# Afterword

If we continue on the path traced out by this book and analyze to what extent contemporary mathematics corresponds to the observability principle, we see that many things in our science are simply conceptually unfounded. This unavoidably leads to serious difficulties, which are usually ignored from force of habit even when they contradict our experience. If, for example, measure theory is the correct theory of integration, then why is it that all attempts to construct the continual integral on its basis have failed, although the existence of such integrals is experimentally verified?

As the result of this, physicists are forced to use "unobservable" mathematics in their theories, which leads to serious difficulties, say, in quantum field theory, which some even regard as an inherent aspect of the theory. It is generally believed that the mathematical basis of quantum mechanics is the theory of self-adjoint operators in Hilbert space. But then why does Dirac write that "physically significant interactions in quantum field theory are so strong that they throw any Schrödinger state vector out of Hilbert space in the shortest possible time interval"?

Having noted this, one must either avoid writing the Schrödinger equation in the context of quantum fields theory or refuse to consider Hilbert spaces as the foundation of quantum mechanics. Dirac reluctantly chose the first alternative, and this refusal was forced, since the mathematics of that time allowed him to talk about solutions of differential equations only in a very limited language (see the quotation at the beginning of the Introduction). On the other hand, since the Hilbert space formalism contains no procedure for distinguishing one vector from another, the observability principle is not followed here. Thus the second alternative seems more appropriate, but it requires specifying many other points, e.g., finding out how one can observe solutions of partial differential equations; this question, however, is outside the sphere of interests of the PDE experts: To them even setting the question would seem strange, to say the least.

Thus the systematic mathematical formalization of the observability principle requires rethinking many branches of mathematics that seemed established once and for all. The main difficult step that must be taken in this direction is to find solutions in the framework of the differential calculus, avoiding the appeal of functional analysis, measure theory, and other purely set-theoretical constructions. In particular, we must refuse measure theory as integration theory in favor of the purely cohomological approach. One page suffices to write out the main rules of measure theory. The number of pages needed to explain de Rham cohomology is much larger. The conceptual distance between the two approaches shows what serious difficulties must be overcome on this road.

The author intends to explain, in the next issues of his infinite series of books, how this road leads to the secondary differential calculus (already mentioned in the Introduction) and its main applications, e.g., cohomological physics. The reader may obtain an idea of what has already been done, and what remains to be done in this direction, by consulting the references appearing below.

# Appendix

## A. M. Vinogradov

## Observability Principle, Set Theory and the "Foundations of Mathematics"

The following general remarks are meant to place the questions discussed in this book in the perspective of observable mathematics.

**Propositional and Boolean algebras.** While the physicist describes nature by means of measuring devices with $\mathbb{R}$-valued scales, the ordinary man or woman does so by means of statements. Using the elementary operations of conjunction, disjunction, and negation, new statements may be constructed from given ones. A system of statements (propositions) closed with respect to these operations is said to be a *propositional algebra*. Thus, the means of observation of an individual not possessing any measuring devices is formalized by the notion of propositional algebra. Let us explain this in more detail.

Let us note, first of all, that the individual observing the world without measuring devices was considered above only as an example of the main, initial mechanism of information processing, which in the sequel we shall call *primitive*. Thus, we identify propositional algebras with primitive means of observation.

Further, let us recall that any propositional algebra $A$ may be transformed into a unital commutative algebra over the field $\mathbb{Z}_2$ of residues modulo 2 by introducing the operations of multiplication and addition as follows:

$$pq \stackrel{\text{def}}{=} p \wedge q,$$

$$p + q \stackrel{\text{def}}{=} (p \wedge \bar{q}) \vee (\bar{p} \wedge q),$$

where $\vee$ and $\wedge$ are the propositional connectives conjunction and disjunction, respectively, while the bar over a letter denotes negation. All elements of the algebra thus obtained are idempotent, i.e., $a^2 = a$. Let us call any unital commutative $\mathbb{Z}_2$-algebra *Boolean* if all its elements are idempotent. Conversely, any Boolean algebra may be regarded as a propositional algebra with respect to the operations

$$\begin{aligned} p \wedge q &\stackrel{\text{def}}{=} pq, \\ p \vee q &\stackrel{\text{def}}{=} p + q + pq, \\ \bar{p} &\stackrel{\text{def}}{=} 1 + p. \end{aligned}$$

This shows that there is no essential difference between propositional and Boolean algebras, and the use of one or the other only specifies what operations are involved in the given context. Thus we can restate the previous remarks about means of observation as follows: *Boolean algebras are primitive means of observation.*

**Boolean spectra.** The advantage of the previous formulation is that it immediately allows us to discern the remarkable analogy with the observation mechanism in classical physics as interpreted in this book. Namely, in this mechanism one must merely replace the $\mathbb{R}$-valued measurement scales by $\mathbb{Z}_2$-valued ones (i.e., those that say either "yes" or "no") and add the idempotence condition. This analogy shows that *what we can observe by means of a Boolean algebra $A$ is its $\mathbb{Z}_2$-spectrum, i.e., the set of all its homomorphisms as a unital $\mathbb{Z}_2$-algebra to the unital $\mathbb{Z}_2$-algebra $\mathbb{Z}_2$.*

Let us denote this spectrum by $\mathrm{Spec}_{\mathbb{Z}_2}$ and endow it with the natural topology, namely the Zariski one. Then we can say, more precisely, that Boolean algebras allow us to observe topological spaces of the form $\mathrm{Spec}_{\mathbb{Z}_2}$, which we shall call, for this reason, *Boolean spaces.*

In connection with the above, one may naturally ask whether the spectra of Boolean algebras possess any structure besides the topological one, say, a smooth structure, as was the case for spectra of $\mathbb{R}$-algebras. The reader who managed to do Exercise 4 from Section 9.45 already knows that the differential calculus over Boolean algebras is trivial in the sense that any differential operator on such an algebra is of order zero, i.e., is a homomorphism of modules over this algebra. This means, in particular, that *the phenomenon of motion cannot be adequately described and studied in mathematical terms by using only logical notions* or, to put it simply, by using everyday language (recall the classical logical paradoxes on this topic).

The Stone theorem stated below, which plays a central role in the theory of Boolean algebras, shows that the spectra of Boolean algebras possess only one independent structure: the topological one. In the statement of the theorem it is assumed that the field $\mathbb{Z}_2$ is supplied with the discrete topology.

**Stone's theorem.** *Any Boolean space is an absolutely disconnected compact Hausdorff space and, conversely, any Boolean algebra coincides with the algebra of open-and-closed sets of its spectrum with respect to the set-theoretic operations of symmetric difference and intersection.*

Recall that the absolute disconnectedness of a topological space means that the open-and-closed sets form a base of its topology. The appearance of these simultaneously open and closed sets in Boolean spaces is explained by the fact that any propositional algebra possesses a natural duality. Namely, the negation operation maps it onto itself and interchanges conjunction and disjunction. Note also that Stone's theorem is an identical twin of the Spectrum theorem (see Sections 7.2 and 7.7). Their proofs are based on the same idea, and differ only in technical details reflecting the specifics of the different classes of algebras under consideration. The reader may try to prove this theorem as an exercise, having in mind that the elements of the given Boolean algebra can be naturally identified with the open-and-closed subsets of its spectrum, while the operations of conjunction, disjunction, and negation then become the set-theoretical operations of intersection, union, and complement, respectively. It is easy to see that the spectrum of a finite Boolean algebra is a finite set supplied with the discrete topology. Thus any finite Boolean algebra turns out to be isomorphic to the algebra of all subsets of a certain set.

**"Eyes" and "ears".** After all these preliminaries, the role of "eyes" and "ears" in the process of observation may be described as follows. First of all, the "crude" data absorbed by our senses are written down by the brain and sent to the corresponding part of our memory. One may think that in the process of writing down, the crude data are split up into elementary blocks, "macros," and so on, which are marked by appropriate expressions of everyday language. These marks are needed for further processing of the stored data. The system of statements constituting some description generates an ideal of the controlling Boolean algebra, thus distinguishing the corresponding closed subset in its spectrum. Supposing that to each point of the spectrum an elementary block is assigned, and this block is marked by the associated maximal ideal, we come to the conclusion that to each closed subset of the spectrum one can associate a certain image, just as a criminalist creates an identikit from individual details described by witnesses. Thus, if we forget about the "material" content of the elementary blocks (they may be "photographs" of an atomic fragment of a visual or an audio image, etc.) that corresponds (according to the above scheme) to points of the spectrum of the controlling Boolean algebra, we may assume that everything that can be observed on the primitive level is tautologically expressed by the points of this spectrum.

**Boolean algebras corresponding to the primitive level.** It is clear that any rigorous mathematical notion of observability must come from some notion of observer, understood as a kind of mechanism for gathering and processing information. In other words, the notion of observability must be formalized approximately in the same way as Turing machines formalize the notion of algorithm. So as not to turn out to be an a priori formalized metaphysical scheme, such a formalization must take into account "experimental data." The latter may be found in the construction and evolution of computer hardware and in the underlying theoretical ideas. Therefore it is useful to regard the individual mathematician, or better still, the mathematical community, in the spirit of the "noosphere" of Vernadskii, as a kind of computer. Then, having in mind that the operational system of any modern computer is a program written in the language of binary codes, we can say that there is no alternative to Boolean algebras as the mechanism describing information on the primitive level. For practical reasons, as well as for considerations of theoretical simplicity, it would be inconvenient to limit the size of this algebra by some concrete number, say the number of elementary particles in the universe. Hence it is natural to choose the free algebra in a countable number of generators. The notion of level of observability is apparently important for the mathematical analysis of the notion of observability itself, and we shall return to it below.

**How set theory appeared in the foundations of mathematics.** As we saw above, any propositional algebra is canonically isomorphic to the algebra of all subsets of the spectrum of the associated Boolean algebra. If this spectrum is finite, then its topology is discrete. So we can forget about the topology without losing anything. Moreover, any concrete individual, especially if he/she is not familiar with Boolean algebras, feels sure that what she/he is observing are just subsets or, more precisely, the identikits which he defined. Therefore, such an immediate "material" feeling leads us to the idea that the initial building blocks of precise abstract thinking are "points" ("elements") grouped together in "families," i.e., sets. Having accepted or rather having experienced this feeling of primitivity of the notion of set under the pressure of our immediate feelings, we are forced to place set theory at the foundation of exact knowledge, i.e., of mathematics. On the primitive level of finite sets, this choice, in view of what was explained above, does not contradict the observability principle, since any finite set can be naturally and uniquely interpreted as the spectrum of some Boolean algebra.

However, if we go beyond the class of finite sets, the situation changes radically: The notion of observable set, i.e., of Boolean space, ceases to coincide with the general notion of a set without any additional structure. Therefore, our respect for the observability principle leads us to abandon the notion of a set as the formal-logical foundation of mathematics and leave the paradise so favored by Hilbert. One of the advantages of such a step, among others, is that it allows us to avoid many of the paradoxes inherent to set theory. For example, the analogue of the "set of all sets" in observable mathematics is the "Boolean space of all Boolean spaces." But this last construction is clearly meaningless, because it defines no topology in the "Boolean space of all Boolean spaces." Or the "observable" version of the "set (not) containing itself as an element," i.e., the "Boolean space (not) containing itself as an element" is so striking that no comment is needed. In this connection we should additionally note that in order to observe Boolean spaces (on the primitive level!) as individual objects, a separate Boolean space that distinguishes them is required.

**Observable mathematical structures (Boole groups).** Now is the time to ask what observable mathematical structures are. If we are talking about groups observable in the "Boolean" sense, then we mean topological groups whose set of elements constitutes a Boolean space. Such a group should be called Boolean. In other words, a *Boole group* is a group structure on the spectrum of some Boolean algebra. If we replace in this definition the notion of Boolean observability by that of classical observability, we come to the notion of Lie group, i.e., of a group structure on the spectrum of the classical algebra of observables.

**Observing observables: different levels of observability.** Just as the operating system in a computer manipulates programs of the next level, one can imagine a Boolean algebra of the primitive level (see above) with the points of its spectrum marking other Boolean algebras. In other words, this is a Boolean algebra observing other Boolean algebras. Iterating this procedure, we come to "observed objects," which, if one forgets the multistep observation scheme, can naively be understood as sets of cardinality higher than finite or countable. For instance, starting from the primitive level, we can introduce into observable mathematics things that in "nonobservable" mathematics are related to sets of continual cardinality. In this direction, one may hope that there is a constructive formalization of the observability of smooth $\mathbb{R}$-algebras, which, in turn, formalize the observation procedure in classical physics.

**Down with set theory?** The numerous failed attempts to construct mathematics on the formal-logical foundations of set theory, together with the considerations related to observability developed above, lead us to refuse this idea altogether. We can note that it also contradicts the physiological basis of human thought, which ideally consists in the harmonious interaction of the left and right hemispheres of the brain. It is known that the left hemisphere is responsible for rational reasoning, computations, logical analysis, and pragmatic decision-making. Dually, the right hemisphere answers for "irrational" thought, i.e., intuition, premonitions, emotions, imagination, and geometry. If the problem under consideration is too hard for direct logical analysis, we ask our intuition what to do. We also know that in order to obtain a satisfactory result, the intuitive solution must be controlled by logical analysis and, possibly, corrected on its basis. Thus, in the process of decision-making, in the search for the solution of a problem, etc., the switching of control from one hemisphere to the other takes place, and such iterations can be numerous.

All this, of course, is entirely relevant to the solution of mathematical problems. The left hemisphere, i.e., the algebro-analytical part of our brain, is incapable of finding the solution to a problem whose complexity is higher than, say, the possibilities of human memory. Indeed, from any assumption one can deduce numerous logically correct consequences. Therefore, in the purely logical approach, the number of chains of inference grows at least exponentially with their length, while those that lead to a correct solution constitute a vanishingly small part of that number. Thus if the correct consequence is chosen haphazardly at each step, and the left hemisphere knows no better, then the propagation of this "logical wave" in all directions will overfill our memory before it reaches the desired haven.

The only way out of this situation is to direct this wave along an appropriate path, i.e., to choose at each step the consequences that can lead in a more or less straight line to the solution. But what do we mean by a "straight line"? This means that an overall picture of the problem must be sketched, a picture on which possible ways of solution could be drawn. The construction of such an overall picture, in other words, of the geometric image of the problem, takes place in the right hemisphere, which was created by nature precisely for such constructions. The basic building blocks for them, at least when we are dealing with mathematics, are sets. These are sets in the naive sense, since they live in the right hemisphere. Hence any attempt to formalize them, moving them from the right hemisphere to the left one, is just an outrage against nature. So let us leave set theory in the right hemisphere in its naive form, thanks to which it is has been so useful.

**Infinitesimal observability.** Above we considered Boolean algebras as analogue of smooth algebras. But we can interchange our priorities and do things the other way around. From this point of view, the operations or, better, the functors of the differential calculus, will appear as the analogue of logical operations, and the calculus itself as a mechanism for manipulating infinitesimal descriptions. In this way we would like to stress the infinitesimal aspect related to observability.

Some of the "primary" functors were described in this book. Their complete list should be understood as the logic algebra of the differential calculus. The work related to the complete formalization of this idea is still to be completed.

In conclusion let us note, expressing ourselves informally, that in our imaginary computer, working with stored knowledge, the program called "differential calculus" is not part of its operating system, and so is located at a higher level than the primitive one (see above). This means that the geometric images built on its basis cannot be interpreted in a material way. They should retain their naive status in the sense explained above.

The constructive differential calculus, developed in the framework of "constructive mathematical logic," illustrates what can happen if this warning is ignored.

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