continued from page ii)

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