A
Appendices

A.1 Notation

$\mathbb{C}$	set of complex numbers
$\mathbb{N}$	set of natural numbers
$\mathbb{N}_0$	$:= \mathbb{N} \cup \{0\}$
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of positive real numbers
$\mathbb{Z}$	set of integers
$\Re z$	real part of the complex number z
$\Im z$	imaginary part of the complex number z
x^T	transpose of the vector $x \in \mathbb{R}^d,\ d \in \mathbb{N}$
$\lvert x\rvert_p$	$:= \left(\sum_{j=1}^d \lvert x_j\rvert^p\right)^{1/p}, x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d,\ d \in \mathbb{N},\ p \in [1, \infty)$
$\lvert x\rvert_\infty$	$:= \max_{j=1,\ldots,d} \lvert x_j\rvert$ maximum norm of the vector $x \in \mathbb{R}^d,\ d \in \mathbb{N}$
$\lvert x\rvert$	$:= \lvert x\rvert_2$ Euclidean norm of the vector $x \in \mathbb{R}^d,\ d \in \mathbb{N}$
$x \cdot y$	$:= x^T y = \sum_{j=1}^d x_j y_j$ scalar product of the vectors $x, y \in \mathbb{R}^d$
$\langle x, y\rangle_A$	$:= y^T A x = y \cdot A x$ energy product of the vectors $x, y \in \mathbb{R}^d$ w.r.t. a symmetric, positive definite matrix A
$\lvert\alpha\rvert$	$:= \lvert\alpha\rvert_1$ order (or length) of the multi-index $\alpha \in \mathbb{N}_0^d,\ d \in \mathbb{N}$
I	identity matrix or identity operator
e_j	jth unit vector in $\mathbb{R}^m$, $j = 1, \ldots, m$
$\mathrm{diag}(\lambda_i)$	$= \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ diagonal matrix in $\mathbb{R}^{m,m}$ with diagonal entries $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$

A^T	transpose of the matrix A
A^{-T}	transpose of the inverse matrix A^{-1}
$\det A$	determinant of the square matrix A
$\lambda_{\min}(A)$	minimum eigenvalue of a matrix A with real eigenvalues
$\lambda_{\max}(A)$	maximum eigenvalue of a matrix A with real eigenvalues
$\sigma(A)$	set of eigenvalues (spectrum) of the square matrix A
$\varrho(A)$	spectral radius of the square matrix A
$m(A)$	bandwidth of the symmetric matrix A
$\mathrm{Env}\,(A)$	hull of the square matrix A
$p(A)$	profile of the square matrix A
$B_\varrho(x_0)$	$:= \{x : \|x - x_0\| < \varrho\}$ open ball in a normed space
$\overline{B}_\varrho(x_0)$	$:= \{x : \|x - x_0\| \le \varrho\}$ closed ball in a normed space
$\mathrm{diam}\,(G)$	diameter of the set $G \subset \mathbb{R}^d$
$\|G\|_n$	n-dimensional (Lebesgue) measure of the $G \subset \mathbb{R}^n$, $n \in \{1, \dots, d\}$
$\|G\|$	$:= \|G\|_d$ d-dimensional (Lebesgue) measure of the set $G \subset \mathbb{R}^d$
$\mathrm{vol}\,(G)$	length ($d = 1$), area ($d = 2$), volume ($d = 3$) of "geometric bodies" $G \subset \mathbb{R}^d$
$\mathrm{int}\, G$	interior of the set G
∂G	boundary of the set G
$\overline{G}$	closure of the set G
$\mathrm{span}\, G$	linear hull of the set G
$\mathrm{conv}\, G$	convex hull of the set G
$\|G\|$	cardinal number of the discrete set G
ν	outer unit normal w.r.t. the set $G \subset \mathbb{R}^d$
Ω	domain of $\mathbb{R}^d$, $d \in \mathbb{N}$
Γ	$:= \partial\Omega$ boundary of the domain $\Omega \subset \mathbb{R}^d$
$\mathrm{supp}\, \varphi$	support of the function φ
f^{-1}	inverse of the mapping f
$f[G]$	image of the set G under the mapping f
$f^{-1}[G]$	preimage of the set G under the mapping f
$f\|_K$	restriction of $f : G \to \mathbb{R}$ to a subset $K \subset G$
$\|v\|_X$	norm of the element v of the normed space X
$\dim X$	dimension of the finite-dimensional linear space X
$L[X, Y]$	set of linear, continuous operators acting from the normed space X in the normed space Y
X'	$:= L[X, \mathbb{R}]$ dual space of the real normed space X
$O(\cdot)$, $o(\cdot)$	Landau symbols of asymptotic analysis
δ_{ij}	($i, j \in \mathbb{N}_0$) Kronecker symbol, i.e., $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$

Differential expressions

∂_l	($l \in \mathbb{N}$) symbol for the partial derivative w.r.t. the lth variable
∂_t	($t \in \mathbb{R}$) symbol for the partial derivative w.r.t. the variable t
∂^α	($\alpha \in \mathbb{N}_0^d$ multi-index) αth partial derivative
∇	$:= (\partial_1, \dots, \partial_d)^T$ Nabla operator (symbolic vector)

Δ Laplace operator
∂_μ $:= \mu \cdot \nabla$ directional derivative w.r.t. the vector μ
$D\Phi$ $:= \frac{\partial \Phi}{\partial x} := (\partial_j \Phi_i)_{i,j=1}^m$ Jacobi matrix or functional matrix of a differentiable mapping $\Phi : \mathbb{R}^m \to \mathbb{R}^m$

Coefficients in differential expressions

K diffusion coefficient (a square matrix function)
c convection coefficient (a vector function)
r reaction coefficient

Discretization methods

V_h ansatz space
X_h extended ansatz space without any homogeneous Dirichlet boundary conditions
a_h approximated bilinear form
b_h approximated linear form

Function spaces (see also Appendix A.5)

$\mathcal{P}_k(G)$ set of polynomials of maximum degree k on $G \subset \mathbb{R}^d$
$C(G) = C^0(G)$ set of continuous functions on G
$C^l(G)$ ($l \in \mathbb{N}$) set of l-times continuously differentiable functions on G
$C^\infty(G)$ set of infinitely often continuously differentiable functions on G
$C(\overline{G}) = C^0(\overline{G})$ set of bounded and uniformly continuous functions on G
$C^l(\overline{G})$ ($l \in \mathbb{N}$) set of functions with bounded and uniformly continuous derivatives up to the order l on G
$C^\infty(\overline{G})$ set of functions, all partial derivatives of which are bounded and uniformly continuous on G
$C_0(G) = C_0^0(G)$ set of continuous functions on G with compact support
$C_0^l(G)$ ($l \in \mathbb{N}$) set of l-times continuously differentiable functions on G with compact support
$C_0^\infty(G)$ set of infinitely often continuously differentiable functions on G with compact support
$L^p(G)$ ($p \in [1, \infty)$) set of Lebesgue-measurable functions whose pth power of their absolute value is Lebesgue-integrable on G
$L^\infty(G)$ set of measurable, essentially bounded functions
$\langle \cdot, \cdot \rangle_{0,G}$ scalar product in $L^2(G)$ †
$\| \cdot \|_{0,G}$ norm in $L^2(G)$ †
$\| \cdot \|_{0,p,G}$ ($p \in [1, \infty]$) norm in $L^p(G)$ †
$\| \cdot \|_{\infty,G}$ norm in $L^\infty(G)$ †
$W_p^l(G)$ ($l \in \mathbb{N}$, $p \in [1, \infty]$) set of l-times weakly differentiable functions from $L_p(G)$, with derivatives in $L^p(G)$
$\| \cdot \|_{l,p,G}$ ($l \in \mathbb{N}$, $p \in [1, \infty]$) norm in $W_p^l(G)$ †
$| \cdot |_{l,p,G}$ ($l \in \mathbb{N}$, $p \in [1, \infty]$) seminorm in $W_p^l(G)$ †

$H^l(G)$ $:= W_2^l(G)$ $(l \in \mathbb{N})$
$\langle \cdot, \cdot \rangle_{l,G}$ $(l \in \mathbb{N})$ scalar product in $H^l(G)$ †
$\| \cdot \|_{l,G}$ $(l \in \mathbb{N})$ norm in $H^l(G)$ †
$| \cdot |_{l,G}$ $(l \in \mathbb{N})$ seminorm in $H^l(G)$ †
$\langle \cdot, \cdot \rangle_{0,h}$ discrete $L^2(\Omega)$-scalar product
$\| \cdot \|_{0,h}$ discrete $L^2(\Omega)$-norm
$L^2(\partial G)$ set of square Lebesgue-integrable functions on the boundary ∂G
$H_0^1(G)$ set of functions from $H^1(G)$ with vanishing trace on ∂G
$C([0,T],X) = C^0([0,T],X)$ set of continuous functions on $[0,T]$ with values in the normed space X
$C^l([0,T],X)$ $(l \in \mathbb{N})$ set of l-times continuously differentiable functions on $[0,T]$ with values in the normed space X
$L^p((0,T),X)$ $(p \in [1,\infty])$ Lebesgue-space of functions on $[0,T]$ with values in the normed space X

† **Convention:** In the case $G = \Omega$, this specification is omitted.

A.2 Basic Concepts of Analysis

A subset $G \subset \mathbb{R}^d$ is called a *set of measure zero* if, for any number $\varepsilon > 0$, a countable family of balls B_j with d-dimensional volume $\varepsilon_j > 0$ exists such that

$$\sum_{j=1}^{\infty} \varepsilon_j < \varepsilon \qquad \text{and} \qquad G \subset \bigcup_{j=1}^{\infty} B_j \,.$$

Two functions $f, g : G \to \mathbb{R}$ are called *equal almost everywhere* (in short: *equal a.e.*, notation: $f \equiv g$) if the set $\{x \in G : f(x) \neq g(x)\}$ is of measure zero.

In particular, a function $f : G \to \mathbb{R}$ is called *vanishing almost everywhere* if it is equal to the constant function zero almost everywhere.

A function $f : G \to \mathbb{R}$ is called *measurable* if there exists a sequence $(f_i)_i$ of step functions $f_i : G \to \mathbb{R}$ such that $f_i \to f$ for $i \to \infty$ almost everywhere.

In what follows, G denotes a subset of $\mathbb{R}^d$, $d \in \mathbb{N}$.

(i) A point $x = (x_1, x_2, \ldots, x_d)^T \in \mathbb{R}^d$ is called a *boundary point* of G if every open neighbourhood (perhaps an open ball) of x contains a point of G as well as a point of the complementary set $\mathbb{R} \setminus G$.

(ii) The collection of all boundary points of G is called the *boundary* of G and is denoted by ∂G.

(iii) The set $\overline{G} := G \cup \partial G$ is called the *closure* of G.

(iv) The set G is called *closed* if $\overline{G} = G$.

(v) The set G is called *open* if $G \cap \partial G = \emptyset$.

(vi) The set $G \setminus \partial G$ is called the *interior* of G and is denoted by $\operatorname{int} G$.

A subset $G \subset \mathbb{R}^d$ is called *connected* if for arbitrary distinct points $x_1, x_2 \in G$ there exists a continuous curve in G connecting them.

The set G is called *convex* if any two points from G can be connected by a straight-line segment in G.

A nonempty, open, and connected set $G \subset \mathbb{R}^d$ is called a *domain* in $\mathbb{R}^d$.

By $\alpha = (\alpha_1, \ldots, \alpha_d)^T \in \mathbb{N}_0^d$ a so-called *multi-index* is denoted. Multi-indices are a popular tool to abbreviate some elaborate notation. For example,

$$\partial^\alpha := \prod_{i=1}^d \partial_i^{\alpha_i}, \quad \alpha! := \prod_{i=1}^d \alpha_i!, \quad |\alpha| := \sum_{i=1}^d \alpha_i .$$

The number $|\alpha|$ is called the *order* (or *length*) of the multi-index α.

For a continuous function $\varphi : G \to \mathbb{R}$, the set $\operatorname{supp} \varphi := \overline{\{x \in G : \varphi(x) \neq 0\}}$ denotes the *support* of φ.

A.3 Basic Concepts of Linear Algebra

A square matrix $A \in \mathbb{R}^{n,n}$ with entries a_{ij} is called *symmetric* if $a_{ij} = a_{ji}$ holds for all $i, j \in \{1, \ldots, n\}$.

A matrix $A \in \mathbb{R}^{n,n}$ is called *positive definite* if $x \cdot Ax > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

Given a polynomial $p \in \mathcal{P}_k$, $k \in \mathbb{N}_0$, of the form

$$p(z) = \sum_{j=0}^k a_j z^j \quad \text{with} \quad a_j \in \mathbb{C},\ j \in \{0, \ldots, k\}$$

and a matrix $A \in \mathbb{C}^{n,n}$, then the following *matrix polynomial* of A can be established:

$$p(A) := \sum_{j=0}^k a_j A^j .$$

Eigenvalues and Eigenvectors

Let $A \in \mathbb{C}^{n,n}$. A number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of A if

$$\det(A - \lambda I) = 0 .$$

If λ is an eigenvalue of A, then any vector $x \in \mathbb{C}^n \setminus \{0\}$ such that

$$Ax = \lambda x \qquad (\Leftrightarrow (A - \lambda I)x = 0)$$

is called an *eigenvector* of A associated with the eigenvalue λ.

The polynomial $p_A(\lambda) := \det(A - \lambda I)$ is called the *characteristic polynomial* of A.

The set of all eigenvalues of a matrix A is called the *spectrum* of A, denoted by $\sigma(A)$.

If all eigenvalues of a matrix A are real, then the numbers $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the largest, respectively smallest, of these eigenvalues.

The number $\varrho(A) = \max_{\lambda\in\sigma(A)} |\lambda|$ is called the *spectral radius* of A.

Norms of Vectors and Matrices

The *norm of a vector* $x \in \mathbb{R}^n$, $n \in \mathbb{N}$, is a real-valued function $x \mapsto |x|$ satisfying the following three properties:

(i) $|x| \geq 0$ for all $x \in \mathbb{R}^n$, $|x| = 0 \Leftrightarrow x = 0$,

(ii) $|\alpha x| = |\alpha|\,|x|$ for all $\alpha \in \mathbb{R}$, $x \in \mathbb{R}^n$,

(iii) $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}^n$.

For example, the most frequently used vector norms are

(a) the *maximum norm*:

$$|x|_\infty := \max_{j=1\ldots n} |x_j| . \tag{A3.1}$$

(b) the ℓ_p-*norm*, $p \in [1, \infty)$:

$$|x|_p := \left\{ \sum_{j=1}^{n} |x_j|^p \right\}^{1/p} . \tag{A3.2}$$

The important case $p = 2$ yields the so-called *Euclidean norm*:

$$|x|_2 := \left\{ \sum_{j=1}^{n} x_j^2 \right\}^{1/2} . \tag{A3.3}$$

The three most important norms (that is, $p = 1, 2, \infty$) in $\mathbb{R}^n$ are *equivalent* in the following sense: The inequalities

$$\begin{aligned} \frac{1}{\sqrt{n}}\,|x|_2 \;&\leq\; |x|_\infty \;\leq\; |x|_2 \;\leq\; \sqrt{n}\,|x|_\infty , \\ \frac{1}{n}\,|x|_1 \;&\leq\; |x|_\infty \;\leq\; |x|_1 \;\leq\; n\,|x|_\infty , \\ \frac{1}{\sqrt{n}}\,|x|_1 \;&\leq\; |x|_2 \;\leq\; |x|_1 \;\leq\; \sqrt{n}\,|x|_2 \end{aligned}$$

are valid for all $x \in \mathbb{R}^n$.

The *norm of the matrix* $A \in \mathbb{R}^{n,n}$ is a real-valued function $A \mapsto \|A\|$ satisfying the following four properties:

(i) $\|A\| \geq 0$ for all $A \in \mathbb{R}^{n,n}$, $\|A\| = 0 \Leftrightarrow A = 0$,

(ii) $\|\alpha A\| = |\alpha|\,\|A\|$ for all $\alpha \in \mathbb{R}$, $A \in \mathbb{R}^{n,n}$,

(iii) $\|A + B\| \leq \|A\| + \|B\|$ for all $A, B \in \mathbb{R}^{n,n}$,

(iv) $\|AB\| \leq \|A\|\,\|B\|$ for all $A, B \in \mathbb{R}^{n,n}$.

In comparison with the definition of a vector norm, we include here an additional property (iv), which is called the *submultiplicative property.* It restricts the general set of matrix norms to the practically important class of *submultiplicative norms.*

The most common matrix norms are
(a) the *total norm*:

$$\|A\|_G := n \max_{1\le i,k\le n} |a_{ik}| \,, \tag{A3.4}$$

(b) the *Frobenius norm*:

$$\|A\|_F := \left\{ \sum_{i,k=1}^{n} a_{ik}^2 \right\}^{1/2} , \tag{A3.5}$$

(c) the *maximum row sum*:

$$\|A\|_\infty := \max_{1\le i\le n} \sum_{k=1}^{n} |a_{ik}| \,, \tag{A3.6}$$

(d) the *maximum column sum*:

$$\|A\|_1 := \max_{1\le k\le n} \sum_{i=1}^{n} |a_{ik}| \,. \tag{A3.7}$$

All these matrix norms are equivalent. For example, we have

$$\frac{1}{n}\|A\|_G \le \|A\|_p \le \|A\|_G \le n\|A\|_p \,, \quad p \in \{1, \infty\} \,,$$

or

$$\frac{1}{n}\|A\|_G \le \|A\|_F \le \|A\|_G \le n\|A\|_F \,.$$

Note that the spectral radius $\varrho(A)$ is not a matrix norm, as the following simple example shows:

For $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have that $A \ne 0$ but $\varrho(A) = 0$.

However, for any matrix norm $\|\cdot\|$ the following relation is valid:

$$\varrho(A) \le \|A\| \,. \tag{A3.8}$$

Very often, matrices and vectors simultaneously appear as a product Ax. In order to be able to handle such situations, there should be a certain correlation between matrix and vector norms.

A matrix norm $\|\cdot\|$ is called *mutually consistent* or *compatible* with the vector norm $|\cdot|$ if the inequality

$$|Ax| \le \|A\| \, |x| \tag{A3.9}$$

is valid for all $x \in \mathbb{R}^n$ and all $A \in \mathbb{R}^{n,n}$.

Examples of mutually consistent norms are

$$\|A\|_G \quad \text{or} \quad \|A\|_\infty \quad \text{with} \quad |x|_\infty \,,$$

$$\|A\|_G \quad \text{or} \quad \|A\|_1 \quad \text{with} \quad |x|_1 \,,$$

$$\|A\|_G \quad \text{or} \quad \|A\|_F \quad \text{with} \quad |x|_2 \,.$$

In many cases, the bound for $|Ax|$ given by (A3.9) is not sharp enough; i.e., for $x \neq 0$ we just have that

$$|Ax| < \|A\| \, |x| \,.$$

Therefore, the question arises of how to find, for a given vector norm, a compatible matrix norm such that in (A3.9) the equality holds for at least one element $x \neq 0$.

Given a vector norm $|x|$, the number

$$\|A\| := \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|Ax|}{|x|} = \sup_{x \in \mathbb{R}^n :\, |x|=1} |Ax|$$

is called the *induced* or *subordinate* matrix norm.

The induced norm is a compatible matrix norm with the given vector norm. It is the smallest norm among all matrix norms that are compatible with the given vector norm $|x|$.

To illustrate the definition of the induced matrix norm, the matrix norm induced by the Euclidean vector norm is derived:

$$\|A\|_2 := \max_{|x|_2=1} |Ax|_2 = \max_{|x|_2=1} \sqrt{x^T(A^TA)x} = \sqrt{\lambda_{\max}(A^TA)} = \sqrt{\varrho(A^TA)} \,. \tag{A3.10}$$

The matrix norm $\|A\|_2$ induced by the Euclidean vector norm is also called the *spectral norm.* This term becomes understandable in the special case of a symmetric matrix A. If $\lambda_1, \ldots, \lambda_n$ denote the real eigenvalues of A, then the matrix $A^TA = A^2$ has the eigenvalues λ_i^2 satisfying

$$\|A\|_2 = |\lambda_{\max}(A)| \,.$$

For symmetric matrices, the spectral norm coincides with the spectral radius. Because of (A3.8), it is the smallest possible matrix norm in that case.

As a further example, the maximum row sum $\|A\|_\infty$ is the matrix norm induced by the maximum norm $|x|_\infty$.

The number

$$\kappa(A) := \|A\| \, \|A^{-1}\|$$

is called the *condition number* of the matrix A with respect to the matrix norm under consideration.

The following relation holds:

$$1 \leq \|I\| = \|AA^{-1}\| \leq \|A\| \, \|A^{-1}\| \, .$$

For $\|\cdot\| = \|\cdot\|_p$, the condition number is also denoted by $\kappa_p(A)$. If all eigenvalues of A are real, the number

$$\kappa(A) := \lambda_{\max}(A)/\lambda_{\min}(A)$$

is called the *spectral condition number.* Hence, for a symmetric matrix A the equality $\kappa(A) = \kappa_2(A)$ is valid.

Occasionally, it is necessary to estimate small perturbations of nonsingular matrices. For this purpose, the following result is useful (*perturbation lemma* or *Neumann's lemma*). Let $A \in \mathbb{R}^{n,n}$ satisfy $\|A\| < 1$ with respect to an arbitrary, but fixed, matrix norm. Then the inverse of $I - A$ exists and can be represented as a convergent power series of the form

$$(I - A)^{-1} = \sum_{j=0}^{\infty} A^j ,$$

with

$$\|(I - A)^{-1}\| \leq \frac{1}{1 - \|A\|} \, . \tag{A3.11}$$

Special Matrices

The matrix $A \in \mathbb{R}^{n,n}$ is called an *upper,* respectively *lower, triangular matrix* if its entries satisfy $a_{ij} = 0$ for $i > j$, respectively $a_{ij} = 0$ for $i < j$.

A matrix $H \in \mathbb{R}^{n,n}$ is called an *(upper) Hessenberg matrix* if it has the following structure:

$$H := \begin{pmatrix} h_{11} & & & & \\ h_{21} & \ddots & & * & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & 0 & & h_{nn-1} & h_{nn} \end{pmatrix}$$

(that is, $h_{ij} = 0$ for $i > j + 1$).

The matrix $A \in \mathbb{R}^{n,n}$ satisfies the *strict row sum criterion* (or is *strictly row diagonally dominant*) if it satisfies

$$\sum_{\substack{j=1\\ j \neq i}}^{n} |a_{ij}| < |a_{ii}| \quad \text{for all } i = 1, \ldots, n \, .$$

It satisfies the *strict column sum criterion* if the following relation holds:

$$\sum_{\substack{i=1\\ i \neq j}}^{n} |a_{ij}| < |a_{jj}| \quad \text{for all } j = 1, \ldots, n \, .$$

The matrix $A \in \mathbb{R}^{n,n}$ satisfies the *weak row sum criterion* (or is *weakly row diagonally dominant*) if

$$\sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}| \leq |a_{ii}| \quad \text{holds for all } i = 1, \ldots, n$$

and the strict inequality "<" is valid for at least one number $i \in \{1, \ldots, n\}$.

The weak column sum criterion is defined similarly.

The matrix $A \in \mathbb{R}^{n,n}$ is called *reducible* if there exist subsets $N_1, N_2 \subset \{1, \ldots, n\}$ with $N_1 \cap N_2 = \emptyset$, $N_1 \neq \emptyset \neq N_2$, and $N_1 \cup N_2 = \{1, \ldots, n\}$ such that the following property is satisfied:

$$\text{For all } i \in N_1,\ j \in N_2 \colon\ a_{ij} = 0\ .$$

A matrix that is not reducible is called *irreducible.*

A matrix $A \in \mathbb{R}^{n,n}$ is called an L_0*-matrix* if for $i, j \in \{1, \ldots, n\}$ the inequalities

$$a_{ii} \geq 0 \quad \text{and} \quad a_{ij} \leq 0 \ (i \neq j)$$

are valid. An L_0-matrix is called an *L-matrix* if all diagonal entries are positive.

A matrix $A \in \mathbb{R}^{n,n}$ is called *monotone* (or *of monotone type*) if the relation $Ax \leq Ay$ for two (otherwise arbitrary) elements $x, y \in \mathbb{R}^n$ implies $x \leq y$. Here the relation sign is to be understood componentwise.

A matrix of monotone type is invertible.

A matrix $A \in \mathbb{R}^{n,n}$ is a matrix of monotone type if it is invertible and all entries of the inverse are nonnegative.

An important subclass of matrices of monotone type is formed by the so-called M-matrices.

A monotone matrix A with $a_{ij} \leq 0$ for $i \neq j$ is called an *M-matrix.*

Let $A \in \mathbb{R}^{n,n}$ be a matrix with $a_{ij} \leq 0$ for $i \neq j$ and $a_{ii} \geq 0$ $(i, j \in \{1, \ldots, n\})$. In addition, let A satisfy one of the following conditions:

(i) A satisfies the strict row sum criterion.

(ii) A satisfies the weak row sum criterion and is irreducible.

Then A is an M-matrix.

A.4 Some Definitions and Arguments of Linear Functional Analysis

Working with vector spaces whose elements are (classical or generalized) functions, it is desirable to have a measure for the "length" or "magnitude" of a function, and, as a consequence, for the distance of two functions.

Let V be a real vector space (in short, an $\mathbb{R}$ vector space) and let $\|\cdot\|$ be a real-valued mapping $\|\cdot\| : V \to \mathbb{R}$.

The pair $(V, \|\cdot\|)$ is called a *normed space* ("V is endowed with the *norm* $\|\cdot\|$") if the following properties hold:

$$\|u\| \geq 0 \quad \text{for all } u \in V\,, \quad \|u\| = 0 \Leftrightarrow u = 0\,, \tag{A4.1}$$

$$\|\alpha u\| = |\alpha|\,\|u\| \quad \text{for all } \alpha \in \mathbb{R}\,,\ u \in V\,, \tag{A4.2}$$

$$\|u + v\| \leq \|u\| + \|v\| \quad \text{for all } u, v \in V\,. \tag{A4.3}$$

The property (A4.1) is called *definiteness*; (A4.3) is called the *triangle inequality.* If a mapping $\|\cdot\| : V \to \mathbb{R}$ satisfies only (A4.2) and (A4.3), it is called a *seminorm.* Due to (A4.2), we still have $\|0\| = 0$, but there may exist elements $u \neq 0$ with $\|u\| = 0$.

A particularly interesting example of a norm can be obtained if the space V is equipped with a so-called *scalar product.* This is a mapping $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ with the following properties:

(1) $\langle\cdot,\cdot\rangle$ is a *bilinear form,* that is,

$$\begin{aligned} \langle u, v_1 + v_2\rangle &= \langle u, v_1\rangle + \langle u, v_2\rangle \quad &&\text{for all} \quad u, v_1, v_2 \in V\,, \\ \langle u, \alpha v\rangle &= \alpha\,\langle u, v\rangle &&\text{for all} \quad u, v \in V,\ \alpha \in \mathbb{R}\,, \end{aligned} \tag{A4.4}$$

and an analogous relation is valid for the first argument.

(2) $\langle\cdot,\cdot\rangle$ is *symmetric*, that is,

$$\langle u, v\rangle = \langle v, u\rangle \quad \text{for all} \quad u, v \in V\,. \tag{A4.5}$$

(3) $\langle\cdot,\cdot\rangle$ is *positive*, that is,

$$\langle u, u\rangle \geq 0 \quad \text{for all} \quad u \in V\,. \tag{A4.6}$$

(4) $\langle\cdot,\cdot\rangle$ is *definite*, that is,

$$\langle u, u\rangle = 0 \Leftrightarrow u = 0\,. \tag{A4.7}$$

A positive and definite bilinear form is called *positive definite.*

A scalar product $\langle\cdot,\cdot\rangle$ defines a norm on V in a natural way if we set

$$\|v\| := \langle v, v\rangle^{1/2}\,. \tag{A4.8}$$

In absence of the definiteness (A4.7), only a seminorm is induced.

A norm (or a seminorm) induced by a scalar product (respectively by a symmetric and positive bilinear form) has some interesting properties. For example, it satisfies the *Cauchy–Schwarz inequality*, that is,

$$|\langle u, v\rangle| \leq \|u\|\,\|v\| \quad \text{for all } u, v \in V\,, \tag{A4.9}$$

and the *parallelogram identity*

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2) \quad \text{for all } u, v \in V\,. \tag{A4.10}$$

Typical examples of normed spaces are the spaces $\mathbb{R}^n$ equipped with one of the ℓ^p-norms (for some fixed $p \in [1, \infty]$). In particular, the Euclidean norm (A3.3) is induced by the *Euclidean scalar product*

$$(x, y) \mapsto x \cdot y \quad \text{for all } x, y \in \mathbb{R}^n . \tag{A4.11}$$

On the other hand, infinite-dimensional function spaces play an important role (see Appendix A.5).

If a vector space V is equipped with a scalar product $\langle \cdot, \cdot \rangle$, then, in analogy to $\mathbb{R}^n$, an element $u \in V$ is said to be *orthogonal* to $v \in V$ if

$$\langle u, v \rangle = 0 . \tag{A4.12}$$

Given a normed space $(V, \|\cdot\|)$, it is easy to define the concept of *convergence* of a sequence $(u_i)_i$ in V to $u \in V$:

$$u_i \to u \quad \text{for } i \to \infty \qquad \Longleftrightarrow \qquad \|u_i - u\| \to 0 \quad \text{for } i \to \infty . \tag{A4.13}$$

Often, it is necessary to consider function spaces endowed with different norms. In such situations, different kinds of convergence may occur. However, if the corresponding norms are equivalent, then there is no change in the type of convergence. Two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ in V are called *equivalent* if there exist constants $C_1, C_2 > 0$ such that

$$C_1 \|u\|_1 \le \|u\|_2 \le C_2 \|u\|_1 \qquad \text{for all } u \in V . \tag{A4.14}$$

If there is only a one-sided inequality of the form

$$\|u\|_2 \le C \|u\|_1 \quad \text{for all } u \in V \tag{A4.15}$$

with a constant $C > 0$, then the norm $\| \cdot \|_1$ is called *stronger* than the norm $\| \cdot \|_2$.

In a finite-dimensional vector space, all norms are equivalent. Examples can be found in Appendix A.3. In particular, it is important to observe that the constants may depend on the dimension n of the finite-dimensional vector space. This observation also indicates that in the case of infinite-dimensional vector spaces, the equivalence of two different norms cannot be expected, in general.

As a consequence of (A4.14), two equivalent norms $\| \cdot \|_1, \| \cdot \|_2$ in V yield the same type of convergence:

$$\begin{aligned} u_i \to u \text{ w.r.t. } \| \cdot \|_1 \quad &\Leftrightarrow \quad \|u_i - u\|_1 \to 0 \\ &\Leftrightarrow \quad \|u_i - u\|_2 \to 0 \quad \Leftrightarrow \quad u_i \to u \text{ w.r.t. } \| \cdot \|_2 . \end{aligned} \tag{A4.16}$$

In this book, the finite-dimensional vector space $\mathbb{R}^n$ is used in two aspects: For $n = d$, it is the basic space of independent variables, and for $n = M$ or $n = m$ it represents the finite-dimensional trial space. In the first case, the equivalence of all norms can be used in all estimates without any side effects, whereas in the second case the aim is to obtain uniform estimates with respect to all M and m, and so the dependence of the equivalence constants on M and m has to be followed thoroughly.

Now we consider two normed spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$. A mapping $f : V \to W$ is called *continuous* in $v \in V$ if for all sequences $(v_i)_i$ in V with $v_i \to v$ for $i \to \infty$ we get

$$f(v_i) \to f(v) \quad \text{for} \quad i \to \infty\,.$$

Note that the first convergence is measured in $\|\cdot\|_V$ and the second one in $\|\cdot\|_W$. Hence a change of the norm may have an influence on the continuity. As in classical analysis, we can say that

$$\begin{aligned} &f \text{ is continuous in all } v \in V \iff \\ &f^{-1}[G] \text{ is closed for each closed } G \subset W\,. \end{aligned} \tag{A4.17}$$

Here, a subset $G \subset W$ of a normed space W is called *closed* if for any sequence $(u_i)_i$ from G such that $u_i \to u$ for $i \to \infty$ the inclusion $u \in G$ follows. Because of (A4.17), the closedness of a set can be verified by showing that it is a continuous preimage of a closed set.

The concept of continuity is a qualitative relation between the preimage and the image. A quantitative relation is given by the stronger notion of Lipschitz continuity:

A mapping $f : V \to W$ is called *Lipschitz continuous* if there exists a constant $L > 0$, the *Lipschitz constant*, such that

$$\|f(u) - f(v)\|_W \le L\|u - v\|_V \quad \text{for all } u, v \in V\,. \tag{A4.18}$$

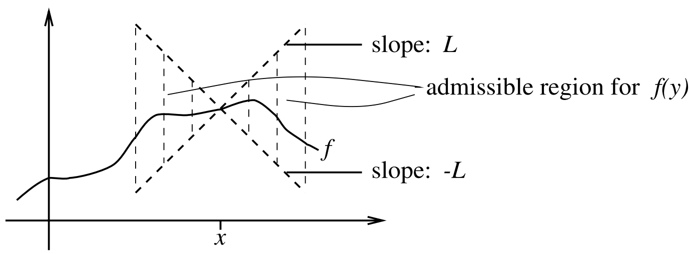

Figure A.1. Lipschitz continuity (for $V = W = \mathbb{R}$).

A Lipschitz continuous mapping with $L < 1$ is called *contractive* or a *contraction*; cf. Figure A.1.

Most of the mappings used are *linear*; that is, they satisfy

$$\left.\begin{aligned} f(u+v) &= f(u) + f(v)\,, \\ f(\lambda u) &= \lambda f(u)\,, \end{aligned}\right\} \quad \text{for all } u, v \in V \text{ and } \lambda \in \mathbb{R}\,. \tag{A4.19}$$

For a linear mapping, the Lipschitz continuity is equivalent to the *boundedness*; that is, there exists a constant $C > 0$ such that

$$\|f(u)\|_W \le C\|u\|_V \quad \text{for all } u \in V\,. \tag{A4.20}$$

In fact, for a linear mapping f, the continuity at one point is equivalent to (A4.20). Linear, continuous mappings acting from V to W are also called (linear, continuous) *operators* and are denoted by capital letters, for example $S, T, \ldots$.

In the case $V = W = \mathbb{R}^n$, the linear, continuous operators in $\mathbb{R}^n$ are the mappings $x \mapsto Ax$ defined by matrices $A \in \mathbb{R}^{n,n}$. Their boundedness, for example with respect to $\|\cdot\|_V = \|\cdot\|_W = \|\cdot\|_\infty$, is an immediate consequence of the compatibility property of the $\|\cdot\|_\infty$-norm. Moreover, since all norms in $\mathbb{R}^n$ are equivalent, these mappings are bounded with respect to any norms in $\mathbb{R}^n$.

Similarly to (A4.20), a bilinear form $f : V \times V \to \mathbb{R}$ is continuous if it is *bounded,* that is, if there exists a constant $C > 0$ such that

$$|f(u,v)| \leq C \, \|u\|_V \|v\|_V \quad \text{for all } u, v \in V \,. \tag{A4.21}$$

In particular, due to (A4.9) any scalar product is continuous with respect to the induced norm of V; that is,

$$u_i \to u \,, \; v_i \to v \quad \Rightarrow \quad \langle u_i, v_i \rangle \to \langle u, v \rangle \,. \tag{A4.22}$$

Now let $(V, \|\cdot\|_V)$ be a normed space and W a subspace that is (additionally to $\|\cdot\|_V$) endowed with the norm $\|\cdot\|_W$. The *embedding* from $(W, \|\cdot\|_W)$ to $(V, \|\cdot\|_V)$, i.e., the linear mapping that assigns any element of W to itself but considered as an element of V, is continuous iff the norm $\|\cdot\|_W$ is stronger than the norm $\|\cdot\|_V$ (cf. (A4.15)).

The collection of linear, continuous operators from $(V, \|\cdot\|_V)$ to $(W, \|\cdot\|_W)$ forms an $\mathbb{R}$ vector space with the following (argumentwise) operations:

$$\begin{aligned} (T+S)(u) &:= T(u) + S(u) \qquad &&\text{for all } u \in V \,, \\ (\lambda T)(u) &:= \lambda T(u) &&\text{for all } u \in V \,, \end{aligned}$$

for all operators T, S and $\lambda \in \mathbb{R}$. This space is denoted by

$$L[V, W] \,. \tag{A4.23}$$

In the special case $W = \mathbb{R}$, the corresponding operators are called linear, continuous *functionals,* and the notation

$$V' := L[V, \mathbb{R}] \tag{A4.24}$$

is used. The $\mathbb{R}$ vector space $L[V, W]$ can be equipped with a norm, the so-called *operator norm*, by

$$\|T\| := \sup \left\{ \|T(u)\|_W \;\middle|\; u \in V \,, \; \|u\|_V \leq 1 \right\} \quad \text{for} \quad T \in L[V, W] \,. \tag{A4.25}$$

Here $\|T\|$ is the smallest constant such that (A4.20) holds. Specifically, for a functional $f \in V'$, we have that

$$\|f\| = \sup \left\{ |f(u)| \;\middle|\; \|u\|_V \leq 1 \right\} \,.$$

For example, in the case $V = W = \mathbb{R}^n$ and $\|u\|_V = \|u\|_W$, the norm of a linear, bounded operator that is represented by a matrix $A \in \mathbb{R}^{n,n}$ coincides with the corresponding induced matrix norm (cf. Appendix A.3).

Let $(V, \|\cdot\|_V)$ be a normed space. A sequence $(u_i)_i$ in V is called a *Cauchy sequence* if for any $\varepsilon > 0$ there exists a number $n_0 \in \mathbb{N}$ such that

$$\|u_i - u_j\|_V \le \varepsilon \quad \text{for all } i, j \in \mathbb{N} \text{ with } i, j \ge n_0 \,.$$

The space V is called *complete* or a *Banach space* if for any Cauchy sequence $(u_i)_i$ in V there exists an element $u \in V$ such that $u_i \to u$ for $i \to \infty$. If the norm $\|\cdot\|_V$ of a Banach space V is induced by a scalar product, then V is called a *Hilbert space*.

A subspace W of a Banach space is complete iff it is closed. A basic problem in the variational treatment of boundary value problems consists in the fact that the space of continuous functions (cf. the preliminary definition (2.7)), which is required to be taken as a basis, is not complete with respect to the norm ($\|\cdot\|_l$, $l = 0$ or $l = 1$). However, if in addition to the normed space $(W, \|\cdot\|)$, a larger space V is given that is complete with respect to the norm $\|\cdot\|$, then that space or the closure

$$\widetilde{W} := \overline{W} \tag{A4.26}$$

(as the smallest Banach space containing W) can be used. Such a *completion* can be introduced for any normed space in an abstract way. The problem is that the "nature" of the limiting elements remains vague.

If the relation (A4.26) is valid for some normed space W, then W is called *dense* in $\widetilde{W}$. In fact, given W, all "essential" elements of $\widetilde{W}$ are already captured. For example, if T is a linear, continuous operator T from $(\widetilde{W}, \|\cdot\|)$ to another normed space, then the identity

$$T(u) = 0 \quad \text{for all} \quad u \in W \tag{A4.27}$$

is sufficient for

$$T(u) = 0 \quad \text{for all} \quad u \in \widetilde{W} \,. \tag{A4.28}$$

The space of linear, bounded operators is complete if the image space is complete. In particular, the space V' of linear, bounded functionals on the normed space V is always complete.

A.5 Function Spaces

In this section $G \subset \mathbb{R}^d$ denotes a bounded domain.

The function space $C(G)$ contains all (real-valued) functions defined on G that are continuous in G. By $C^l(G)$, $l \in \mathbb{N}$, the set of l-times continuously differentiable functions on G is denoted. Usually, for the sake of consistency, the conventions $C^0(G) := C(G)$ and $C^\infty(G) := \bigcap_{l=0}^{\infty} C^l(G)$ are used.

Functions from $C^l(G)$, $l \in \mathbb{N}_0$, and $C^\infty(G)$ need not be bounded, as for $d = 1$ the example $f(x) := x^{-1}$, $x \in (0,1)$ shows.

To overcome this difficulty, further spaces of continuous functions are introduced. The space $C(\overline{G})$ contains all bounded and uniformly continuous functions on G, whereas $C^l(\overline{G})$, $l \in \mathbb{N}$, consists of functions with bounded and uniformly continuous derivatives up to order l on G. Here the conventions $C^0(\overline{G}) := C(\overline{G})$ and $C^\infty(\overline{G}) := \bigcap_{l=0}^\infty C^l(\overline{G})$ are used, too.

The space $C_0(G)$, respectively $C_0^l(G)$, $l \in \mathbb{N}$, denotes the set of all those continuous, respectively l-times continuously differentiable, functions, the supports of which are contained in G. Often this set is called the set of functions with compact support in G. Since G is bounded, this means that the supports do not intersect boundary points of G. We also set $C_0^0(G) := C_0(G)$ and $C_0^\infty(G) := C_0(G) \cap C^\infty(G)$.

The linear space $L^p(G)$, $p \in [1, \infty)$, contains all Lebesgue measurable functions defined on G whose pth power of their absolute value is Lebesgue integrable on G. The norm in $L^p(G)$ is defined as follows:

$$\|u\|_{0,p,G} := \left\{ \int_G |u|^p \, dx \right\}^{1/p}, \quad p \in [1, \infty) .$$

In the case $p = 2$, the specification of p is frequently omitted; that is, $\|u\|_{0,G} = \|u\|_{0,2,G}$. The $L^2(G)$-scalar product

$$\langle u, v \rangle_{0,G} := \int_G uv \, dx , \quad u, v \in L^2(G) ,$$

induces the $L^2(G)$-norm by setting $\|u\|_{0,G} := \sqrt{\langle u, u \rangle_{0,G}}$.

The space $L^\infty(G)$ contains all measurable, essentially bounded functions on G, where a function $u : G \to \mathbb{R}$ is called *essentially bounded* if the quantity

$$\|u\|_{\infty,G} := \inf_{G_0 \subset G : \, |G_0|_d = 0} \ \sup_{x \in G \setminus G_0} |u(x)|$$

is finite. For continuous functions, this norm coincides with the usual maximum norm:

$$\|u\|_{\infty,G} = \max_{x \in \overline{G}} |u(x)| , \quad u \in C(\overline{G}) .$$

For $1 \le q \le p \le \infty$, we have $L^p(G) \subset L^q(G)$, and the embedding is continuous.

The space $W_p^l(G)$, $l \in \mathbb{N}$, $p \in [1, \infty]$, consists of all l-times weakly differentiable functions from $L_p(G)$ with derivatives in $L^p(G)$. In the special case $p = 2$, we also write $H^l(G) := W_2^l(G)$. In analogy to the case of continuous functions, the convention $H^0(G) := L^2(G)$ is used. The norm in $W_p^l(G)$ is defined as follows:

$$
\begin{aligned}
\|u\|_{l,p,G} &:= \left\{ \sum_{|\alpha|\le l} \int_G |\partial^\alpha u|^p \, dx \right\}^{1/p}, \quad p \in [1,\infty)\,, \\
\|u\|_{l,\infty,G} &:= \max_{|\alpha|\le l} |\partial^\alpha u|_{\infty,G}\,.
\end{aligned}
$$

In $H^l(G)$ a scalar product can be defined by

$$
\langle u, v\rangle_{l,G} := \sum_{|\alpha|\le l} \int_G \partial^\alpha u \partial^\alpha v \, dx\,, \quad u, v \in H^l(G)\,.
$$

The norm induced by this scalar product is denoted by $\|\cdot\|_{l,G}$, $l \in \mathbb{N}$:

$$
\|u\|_{l,G} := \sqrt{\langle u, u\rangle_{l,G}}\,.
$$

For $l \in \mathbb{N}$, the symbol $|\cdot|_{l,G}$ stands for the corresponding $H^l(G)$-*seminorm*:

$$
|u|_{l,G} := \sqrt{\sum_{|\alpha|=l} \int_G |\partial^\alpha u|^2 \, dx}\,.
$$

The space $H^1_0(G)$ is defined as the closure (or completion) of $C_0^\infty(G)$ in the norm $\|\cdot\|_1$ of $H^1(G)$.

Convention: Usually, in the case $G = \Omega$ the specification of the domain in the above norms and scalar products is omitted.

In the study of partial differential equations, it is often desirable to speak of boundary values of functions defined on the domain G. In this respect, the Lebesgue spaces of functions that are square integrable at the boundary of G are important. To introduce these spaces, some preparations are necessary.

In what follows, a point $x \in \mathbb{R}^d$ is written in the form $x = \binom{x'}{x_d}$ with $x' = (x_1, \ldots, x_{d-1})^T \in \mathbb{R}^{d-1}$.

A domain $G \subset \mathbb{R}^d$ is said to be *located at one side of* ∂G if for any $x \in \partial G$ there exist an open neighbourhood $U_x \subset \mathbb{R}^d$ and an orthogonal mapping Q_x in $\mathbb{R}^d$ such that the point x is mapped to a point $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_d)^T$, and so U_x is mapped onto a neighbourhood $U_{\hat{x}} \subset \mathbb{R}^d$ of $\hat{x}$, where in the neighbourhood $U_{\hat{x}}$ the following properties hold:

(1) The image of $U_x \cap \partial G$ is the graph of some function $\Psi_x : Y_x \subset \mathbb{R}^{d-1} \to \mathbb{R}$; that is, $\hat{x}_d = \Psi_x(\hat{x}_1, \ldots, \hat{x}_{d-1}) = \Psi_x(\hat{x}')$ for $\hat{x}' \in Y_x$.

(2) The image of $U_x \cap G$ is "above this graph" (i.e., the points in $U_x \cap G$ correspond to $\hat{x}_d > 0$).

(3) The image of $U_x \cap (\mathbb{R}^d \setminus \overline{G})$ is "below this graph" (i.e., the points in $U_x \cap (\mathbb{R}^d \setminus \overline{G})$ correspond to $\hat{x}_d < 0$).

A domain G that is located at one side of ∂G is called a C^l *domain*, $l \in \mathbb{N}$, respectively a *Lipschitz(ian) domain*, if all Ψ_x are l-times continuously differentiable, respectively Lipschitz continuous, in Y_x.

Bounded Lipschitz domains are also called *strongly Lipschitz.*

For bounded domains located at one side of ∂G, it is well known (cf., e.g. [37]) that from the whole set of neighbourhoods $\{U_x\}_{x\in\partial G}$ there can be selected a family $\{U_i\}_{i=1}^n$ of finitely many neighbourhoods covering ∂G, i.e., $n \in \mathbb{N}$ and $\partial G \subset \bigcup_{i=1}^n U_i$. Furthermore, for any such family there exists a system of functions $\{\varphi_i\}_{i=1}^n$ with the properties $\varphi_i \in C_0^\infty(U_i)$, $\varphi_i(x) \in [0,1]$ for all $x \in U_i$ and $\sum_{i=1}^n \varphi_i(x) = 1$ for all $x \in \partial G$. Such a system is called a *partition of unity.*

If the domain G is at least Lipschitzian, then Lebesgue's integral over the boundary of G is defined by means of those partitions of unity. In correspondence to the definition of a Lipschitz domain, Q_i, Ψ_i, and Y_i denote the orthogonal mapping on U_i, the function describing the corresponding local boundary, and the preimage of $Q_i(U_i \cap \partial G)$ with respect to Ψ_i.

A function $v : \partial G \to \mathbb{R}$ is called *Lebesgue integrable over* ∂G if the composite functions $\hat{x}' \mapsto v\left(Q_i^T\binom{\hat{x}'}{\Psi_i(\hat{x}')}\right)$ belong to $L^1(Y_i)$. The integral is defined as follows:

$$\begin{aligned}\int_{\partial G} v(s)\,ds &:= \sum_{i=1}^n \int_{\partial G} v(s)\varphi_i(s)\,ds\\ &:= \sum_{i=1}^n \int_{Y_i} v\left(Q_i^T\binom{\hat{x}'}{\Psi_i(\hat{x}')}\right)\varphi_i\left(Q_i^T\binom{\hat{x}'}{\Psi_i(\hat{x}')}\right)\\ &\qquad\times\sqrt{|\det(\partial_j\Psi_i(\hat{x}')\partial_k\Psi_i(\hat{x}'))_{j,k=1}^{d-1}|}\,d\hat{x}'\,.\end{aligned}$$

A function $v : \partial G \to \mathbb{R}$ belongs to $L^2(\partial G)$ iff both v and v^2 are Lebesgue integrable over ∂G.

In the investigation of time-dependent partial differential equations, linear spaces whose elements are functions of the time variable $t \in [0,T]$, $T > 0$, with values in a normed space X are of interest.

A function $v : [0,T] \to X$ is called *continuous on* $[0,T]$ if for all $t \in [0,T]$ the convergence $\|v(t+k) - v(t)\|_X \to 0$ as $k \to 0$ holds.

The space $C([0,T],X) = C^0([0,T],X)$ consists of all continuous functions $v : [0,T] \to X$ such that

$$\sup_{t\in(0,T)} \|v(t)\|_X < \infty\,.$$

The space $C^l([0,T],X)$, $l \in \mathbb{N}$, consists of all continuous functions $v : [0,T] \to X$ that have continuous derivatives up to order l on $[0,T]$ with the norm

$$\sum_{i=0}^{l} \sup_{t \in (0,T)} \|v^{(i)}(t)\|_X \,.$$

The space $L^p((0,T),X)$ with $1 \le p \le \infty$ consists of all functions on $(0,T)\times \Omega$ for which

$$v(t,\cdot) \in X \text{ for any } t \in (0,T)\,, \quad F \in L^p(0,T) \quad \text{with } F(t) := \|v(t,\cdot)\|_X \,.$$

Furthermore,

$$\|v\|_{L^p((0,T),X)} := \|F\|_{L^p(0,T)} \,.$$

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