A

Numerical Techniques for Maximization

Many of the procedures discussed in this volume require maximizing the log likelihood or partial log likelihood function. For many models, it is impossible to perform this maximization analytically, so, numerical methods must be employed. In this appendix, we shall summarize some techniques which can be used in both univariate and multivariate cases. The reader is referred to a text on statistical computing, such as Thisted (1988), for a more detailed discussion of these techniques.

A.1 Univariate Methods

Suppose we wish to find the value x which maximizes a function $f(\,)$ of a single variable. Under some mild regularity conditions, x maximizes f if the score equation $f'(x)$ equals 0 and $f''(x) < 0$. We present three numerical methods which attempt to find the maximum of $f(\,)$ by solving the score equation. Some care must be taken when using these routines because they do not ensure that the second derivative of f is negative at the solution we find.

The first technique is the bisection method. Here, the algorithm starts with two initial values, x_L and x_U, which bracket the root of $f'(x) = 0$, that is, $f'(x_L) \cdot f'(x_U) < 0$. A new guess at the root is taken to be the midpoint of the interval (x_L, x_U), namely, $x_N = (x_L + x_U)/2$. If $f'(x_L)$ and $f'(x_N)$ have the same sign, x_L is replaced by x_N, otherwise x_U is replaced by x_N. In either case, the algorithm continues with the new values of x_L and x_U until the desired accuracy is achieved. At each step, the length of the interval $(x_U - x_L)$ is a measure of the largest possible difference between our updated guess at the root of $f'(\,)$ and the actual value of the root.

A second method, of use when one has good initial starting values and a complicated second derivative of $f(\,)$, is the secant method or *regula falsi*. Again, we start with two initial guesses at the root, x_o and x_1. These guesses need not bracket the root. After i steps of the algorithm, the new guess at the root of $f'(x)$ is given by

$$x_{i+1} = x_i - f'(x_i)(x_i - x_{i-1})/[f'(x_i) - f'(x_{i-1})]. \tag{A.1}$$

Iterations continue until convergence. Typical stopping criteria are

$$|x_{i+1} - x_i| < \gamma,\ |f'(x_{i+1})| < \gamma$$

or

$$|(x_{i+1} - x_i)/x_i| < \gamma,$$

where γ is some small number.

The third method is the Newton–Raphson technique. Here, a single initial guess, x_o, of the root is made. After i steps of the algorithm, the updated guess is given by

$$x_{i+1} = x_i - f'(x_i)/f''(x_i). \tag{A.2}$$

Again, the iterative procedure continues until the desired level of accuracy is met. Compared to the secant method, this technique has the advantage of requiring a single starting value, and convergence is quicker than the secant method when the starting values are good. Both the secant and Newton–Raphson techniques may fail to converge when the starting values are not close to the maximum.

EXAMPLE A.1 Suppose we have the following 10 uncensored observations from a Weibull model with scale parameter $\lambda = 1$ and shape parameter α, that is, $h(t) = \alpha t^{\alpha-1} e^{-t^{\alpha}}$.

Data: 2.57, 0.58, 0.82, 1.02, 0.78, 0.46, 1.04, 0.43, 0.69, 1.37

To find the maximum likelihood estimator of α, we need to maximize the log likelihood $f(\alpha) = \ln L(\alpha) = n\ln(\alpha) + (\alpha - 1)\sum \ln(t_j) - \sum t_j^{\alpha}$.

Here, $f'(\alpha) = n/\alpha + \sum \ln(t_j) - \sum t_j^{\alpha}\ln(t_j)$, and $f''(\alpha) = -n/\alpha^2 - \sum t_j^{\alpha}[\ln(t_j)]^2$.

Applying the bisection method with $\alpha_L = 1.5$ and $\alpha_U = 2$ and stopping the algorithm when $|f'(\alpha)| < 0.01$, we have the following values:

Step	α_L	α_U	α_N	$f'(\alpha_L)$	$f'(\alpha_U)$	$f'(\alpha_N)$
1	1.5	2	1.75	1.798	−2.589	−0.387
2	1.5	1.75	1.625	1.798	−0.387	0.697
3	1.625	1.75	1.6875	0.697	−0.387	0.154
4	1.6875	1.75	1.71875	0.154	−0.387	−0.116
5	1.6875	1.71875	1.70313	0.154	−0.116	0.019
6	1.70313	1.71875	1.71094	0.019	−0.116	−0.049
7	1.70313	1.71094	1.70704	0.019	−0.049	−0.015
8	1.70313	1.70704	1.70509	0.019	−0.015	0.002

So, after eight steps the algorithm stops with $\hat{\alpha} = 1.705$.

For the secant method, we shall start the algorithm with $\alpha_o = 1$ and $\alpha_1 = 1.5$. The results are in the following table:

Step	α_{i-1}	α_i	$f'(\alpha_{i-1})$	$f'(\alpha_i)$	α_{i+1}	$f'(\alpha_{i+1})$
1	1	1.5	7.065	1.798	1.671	0.300
2	1.5	1.671	1.798	0.300	1.705	0.004

Here, using the same stopping rule $|f'(\alpha)| < 0.01$, the algorithm stops after two steps with $\hat{\alpha} = 1.705$.

For the Newton–Raphson procedure, we use an initial value of $\alpha_o = 1.5$. The results of the algorithm are in the following table.

i	α_{i-1}	$f'(\alpha_{i-1})$	$f''(\alpha_{i-1})$	α_i	$f'(\alpha_i)$
1	1.5	1.798	−8.947	1.701	0.038
2	1.701	0.038	−8.655	1.705	2×10^{-6}

Again, using the same stopping rule $|f'(\alpha)| < 0.01$, the algorithm stops after two steps with $\hat{\alpha} = 1.705$. Notice the first step of the Newton–Raphson algorithm moves closer to the root than the secant method.

A.2 Multivariate Methods

We present three methods to maximize a function of more than one variable. The first is the method of steepest ascent which requires only the vector of first derivatives of the function. This method is robust to the starting values used in the iterative scheme, but may require a large number of steps to converge to the maximum. The second is the multivariate extension of the Newton–Raphson method. This method, which requires both the first and second derivatives of the function, converges quite rapidly when the starting values are close to the root, but may not converge when the starting values are poorly chosen. The third, called Marquardt's (1963) method, is a compromise between these two methods. It uses a blending constant which controls how closely the algorithm resembles either the method of steepest ascent or the Newton–Raphson method.

Some notation is needed before presenting the three methods. Let $f(\mathbf{x})$ be a function of the p-dimensional vector $\mathbf{x} = (x_1, \ldots, x_p)^t$. Let $\mathbf{u}(\mathbf{x})$ be the p-vector of first order partial derivatives of $f(\mathbf{x})$, that is,

$$\mathbf{u}(\mathbf{x}) = [u_1(\mathbf{x}), \ldots, u_p(\mathbf{x})]^t, \tag{A.3}$$

where

$$u_j(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial x_j}, j = 1, \ldots, p.$$

Let $\mathbf{H}(\mathbf{x})$ be the $p \times p$ Hessian matrix of mixed second partial derivatives of $f(\mathbf{x})$, defined by

$$\mathbf{H}(\mathbf{x}) = (H_{ij}(\mathbf{x})),\ i, j = 1, \ldots, p \text{ where } H_{ij}(\mathbf{x}) = \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}. \tag{A.4}$$

The method of steepest ascent starts with an initial guess, $\mathbf{x}_o$, of the point which maximizes $f(\mathbf{x})$. At any point, the gradient vector $\mathbf{u}(\mathbf{x})$ points the direction of steepest ascent of the function $f(\mathbf{x})$. The algorithm moves along this direction by an amount d to a new estimate of the maximum from the current estimate. The step size d is chosen to maximize the function in this direction, that is, we pick d to maximize $f[\mathbf{x}_k + d\mathbf{u}(\mathbf{x}_k)]$. This requires maximizing a function of a single variable, so that any of the techniques discussed earlier can be employed.

The updated guess at the point which maximizes $f(\mathbf{x})$ is given by

$$\mathbf{x}_{k+1} = \mathbf{x}_k + d\mathbf{u}(\mathbf{x}_k). \tag{A.5}$$

The second method is the Newton–Raphson method which, like the method of steepest ascent, starts with an initial guess at the point which maximizes $f(\mathbf{x})$. After k steps of the algorithm, the updated estimate of the point which maximizes $f(\mathbf{x})$ is given by

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \mathbf{H}(\mathbf{x}_k)^{-1}\mathbf{u}(\mathbf{x}_k). \tag{A.6}$$

The Newton–Raphson algorithm converges quite rapidly when the initial guess is not too far from the maximum. When the initial guess is poor, the algorithm may move in the wrong direction or may take a step in the correct direction, but overshoot the root. The value of the function should be computed at each step to ensure that the algorithm is moving in the correct direction. If $f(\mathbf{x}_k)$ is smaller than $f(\mathbf{x}_{k+1})$, one option is to cut the step size in half and try $\mathbf{x}_{k+1} = \mathbf{x}_k - \mathbf{H}(\mathbf{x}_k)^{-1}\mathbf{u}(\mathbf{x}_k)/2$. This procedure is used in SAS and BMDP in the Cox regression procedure.

The third method is Marquardt's (1963) compromise between the method of steepest ascent and the Newton–Raphson method. This method uses a constant, γ, which blends the two methods together. When γ is zero, the method reduces to the Newton–Raphson method, and, as $\gamma \to \infty$, the method approaches the method of steepest ascent. Again, the method starts with an initial guess, $\mathbf{x}_o$. Let $\mathbf{S}_k$ be the $p \times p$ diagonal scaling matrix with diagonal element $(|\mathbf{H}_{ii}(\mathbf{x}_k)|^{-1/2})$. The updated estimate of the maximum is given by

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \mathbf{S}_k(\mathbf{S}_k\mathbf{H}(\mathbf{x}_k)\mathbf{S}_k + \gamma\mathbf{I})^{-1}\mathbf{S}_k\mathbf{u}(\mathbf{x}_k),$$

where $\mathbf{I}$ is the identity matrix. Typically, the algorithm is implemented with a small value of γ for the first iteration. If $f(\mathbf{x}_1) < f(\mathbf{x}_o)$, then, we are having difficulty approaching the maximum and the value of γ is increased until $f(\mathbf{x}_1) > f(\mathbf{x}_o)$. This procedure is iterated until convergence is attained. For the final step of the algorithm, a "Newton–Raphson" step with $\gamma = 0$ is taken to ensure convergence.

In the multivariate maximization problem, there are several suggestions for declaring convergence of these algorithms. These include stopping when $f(\mathbf{x}_{k+1}) - f(\mathbf{x}_k) < \epsilon$ (or $|[f(\mathbf{x}_{k+1}) - f(\mathbf{x}_k)]/f(\mathbf{x}_k)| < \epsilon$); when $\sum \mathbf{u}_j(\mathbf{x}_{k+1})^2 < \epsilon$ (or $\max[|\mathbf{u}_1(\mathbf{x}_{k+1})|, \ldots, |\mathbf{u}_p(\mathbf{x}_{k+1})|] < \epsilon$) or when $\sum(x_{k+1,j} - x_{k,j})^2 < \epsilon$ (or $\max[|x_{k+1,1} - x_{k,1}|, \ldots, |x_{k+1,p} - x_{k,p}| < \epsilon$).

EXAMPLE A.2

We shall fit a two-parameter Weibull model with survival function $S(t) = \exp(-\lambda t^{\alpha})$ to the ten observations in Example A.2. Here the log likelihood function is given by

$$L(\lambda, \alpha) = n \ln \lambda + n \ln \alpha + (\alpha - 1)\sum \ln t_i - \lambda \sum t_i^{\alpha}.$$

The score vector $\mathbf{u}(\lambda, \alpha)$ is expressed by

$$u_\lambda(\lambda, \alpha) = \frac{\partial L(\lambda, \alpha)}{\partial \lambda} = \frac{n}{\lambda} - \sum t_i^{\alpha}$$

$$u_\alpha(\lambda, \alpha) = \frac{\partial L(\lambda, \alpha)}{\partial \alpha} = \frac{n}{\alpha} + \sum \ln t_i - \lambda \sum t_i^{\alpha} \ln t_i$$

and the Hessian matrix is

$$\mathbf{H}(\lambda, \alpha) = \begin{pmatrix} -\dfrac{n}{\lambda^2} & -\sum t_i^{\alpha} \ln t_i \\ -\sum t_i^{\alpha} \ln t_i & -\dfrac{n}{\alpha^2} - \lambda \sum t_i^{\alpha}(\ln t_i^2) \end{pmatrix}$$

To apply the method of steepest ascent, we must find the value of d_k which maximizes $L[(\lambda_k + d_k u_\lambda[\lambda_k, \alpha_k]), (\alpha_k + d_k u_\alpha[\lambda_k, \alpha_k])]$. This needs to be done numerically and this example uses a Newton–Raphson algorithm. Convergence of the algorithm is declared when the maximum of $|u_\lambda|$ and $|u_\alpha|$ is less than 0.1. Starting with an initial guess of $\alpha = 1$ and $\lambda = 10/\sum t_i = 1.024$, which leads to a log likelihood of -9.757, we have the following results:

Step k	λ_k	α_k	$L(\lambda, \alpha)$	u_λ	u_α	d_k
0	1.024	1.000	−9.757	0.001	7.035	0.098
1	1.025	1.693	−7.491	0.001	−1.80	0.089
2	0.865	1.694	−7.339	0.661	0.001	0.126
3	0.865	1.777	−7.311	0.000	−0.363	0.073
4	0.839	1.777	−7.307	0.121	0.000	0.128
5	0.839	1.792	−7.306	0.000	−0.072	0.007

Thus the method of steepest ascent yields maximum likelihood estimates of $\hat{\lambda} = 0.839$ and $\hat{\alpha} = 1.792$ after 5 iterations of the algorithm.

Applying the Newton–Raphson algorithm with the same starting values and convergence criterion yields

Step k	λ_k	α_k	u_λ	u_α	$H_{\lambda\lambda}$	$H_{\alpha\alpha}$	$H_{\alpha\lambda}$
0	1.024	1.000	0.001	7.035	−9.537	−13.449	−1.270
1	0.954	1.530	−0.471	1.684	−10.987	−8.657	−3.34
2	0.838	1.769	0.035	0.181	−14.223	−7.783	−1.220
3	0.832	1.796	−0.001	0.001	−14.431	−7.750	−4.539

This method yields maximum likelihood estimates of $\hat{\lambda} = 0.832$ and $\hat{\alpha} = 1.796$ after three iterations.

Using $\gamma = 0.5$ in Marquardt's method yields

Step k	λ_k	α_k	u_λ	u_α	$H_{\lambda\lambda}$	$H_{\alpha\alpha}$	$H_{\alpha\lambda}$
0	1.024	1.000	0.001	7.035	−9.537	−13.449	−1.270
1	0.993	1.351	−0.357	3.189	−10.136	−9.534	−2.565
2	0.930	1.585	−0.394	1.295	−11.557	−8.424	−3.599
3	0.883	1.701	−0.275	0.523	−12.813	−8.049	−4.176
4	0.858	1.753	−0.162	0.218	−13.591	−7.891	−4.453
5	0.845	1.777	−0.087	0.094	−14.013	−7.817	−4.581

Here, the algorithm converges in five steps to estimates of $\hat{\lambda} = 0.845$ and $\hat{\alpha} = 1.777$.

B

Large-Sample Tests Based on Likelihood Theory

Many of the test procedures used in survival analysis are based on the asymptotic properties of the likelihood or the partial likelihood. These test procedures are based on either the maximized likelihood itself (likelihood ratio tests), on the estimators standardized by use of the information matrix (Wald tests), or on the first derivatives of the log likelihood (score tests). In this appendix, we will review how these tests are constructed. See Chapter 9 of Cox and Hinkley (1974) for a more detailed reference.

Let $\mathbf{Y}$ denote the data and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_p)$ be the parameter vector. Let $L(\boldsymbol{\theta} : \mathbf{Y})$ denote either the likelihood or partial likelihood function. The maximum likelihood estimator of $\boldsymbol{\theta}$ is the function of the data which maximizes the likelihood, that is, $\hat{\boldsymbol{\theta}}(\mathbf{Y}) = \hat{\boldsymbol{\theta}}$ is the value of $\boldsymbol{\theta}$ which maximizes $L(\boldsymbol{\theta} : \mathbf{Y})$ or, equivalently, maximizes $\log L(\boldsymbol{\theta} : \mathbf{Y})$.

Associated with the likelihood function is the efficient score vector $\mathbf{U}(\boldsymbol{\theta}) = [U_1(\boldsymbol{\theta}), \dots, U_p(\boldsymbol{\theta})]$ defined by

$$U_j(\boldsymbol{\theta}) = \frac{\delta}{\delta\theta_j} \ln L(\boldsymbol{\theta} : \mathbf{Y}). \tag{B.1}$$

In most regular cases, the maximum likelihood estimator is the solution to the equation $\mathbf{U}(\boldsymbol{\theta}) = \mathbf{0}$. The efficient score vector has the property that its expected value is zero when the expectation is taken with respect to the true value of $\boldsymbol{\theta}$.

A second key quantity in large-sample likelihood theory is the Fisher information matrix defined by

$$\mathbf{i}(\boldsymbol{\theta}) = E_{\boldsymbol{\theta}}[\mathbf{U}(\boldsymbol{\theta})^t\mathbf{U}(\boldsymbol{\theta})] = E_{\boldsymbol{\theta}}\left[-\frac{\delta}{\delta\boldsymbol{\theta}}\mathbf{U}(\boldsymbol{\theta})\right]$$

$$= \left\{-E_{\boldsymbol{\theta}}\left[\frac{\delta^2}{\delta\theta_j\delta\theta_k}\ln L(\boldsymbol{\theta}:\mathbf{Y})\right]\right\},\ j = 1,\ldots,p,\ k = 1,\ldots,p. \quad \text{(B.2)}$$

Computation of the expectation in (B.2) is very difficult in most applications of likelihood theory, so a consistent estimator of $\mathbf{i}$ is used. This estimator is the observed information, $\mathbf{I}(\boldsymbol{\theta})$, whose (j, k)th element is given by

$$I_{j,k}(\boldsymbol{\theta}) = -\frac{\delta^2\ln L(\boldsymbol{\theta}:\mathbf{Y})}{\delta\theta_j\delta\theta_k},\ j, k = 1,\ldots,p. \quad \text{(B.3)}$$

The first set of tests based on the likelihood are for the simple null hypothesis, $H_o : \boldsymbol{\theta} = \boldsymbol{\theta}_o$. The first test is the *likelihood ratio test* based on the statistic

$$\chi^2_{\mathrm{LR}} = -2[\ln L(\boldsymbol{\theta}_o:\mathbf{Y}) - \ln L(\hat{\boldsymbol{\theta}}:\mathbf{Y})] \quad \text{(B.4)}$$

This statistic has an asymptotic chi-squared distribution with p degrees of freedom under the null hypothesis.

A second test, called the *Wald test*, is based on the large-sample distribution of the maximum likelihood estimator. For large samples, $\hat{\boldsymbol{\theta}}$ has a multivariate normal distribution with mean $\boldsymbol{\theta}$ and covariance matrix $\mathbf{i}^{-1}(\boldsymbol{\theta})$ so the quadratic form $(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o)\mathbf{i}(\hat{\boldsymbol{\theta}})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o)^t$ has a chi-squared distribution with p degrees of freedom for large samples. Using the observed information as an estimator of the Fisher information, the Wald statistic is expressed as

$$\chi^2_W = (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o)\mathbf{I}(\hat{\boldsymbol{\theta}})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o)^t \quad \text{(B.5)}$$

which has a chi-squared distribution with p degrees of freedom for large samples when H_o is true.

The third test, called the *score or Rao test*, is based on the efficient score statistics. When $\boldsymbol{\theta} = \boldsymbol{\theta}_o$, the score vector $\mathbf{U}(\boldsymbol{\theta}_o)$ has a large-sample multivariate normal distribution with mean $\mathbf{0}$ and covariance matrix $\mathbf{i}(\boldsymbol{\theta})$. This leads to a test statistic given by

$$\chi^2_S = \mathbf{U}(\theta_o)\mathbf{i}^{-1}(\boldsymbol{\theta}_o)\mathbf{U}^t(\boldsymbol{\theta}_o).$$

As for the Wald test, the Fisher information is replaced in most applications by the observed information, so the test statistic is given by

$$\chi^2_S = \mathbf{U}(\boldsymbol{\theta}_o)\mathbf{I}^{-1}(\boldsymbol{\theta}_o)\mathbf{U}^t(\boldsymbol{\theta}_o). \quad \text{(B.6)}$$

Again, this statistic has an asymptotic chi-squared distribution with p degrees of freedom when H_o is true. The score test has an advantage in many applications in that the maximum likelihood estimates need not be calculated.

EXAMPLE B.1 Suppose we have a censored sample of size n from an exponential population with hazard rate λ. We wish to test the hypothesis that $\lambda = 1$. Let $(T_i, \delta_i), i = 1, \ldots, n$, so that the likelihood, $L(\lambda; (T_i, \delta_i), i = 1, \ldots, n)$, is given by $\prod_{i=1}^{n} \lambda^{\delta_i} e^{-\lambda T_i} = \lambda^D e^{-\lambda S}$ where $D = \sum_{i=1}^{n} \delta_i$ is the observed number of deaths and $S = \sum_{i=1}^{n} T_i$ is the total time on test (see Section 3.5). Thus,

$$\ln L(\lambda) = D \ln \lambda - \lambda S, \tag{B.7}$$

$$U(\lambda) = \frac{d}{d\lambda} \ln L(\lambda) = \frac{D}{\lambda} - S, \tag{B.8}$$

and

$$I(\lambda) = -\frac{d^2}{d\lambda^2} \ln L(\lambda) = \frac{D}{\lambda^2}. \tag{B.9}$$

Solving B.8 for λ gives us the maximum likelihood estimator, $\hat{\lambda} = D/S$. Using these statistics,

$$\chi_S^2 = \left(\frac{D}{1} - S\right)^2 \cdot \left(\frac{1^2}{D}\right) = \frac{(D - S)^2}{D},$$

$$\chi_W^2 = \left(\frac{D}{S} - 1\right)^2 \cdot \frac{D}{(D/S)^2} = \frac{(D - S)^2}{D}$$

$$\begin{aligned} \chi_{LR}^2 &= -2\{(D \ln 1 - 1 \cdot S) - [D \ln(D/S) - (D/S) \cdot S]\} \\ &= 2[S - D + D \ln(D/S)] \end{aligned}$$

In this case, note that the Wald and Rao tests are identical. All three of these statistics have asymptotic chi-squared distributions with one degree of freedom.

All three test statistics can be used to test composite hypotheses. Suppose the parameter vector $\boldsymbol{\theta}$ is divided into two vectors $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ of lengths p_1, and p_2, respectively. We would like to test the hypothesis $H_o : \boldsymbol{\psi} = \boldsymbol{\psi}_o$. Here $\boldsymbol{\phi}$ is a nuisance parameter. Let $\hat{\boldsymbol{\phi}}(\boldsymbol{\psi}_o)$ be the maximum likelihood estimates of $\boldsymbol{\phi}$ obtained by maximizing the likelihood with respect to $\boldsymbol{\phi}$, with $\boldsymbol{\psi}$ fixed at $\boldsymbol{\psi}_o$. That is, $\hat{\boldsymbol{\phi}}(\boldsymbol{\psi}_o)$ maximizes $\ln L[(\boldsymbol{\psi}_o, \boldsymbol{\phi}) : \mathbf{Y}]$ with respect to $\boldsymbol{\phi}$. We also partition the information matrix $\mathbf{I}$ into

$$\mathbf{I} = \begin{pmatrix} \mathbf{I}_{\boldsymbol{\psi}\boldsymbol{\psi}} & \mathbf{I}_{\boldsymbol{\psi}\boldsymbol{\phi}} \\ \mathbf{I}_{\boldsymbol{\phi}\boldsymbol{\psi}} & \mathbf{I}_{\boldsymbol{\phi}\boldsymbol{\phi}} \end{pmatrix}, \tag{B.10}$$

where $\mathbf{I}_{\boldsymbol{\psi}\boldsymbol{\psi}}$ is of dimension $p_1 \times p_1$, $\mathbf{I}_{\boldsymbol{\phi}\boldsymbol{\phi}}$ is of dimension $p_2 \times p_2$, $\mathbf{I}_{\boldsymbol{\psi}\boldsymbol{\phi}}$ is $p_1 \times p_2$, and $\mathbf{I}_{\boldsymbol{\phi}\boldsymbol{\psi}}^t = \mathbf{I}_{\boldsymbol{\psi}\boldsymbol{\phi}}$. Notice that a partitioned information matrix has an inverse which is also a partitioned matrix with

$$\mathbf{I}^{-1} = \begin{pmatrix} \mathbf{I}^{\psi\psi} & \mathbf{I}^{\psi\phi} \\ \mathbf{I}^{\phi\psi} & \mathbf{I}^{\phi\phi} \end{pmatrix}, \tag{B.11}$$

With these refinements, the three statistics for testing $H_o : \boldsymbol{\psi} = \boldsymbol{\psi}_o$ are given by
Likelihood ratio test:

$$\chi^2_{LR} = -2\{\ln L[(\boldsymbol{\psi}_o, \hat{\boldsymbol{\phi}}(\boldsymbol{\psi}_o) : \mathbf{Y})] - \ln L(\hat{\boldsymbol{\theta}} : \mathbf{Y})\}, \tag{B.12}$$

Wald test:

$$\chi^2_W = (\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}_o)[\mathbf{I}^{\psi\psi}(\hat{\boldsymbol{\psi}}, \hat{\boldsymbol{\phi}})]^{-1}(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}_o)^t, \tag{B.13}$$

and score test:

$$\chi^2_S = \mathbf{U}_{\psi}[\boldsymbol{\psi}_o, \hat{\boldsymbol{\phi}}(\boldsymbol{\psi}_o)][\mathbf{I}^{\psi\psi}(\boldsymbol{\psi}_o, \hat{\boldsymbol{\phi}}(\boldsymbol{\psi}_o))]\mathbf{U}^t[\boldsymbol{\psi}_o, \hat{\boldsymbol{\phi}}(\boldsymbol{\psi}_o)]. \tag{B.14}$$

All three statistics have an asymptotic chi-squared distribution with p_1 degrees of freedom when the null hypothesis is true.

EXAMPLE B.2 Consider the problem of comparing two treatments, where the time to event in each group has an exponential distribution. For population one, we assume that the hazard rate is λ whereas for population two, we assume that the hazard rate is $\lambda\beta$. We shall test $H_o : \beta = 1$ treating λ as a nuisance parameter. The likelihood function is given by

$$L[(\lambda, \beta) : D_1, D_2, S_1, S_2] = \lambda^{D_1+D_2}\beta^{D_2}\exp(-\lambda S_1 - \lambda\beta S_2) \tag{B.15}$$

where D_i is the number of events and S_i is the total time on test in the ith sample, $i = 1, 2$. From (B.15),

$$\ln L[(\beta, \lambda)] = (D_1 + D_2)\ln\lambda + D_2\ln\beta - \lambda S_1 - \lambda\beta S_2, \tag{B.16}$$

$$U_\beta(\beta, \lambda) = \frac{\delta}{\delta\beta}\ln L(\beta, \lambda) = \frac{D_2}{\beta} - \lambda S_2, \tag{B.17}$$

$$U_\lambda(\beta, \lambda) = \frac{\delta}{\delta\lambda}\ln L(\beta, \lambda) = \frac{D_1 + D_2}{\lambda} - S_1 - \beta S_2, \tag{B.18}$$

$$I_{\beta\beta}(\beta, \lambda) = -\frac{\delta^2 \ln L(\beta, \lambda)}{\delta\beta^2} = \frac{D_2}{\beta^2}, \tag{B.19}$$

$$I_{\lambda\lambda}(\beta, \lambda) = -\frac{\delta^2 \ln L(\beta, \lambda)}{\delta\lambda^2} = \frac{D_1 + D_2}{\lambda^2}, \tag{B.20}$$

and

$$I_{\beta\lambda} = -\frac{\delta^2 \ln L(\beta, \lambda)}{\delta\lambda\delta\beta} = S_2. \tag{B.21}$$

Solving the system of equations $U_\beta(\beta, \lambda) = 0$, $U_\lambda(\beta, \lambda) = 0$ yields the global maximum likelihood estimators $\hat{\beta} = S_1D_2/(S_2D_1)$ and $\hat{\lambda} = D_1/S_1$.

Solving $U_\lambda(\beta, \lambda) = 0$, for β fixed at its value under H_o, yields $\hat{\lambda}(\beta = 1)$, denoted by $\hat{\lambda}(1), = (D_1 + D_2)/(S_1 + S_2)$. Thus, we have from B.12 a likelihood ratio test statistic of

$$\begin{aligned}\chi^2_{LR} &= -2\{[(D_1 + D_2)\ln[\hat{\lambda}(1)] - \hat{\lambda}(1)(S_1 + S_2)] \\ &\qquad - [(D_1 + D_2)\ln[\hat{\lambda}] + D_2 \ln[\hat{\beta}] - \hat{\lambda} S_1 - \hat{\lambda}\hat{\beta} S_2]\}. \\ &= 2D_1 \ln\left[\frac{D_1(S_1 + S_2)}{S_1(D_1 + D_2)}\right] + 2D_2 \ln\left[\frac{D_2(S_1 + S_2)}{S_2(D_1 + D_2)}\right].\end{aligned}$$

From B.19–B.21,

$$I^{\beta\beta}(\beta, \lambda) = \frac{\beta^2(D_1 + D_2)}{\left[D_2(D_1 + D_2) - (\lambda\beta S_2)^2\right]}$$

so, the Wald test is given by

$$\begin{aligned}\chi^2_W &= (\hat{\beta} - 1)^2 \left\{\frac{\hat{\beta}^2(D_1 + D_2)}{[D_2(D_1 + D_2) - (\hat{\lambda}\hat{\beta} S_2)^2]}\right\}^{-1} \\ &= \frac{D_1^2(S_1 D_2 - S_2 D_1)^2}{D_2 S_1^2(D_1 + D_2)}\end{aligned}$$

The score test is given by

$$\begin{aligned}\chi^2_S &= (D_2 - \hat{\lambda}(1)S_2)^2 \frac{(D_1 + D_2)}{[D_2(D_1 + D_2) - (\hat{\lambda}(1)S_2)^2]} \\ &= \frac{[D_2(S_1 + S_2) - (D_1 + D_2)S_2]^2}{D_2(S_1 + S_2)^2 - (D_1 + D_2)S_1^2}\end{aligned}$$

If, for example, $D_1 = 10$, $D_2 = 12$, $S_1 = 25$, and $S_2 = 27$, then $\chi^2_{LR} = 0.0607$, $\chi^2_W = 0.0545$ and $\chi^2_S = 0.0448$, all nonsignificant when compared to a chi-square with one degree of freedom.

C

Statistical Tables

TABLE C.1

Standard Normal Survival Function $P[Z \geq z]$

z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
3.5	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002
3.6	0.0002	0.0002	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
3.7	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
3.8	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
3.9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

TABLE C.2

Upper Percentiles of a Chi-Square Distribution

Degrees of Freedom	*Upper Percentile*				
	0.1	*0.05*	*0.01*	*0.005*	*0.001*
1	2.70554	3.84146	6.63489	7.87940	10.82736
2	4.60518	5.99148	9.21035	10.59653	13.81500
3	6.25139	7.81472	11.34488	12.83807	16.26596
4	7.77943	9.48773	13.27670	14.86017	18.46623
5	9.23635	11.07048	15.08632	16.74965	20.51465
6	10.64464	12.59158	16.81187	18.54751	22.45748
7	12.01703	14.06713	18.47532	20.27774	24.32130
8	13.36156	15.50731	20.09016	21.95486	26.12393
9	14.68366	16.91896	21.66605	23.58927	27.87673
10	15.98717	18.30703	23.20929	25.18805	29.58789
11	17.27501	19.67515	24.72502	26.75686	31.26351
12	18.54934	21.02606	26.21696	28.29966	32.90923
13	19.81193	22.36203	27.68818	29.81932	34.52737
14	21.06414	23.68478	29.14116	31.31943	36.12387
15	22.30712	24.99580	30.57795	32.80149	37.69777
16	23.54182	26.29622	31.99986	34.26705	39.25178
17	24.76903	27.58710	33.40872	35.71838	40.79111
18	25.98942	28.86932	34.80524	37.15639	42.31195
19	27.20356	30.14351	36.19077	38.58212	43.81936
20	28.41197	31.41042	37.56627	39.99686	45.31422
21	29.61509	32.67056	38.93223	41.40094	46.79627
22	30.81329	33.92446	40.28945	42.79566	48.26762
23	32.00689	35.17246	41.63833	44.18139	49.72764
24	33.19624	36.41503	42.97978	45.55836	51.17897
25	34.38158	37.65249	44.31401	46.92797	52.61874
26	35.56316	38.88513	45.64164	48.28978	54.05114
27	36.74123	40.11327	46.96284	49.64504	55.47508
28	37.91591	41.33715	48.27817	50.99356	56.89176
29	39.08748	42.55695	49.58783	52.33550	58.30064
30	40.25602	43.77295	50.89218	53.67187	59.70221
31	41.42175	44.98534	52.19135	55.00248	61.09799
32	42.58473	46.19424	53.48566	56.32799	62.48728
33	43.74518	47.39990	54.77545	57.64831	63.86936
34	44.90316	48.60236	56.06085	58.96371	65.24710
35	46.05877	49.80183	57.34199	60.27459	66.61917
36	47.21217	50.99848	58.61915	61.58107	67.98495
37	48.36339	52.19229	59.89256	62.88317	69.34759
38	49.51258	53.38351	61.16202	64.18123	70.70393
39	50.65978	54.57224	62.42809	65.47532	72.05504

(continued)

TABLE C.2
(continued)

Degrees of Freedom	*Upper Percentile*				
	0.1	*0.05*	*0.01*	*0.005*	*0.001*
40	51.80504	55.75849	63.69077	66.76605	73.40290
41	52.94850	56.94240	64.94998	68.05263	74.74412
42	54.09019	58.12403	66.20629	69.33604	76.08420
43	55.23018	59.30352	67.45929	70.61573	77.41841
44	56.36852	60.48090	68.70964	71.89234	78.74870
45	57.50529	61.65622	69.95690	73.16604	80.07755
46	58.64053	62.82961	71.20150	74.43671	81.39979
47	59.77429	64.00113	72.44317	75.70385	82.71984
48	60.90661	65.17076	73.68256	76.96892	84.03680
49	62.03753	66.33865	74.91939	78.23055	85.34987
50	63.16711	67.50481	76.15380	79.48984	86.66031
51	64.29539	68.66932	77.38601	80.74645	87.96700
52	65.42242	69.83216	78.61563	82.00062	89.27187
53	66.54818	70.99343	79.84336	83.25251	90.57257
54	67.67277	72.15321	81.06878	84.50176	91.87140
55	68.79621	73.31148	82.29198	85.74906	93.16708
56	69.91852	74.46829	83.51355	86.99398	94.46187
57	71.03970	75.62372	84.73265	88.23656	95.74998
58	72.15983	76.77778	85.95015	89.47699	97.03806
59	73.27891	77.93049	87.16583	90.71533	98.32425
60	74.39700	79.08195	88.37943	91.95181	99.60783
61	75.51409	80.23209	89.59122	93.18622	100.88685
62	76.63020	81.38098	90.80150	94.41853	102.16522
63	77.74539	82.52872	92.00989	95.64919	103.44210
64	78.85965	83.67524	93.21670	96.87794	104.71685
65	79.97299	84.82064	94.42200	98.10492	105.98766
66	81.08547	85.96494	95.62559	99.33027	107.25660
67	82.19711	87.10804	96.82768	100.55377	108.52505
68	83.30788	88.25017	98.02832	101.77574	109.79265
69	84.41787	89.39119	99.22741	102.99614	111.05540
70	85.52704	90.53126	100.42505	104.21477	112.31669
71	86.63543	91.67026	101.62144	105.43228	113.57693
72	87.74306	92.80827	102.81634	106.64732	114.83388
73	88.84994	93.94533	104.00977	107.86186	116.09165
74	89.95605	95.08146	105.20193	109.07417	117.34687
75	91.06145	96.21666	106.39285	110.28543	118.59895
76	92.16615	97.35097	107.58244	111.49537	119.85018
77	93.27017	98.48438	108.77089	112.70374	121.10075
78	94.37351	99.61696	109.95822	113.91069	122.34713
79	95.47617	100.74861	111.14403	115.11631	123.59471
80	96.57820	101.87947	112.32879	116.32093	124.83890

TABLE C.3a

Confidence Coefficients $c_{10}(a_{\text{L}}, a_{\text{U}})$ *for 90% EP Confidence Bands*

	a_{L}									
a_{U}	*0.02*	*0.04*	*0.06*	*0.08*	*0.10*	*0.12*	*0.14*	*0.16*	*0.18*	*0.20*
0.10	2.4547	2.3049	2.1947	2.1054						
0.12	2.4907	2.3521	2.2497	2.1654	2.0933					
0.14	2.5198	2.3901	2.2942	2.2147	2.1458	2.0849				
0.16	2.5441	2.4217	2.3313	2.2561	2.1905	2.1318	2.0788			
0.18	2.5650	2.4486	2.3630	2.2917	2.2291	2.1728	2.1214	2.0742		
0.20	2.5833	2.4721	2.3906	2.3227	2.2630	2.2090	2.1594	2.1134	2.0706	
0.22	2.5997	2.4929	2.4150	2.3501	2.2930	2.2412	2.1934	2.1489	2.1071	2.0677
0.24	2.6144	2.5116	2.4368	2.3747	2.3200	2.2703	2.2242	2.1811	2.1405	2.1019
0.26	2.6278	2.5286	2.4567	2.3970	2.3445	2.2967	2.2523	2.2107	2.1712	2.1336
0.28	2.6402	2.5441	2.4748	2.4174	2.3668	2.3208	2.2781	2.2378	2.1996	2.1631
0.30	2.6517	2.5586	2.4915	2.4361	2.3874	2.3431	2.3019	2.2630	2.2260	2.1905
0.32	2.6625	2.5720	2.5071	2.4536	2.4065	2.3638	2.3240	2.2864	2.2506	2.2162
0.34	2.6727	2.5846	2.5217	2.4698	2.4244	2.3831	2.3446	2.3083	2.2736	2.2403
0.36	2.6823	2.5965	2.5354	2.4852	2.4412	2.4012	2.3640	2.3289	2.2953	2.2630
0.38	2.6915	2.6078	2.5484	2.4997	2.4570	2.4183	2.3824	2.3484	2.3159	2.2845
0.40	2.7003	2.6186	2.5608	2.5134	2.4721	2.4346	2.3997	2.3668	2.3353	2.3049
0.42	2.7088	2.6290	2.5726	2.5266	2.4865	2.4501	2.4163	2.3844	2.3539	2.3244
0.44	2.7170	2.6390	2.5840	2.5392	2.5002	2.4649	2.4321	2.4012	2.3716	2.3431
0.46	2.7249	2.6486	2.5950	2.5514	2.5134	2.4792	2.4474	2.4174	2.3887	2.3610
0.48	2.7326	2.6579	2.6056	2.5631	2.5262	2.4929	2.4620	2.4329	2.4051	2.3782
0.50	2.7402	2.6671	2.6160	2.5745	2.5386	2.5062	2.4762	2.4480	2.4210	2.3949
0.52	2.7476	2.6760	2.6261	2.5857	2.5507	2.5192	2.4900	2.4626	2.4364	2.4111
0.54	2.7548	2.6847	2.6359	2.5965	2.5625	2.5318	2.5035	2.4768	2.4514	2.4269
0.56	2.7620	2.6933	2.6456	2.6072	2.5740	2.5441	2.5166	2.4907	2.4660	2.4422
0.58	2.7691	2.7018	2.6552	2.6177	2.5853	2.5563	2.5295	2.5043	2.4804	2.4573
0.60	2.7762	2.7103	2.6647	2.6281	2.5965	2.5682	2.5422	2.5178	2.4945	2.4721
0.62	2.7833	2.7186	2.6741	2.6384	2.6076	2.5801	2.5548	2.5310	2.5084	2.4867
0.64	2.7904	2.7270	2.6835	2.6486	2.6186	2.5918	2.5672	2.5441	2.5222	2.5011
0.66	2.7975	2.7354	2.6929	2.6588	2.6296	2.6036	2.5796	2.5572	2.5359	2.5155
0.68	2.8046	2.7439	2.7023	2.6691	2.6407	2.6153	2.5920	2.5703	2.5496	2.5298
0.70	2.8119	2.7524	2.7118	2.6794	2.6517	2.6271	2.6045	2.5833	2.5633	2.5441
0.72	2.8193	2.7611	2.7214	2.6899	2.6629	2.6390	2.6170	2.5965	2.5771	2.5586
0.74	2.8269	2.7700	2.7313	2.7005	2.6743	2.6510	2.6297	2.6099	2.5911	2.5731
0.76	2.8347	2.7790	2.7413	2.7114	2.6859	2.6633	2.6427	2.6235	2.6053	2.5879
0.78	2.8428	2.7884	2.7517	2.7226	2.6979	2.6760	2.6560	2.6374	2.6198	2.6031
0.80	2.8512	2.7982	2.7624	2.7342	2.7103	2.6890	2.6697	2.6517	2.6348	2.6186
0.82	2.8601	2.8085	2.7737	2.7463	2.7232	2.7026	2.6840	2.6667	2.6504	2.6348
0.84	2.8695	2.8193	2.7856	2.7592	2.7367	2.7170	2.6990	2.6823	2.6667	2.6517
0.86	2.8797	2.8310	2.7984	2.7728	2.7513	2.7322	2.7150	2.6990	2.6840	2.6697
0.88	2.8908	2.8437	2.8123	2.7877	2.7670	2.7487	2.7322	2.7170	2.7026	2.6890
0.90	2.9032	2.8579	2.8277	2.8042	2.7844	2.7670	2.7513	2.7367	2.7232	2.7103
0.92	2.9176	2.8741	2.8454	2.8230	2.8042	2.7877	2.7728	2.7592	2.7463	2.7342
0.94	2.9348	2.8936	2.8664	2.8454	2.8277	2.8123	2.7984	2.7856	2.7737	2.7624
0.96	2.9573	2.9188	2.8936	2.8741	2.8579	2.8437	2.8310	2.8193	2.8085	2.7982
0.98	2.9919	2.9573	2.9348	2.9176	2.9032	2.8908	2.8797	2.8695	2.8601	2.8512

(continued)

TABLE C.3a
(continued)

	a_{L}									
a_{U}	0.22	0.24	0.26	0.28	0.30	0.32	0.34	0.36	0.38	0.40
0.24	2.0654									
0.26	2.0978	2.0634								
0.28	2.1280	2.0943	2.0618							
0.30	2.1563	2.1233	2.0914	2.0605						
0.32	2.1829	2.1507	2.1194	2.0890	2.0594					
0.34	2.2080	2.1766	2.1460	2.1162	2.0870	2.0585				
0.36	2.2316	2.2011	2.1713	2.1421	2.1134	2.0853	2.0577			
0.38	2.2541	2.2244	2.1953	2.1668	2.1387	2.1111	2.0839	2.0570		
0.40	2.2754	2.2466	2.2183	2.1905	2.1631	2.1360	2.1092	2.0827	2.0565	
0.42	2.2958	2.2678	2.2403	2.2132	2.1865	2.1600	2.1337	2.1076	2.0818	2.0561
0.44	2.3153	2.2881	2.2614	2.2351	2.2090	2.1831	2.1574	2.1318	2.1064	2.0810
0.46	2.3341	2.3077	2.2818	2.2561	2.2307	2.2055	2.1804	2.1554	2.1304	2.1054
0.48	2.3521	2.3265	2.3014	2.2765	2.2518	2.2272	2.2027	2.1783	2.1538	2.1293
0.50	2.3696	2.3448	2.3203	2.2962	2.2722	2.2483	2.2244	2.2006	2.1766	2.1526
0.52	2.3865	2.3625	2.3388	2.3153	2.2920	2.2688	2.2456	2.2223	2.1990	2.1755
0.54	2.4030	2.3797	2.3567	2.3339	2.3113	2.2888	2.2662	2.2436	2.2208	2.1979
0.56	2.4191	2.3965	2.3742	2.3521	2.3302	2.3083	2.2864	2.2644	2.2423	2.2200
0.58	2.4349	2.4129	2.3913	2.3699	2.3487	2.3275	2.3062	2.2848	2.2633	2.2416
0.60	2.4503	2.4291	2.4081	2.3874	2.3668	2.3463	2.3257	2.3049	2.2841	2.2630
0.62	2.4656	2.4450	2.4247	2.4046	2.3847	2.3648	2.3448	2.3248	2.3045	2.2841
0.64	2.4807	2.4607	2.4411	2.4217	2.4023	2.3831	2.3638	2.3443	2.3248	2.3049
0.66	2.4957	2.4763	2.4573	2.4385	2.4198	2.4012	2.3825	2.3638	2.3448	2.3257
0.68	2.5106	2.4919	2.4735	2.4553	2.4373	2.4192	2.4012	2.3831	2.3648	2.3463
0.70	2.5256	2.5075	2.4897	2.4721	2.4547	2.4373	2.4198	2.4023	2.3847	2.3668
0.72	2.5406	2.5231	2.5059	2.4889	2.4721	2.4553	2.4385	2.4217	2.4046	2.3874
0.74	2.5558	2.5388	2.5222	2.5059	2.4897	2.4735	2.4573	2.4411	2.4247	2.4081
0.76	2.5712	2.5548	2.5388	2.5231	2.5075	2.4919	2.4763	2.4607	2.4450	2.4291
0.78	2.5869	2.5712	2.5558	2.5406	2.5256	2.5106	2.4957	2.4807	2.4656	2.4503
0.80	2.6031	2.5879	2.5731	2.5586	2.5441	2.5298	2.5155	2.5011	2.4867	2.4721
0.82	2.6198	2.6053	2.5911	2.5771	2.5633	2.5496	2.5359	2.5222	2.5084	2.4945
0.84	2.6374	2.6235	2.6099	2.5965	2.5833	2.5703	2.5572	2.5441	2.5310	2.5178
0.86	2.6560	2.6427	2.6297	2.6170	2.6045	2.5920	2.5796	2.5672	2.5548	2.5422
0.88	2.6760	2.6633	2.6510	2.6390	2.6271	2.6153	2.6036	2.5918	2.5801	2.5682
0.90	2.6979	2.6859	2.6743	2.6629	2.6517	2.6407	2.6296	2.6186	2.6076	2.5965
0.92	2.7226	2.7114	2.7005	2.6899	2.6794	2.6691	2.6588	2.6486	2.6384	2.6281
0.94	2.7517	2.7413	2.7313	2.7214	2.7118	2.7023	2.6929	2.6835	2.6741	2.6647
0.96	2.7884	2.7790	2.7700	2.7611	2.7524	2.7439	2.7354	2.7270	2.7186	2.7103
0.98	2.8428	2.8347	2.8269	2.8193	2.8119	2.8046	2.7975	2.7904	2.7833	2.7762

TABLE C.3a
(continued)

	a_{L}									
a_{U}	0.42	0.44	0.46	0.48	0.50	0.52	0.54	0.56	0.58	0.60
0.44	2.0557									
0.46	2.0804	2.0555								
0.48	2.1047	2.0800	2.0553							
0.50	2.1285	2.1042	2.0798	2.0553						
0.52	2.1518	2.1280	2.1040	2.0797	2.0553					
0.54	2.1748	2.1515	2.1279	2.1040	2.0798	2.0553				
0.56	2.1974	2.1746	2.1515	2.1280	2.1042	2.0800	2.0555			
0.58	2.2197	2.1974	2.1748	2.1518	2.1285	2.1047	2.0804	2.0557		
0.60	2.2416	2.2200	2.1979	2.1755	2.1526	2.1293	2.1054	2.0810	2.0561	
0.62	2.2633	2.2423	2.2208	2.1990	2.1766	2.1538	2.1304	2.1064	2.0818	2.0565
0.64	2.2848	2.2644	2.2436	2.2223	2.2006	2.1783	2.1554	2.1318	2.1076	2.0827
0.66	2.3062	2.2864	2.2662	2.2456	2.2244	2.2027	2.1804	2.1574	2.1337	2.1092
0.68	2.3275	2.3083	2.2888	2.2688	2.2483	2.2272	2.2055	2.1831	2.1600	2.1360
0.70	2.3487	2.3302	2.3113	2.2920	2.2722	2.2518	2.2307	2.2090	2.1865	2.1631
0.72	2.3699	2.3521	2.3339	2.3153	2.2962	2.2765	2.2561	2.2351	2.2132	2.1905
0.74	2.3913	2.3742	2.3567	2.3388	2.3203	2.3014	2.2818	2.2614	2.2403	2.2183
0.76	2.4129	2.3965	2.3797	2.3625	2.3448	2.3265	2.3077	2.2881	2.2678	2.2466
0.78	2.4349	2.4191	2.4030	2.3865	2.3696	2.3521	2.3341	2.3153	2.2958	2.2754
0.80	2.4573	2.4422	2.4269	2.4111	2.3949	2.3782	2.3610	2.3431	2.3244	2.3049
0.82	2.4804	2.4660	2.4514	2.4364	2.4210	2.4051	2.3887	2.3716	2.3539	2.3353
0.84	2.5043	2.4907	2.4768	2.4626	2.4480	2.4329	2.4174	2.4012	2.3844	2.3668
0.86	2.5295	2.5166	2.5035	2.4900	2.4762	2.4620	2.4474	2.4321	2.4163	2.3997
0.88	2.5563	2.5441	2.5318	2.5192	2.5062	2.4929	2.4792	2.4649	2.4501	2.4346
0.90	2.5853	2.5740	2.5625	2.5507	2.5386	2.5262	2.5134	2.5002	2.4865	2.4721
0.92	2.6177	2.6072	2.5965	2.5857	2.5745	2.5631	2.5514	2.5392	2.5266	2.5134
0.94	2.6552	2.6456	2.6359	2.6261	2.6160	2.6056	2.5950	2.5840	2.5726	2.5608
0.96	2.7018	2.6933	2.6847	2.6760	2.6671	2.6579	2.6486	2.6390	2.6290	2.6186
0.98	2.7691	2.7620	2.7548	2.7476	2.7402	2.7326	2.7249	2.7170	2.7088	2.7003

TABLE C.3b

Confidence Coefficients $c_{05}(a_L, a_U)$ for 95% EP Confidence Bands

					a_L					
a_U	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
0.10	2.7500	2.6033	2.4874	2.3859						
0.12	2.7841	2.6506	2.5463	2.4548	2.3715					
0.14	2.8114	2.6879	2.5924	2.5090	2.4327	2.3615				
0.16	2.8341	2.7184	2.6299	2.5530	2.4827	2.4167	2.3542			
0.18	2.8535	2.7442	2.6614	2.5898	2.5245	2.4632	2.4047	2.3486		
0.20	2.8704	2.7666	2.6884	2.6213	2.5602	2.5029	2.4481	2.3953	2.3442	
0.22	2.8855	2.7862	2.7120	2.6487	2.5912	2.5374	2.4859	2.4362	2.3879	2.3407
0.24	2.8990	2.8037	2.7330	2.6729	2.6186	2.5678	2.5193	2.4724	2.4266	2.3818
0.26	2.9114	2.8196	2.7519	2.6946	2.6430	2.5949	2.5490	2.5047	2.4614	2.4188
0.28	2.9227	2.8341	2.7691	2.7143	2.6651	2.6194	2.5758	2.5338	2.4927	2.4523
0.30	2.9333	2.8475	2.7849	2.7323	2.6853	2.6417	2.6002	2.5602	2.5211	2.4827
0.32	2.9432	2.8599	2.7995	2.7490	2.7039	2.6621	2.6225	2.5844	2.5472	2.5106
0.34	2.9525	2.8716	2.8131	2.7644	2.7211	2.6811	2.6432	2.6068	2.5712	2.5363
0.36	2.9613	2.8826	2.8260	2.7789	2.7372	2.6987	2.6624	2.6275	2.5936	2.5602
0.38	2.9696	2.8930	2.8381	2.7925	2.7523	2.7153	2.6804	2.6469	2.6144	2.5825
0.40	2.9777	2.9029	2.8495	2.8055	2.7666	2.7309	2.6973	2.6651	2.6339	2.6033
0.42	2.9854	2.9124	2.8605	2.8178	2.7801	2.7456	2.7133	2.6824	2.6524	2.6230
0.44	2.9928	2.9216	2.8710	2.8295	2.7931	2.7597	2.7285	2.6987	2.6699	2.6417
0.46	3.0001	2.9304	2.8812	2.8408	2.8055	2.7732	2.7431	2.7143	2.6865	2.6594
0.48	3.0071	2.9390	2.8910	2.8517	2.8174	2.7862	2.7570	2.7293	2.7025	2.6764
0.50	3.0140	2.9473	2.9005	2.8623	2.8290	2.7987	2.7705	2.7436	2.7178	2.6926
0.52	3.0207	2.9554	2.9097	2.8726	2.8402	2.8108	2.7835	2.7575	2.7326	2.7083
0.54	3.0273	2.9634	2.9188	2.8826	2.8511	2.8226	2.7961	2.7710	2.7469	2.7234
0.56	3.0338	2.9713	2.9277	2.8924	2.8618	2.8341	2.8084	2.7841	2.7608	2.7382
0.58	3.0403	2.9790	2.9365	2.9021	2.8723	2.8454	2.8205	2.7969	2.7744	2.7525
0.60	3.0468	2.9867	2.9451	2.9116	2.8826	2.8565	2.8323	2.8095	2.7877	2.7666
0.62	3.0532	2.9944	2.9537	2.9210	2.8928	2.8674	2.8440	2.8219	2.8007	2.7803
0.64	3.0596	3.0020	2.9623	2.9304	2.9029	2.8783	2.8555	2.8341	2.8137	2.7939
0.66	3.0661	3.0096	2.9709	2.9398	2.9130	2.8891	2.8670	2.8462	2.8264	2.8074
0.68	3.0726	3.0173	2.9795	2.9492	2.9231	2.8998	2.8784	2.8583	2.8392	2.8207
0.70	3.0792	3.0251	2.9881	2.9586	2.9333	2.9107	2.8899	2.8704	2.8519	2.8341
0.72	3.0859	3.0330	2.9969	2.9682	2.9435	2.9216	2.9014	2.8826	2.8647	2.8475
0.74	3.0928	3.0411	3.0059	2.9779	2.9539	2.9326	2.9131	2.8949	2.8776	2.8610
0.76	3.0999	3.0493	3.0150	2.9878	2.9646	2.9439	2.9250	2.9074	2.8907	2.8746
0.78	3.1072	3.0579	3.0244	2.9980	2.9755	2.9554	2.9372	2.9201	2.9040	2.8886
0.80	3.1149	3.0667	3.0342	3.0086	2.9867	2.9674	2.9497	2.9333	2.9178	2.9029
0.82	3.1230	3.0761	3.0445	3.0196	2.9985	2.9798	2.9628	2.9469	2.9320	2.9178
0.84	3.1315	3.0859	3.0553	3.0312	3.0109	2.9928	2.9765	2.9613	2.9469	2.9333
0.86	3.1408	3.0965	3.0669	3.0437	3.0241	3.0067	2.9910	2.9765	2.9628	2.9497
0.88	3.1509	3.1081	3.0795	3.0572	3.0384	3.0218	3.0067	2.9928	2.9798	2.9674
0.90	3.1622	3.1209	3.0935	3.0722	3.0542	3.0384	3.0241	3.0109	2.9985	2.9867
0.92	3.1752	3.1357	3.1096	3.0892	3.0722	3.0572	3.0437	3.0312	3.0196	3.0086
0.94	3.1909	3.1534	3.1287	3.1096	3.0935	3.0795	3.0669	3.0553	3.0445	3.0342
0.96	3.2113	3.1763	3.1534	3.1357	3.1209	3.1081	3.0965	3.0859	3.0761	3.0667
0.98	3.2428	3.2113	3.1909	3.1752	3.1622	3.1509	3.1408	3.1315	3.1230	3.1149

TABLE C.3b
(continued)

					a_L					
a_U	0.22	0.24	0.26	0.28	0.30	0.32	0.34	0.36	0.38	0.40
0.24	2.3378									
0.26	2.3769	2.3355								
0.28	2.4123	2.3727	2.3335							
0.30	2.4447	2.4069	2.3693	2.3319						
0.32	2.4744	2.4383	2.4024	2.3664	2.3305					
0.34	2.5018	2.4673	2.4330	2.3985	2.3640	2.3293				
0.36	2.5272	2.4943	2.4614	2.4285	2.3953	2.3620	2.3284			
0.38	2.5509	2.5194	2.4880	2.4564	2.4247	2.3926	2.3603	2.3276		
0.40	2.5731	2.5430	2.5129	2.4827	2.4523	2.4215	2.3904	2.3589	2.3269	
0.42	2.5940	2.5652	2.5364	2.5074	2.4783	2.4488	2.4189	2.3885	2.3577	2.3264
0.44	2.6138	2.5862	2.5586	2.5308	2.5029	2.4746	2.4459	2.4167	2.3871	2.3568
0.46	2.6327	2.6061	2.5796	2.5530	2.5262	2.4991	2.4716	2.4436	2.4150	2.3859
0.48	2.6506	2.6251	2.5997	2.5742	2.5485	2.5225	2.4961	2.4692	2.4418	2.4138
0.50	2.6679	2.6433	2.6189	2.5944	2.5697	2.5448	2.5195	2.4937	2.4674	2.4405
0.52	2.6844	2.6609	2.6374	2.6138	2.5902	2.5662	2.5419	2.5172	2.4920	2.4661
0.54	2.7005	2.6777	2.6552	2.6325	2.6098	2.5868	2.5636	2.5398	2.5156	2.4908
0.56	2.7160	2.6941	2.6724	2.6506	2.6288	2.6068	2.5844	2.5617	2.5385	2.5147
0.58	2.7311	2.7100	2.6891	2.6682	2.6472	2.6260	2.6046	2.5828	2.5606	2.5378
0.60	2.7459	2.7256	2.7054	2.6853	2.6651	2.6448	2.6242	2.6033	2.5820	2.5602
0.62	2.7604	2.7408	2.7214	2.7020	2.6826	2.6631	2.6434	2.6233	2.6029	2.5820
0.64	2.7747	2.7558	2.7371	2.7184	2.6998	2.6811	2.6621	2.6429	2.6233	2.6033
0.66	2.7888	2.7706	2.7525	2.7346	2.7167	2.6987	2.6805	2.6621	2.6434	2.6242
0.68	2.8028	2.7852	2.7679	2.7506	2.7334	2.7161	2.6987	2.6811	2.6631	2.6448
0.70	2.8168	2.7998	2.7831	2.7666	2.7500	2.7334	2.7167	2.6998	2.6826	2.6651
0.72	2.8308	2.8145	2.7984	2.7824	2.7666	2.7506	2.7346	2.7184	2.7020	2.6853
0.74	2.8449	2.8292	2.8137	2.7984	2.7831	2.7679	2.7525	2.7371	2.7214	2.7054
0.76	2.8591	2.8440	2.8292	2.8145	2.7998	2.7852	2.7706	2.7558	2.7408	2.7256
0.78	2.8737	2.8591	2.8449	2.8308	2.8168	2.8028	2.7888	2.7747	2.7604	2.7459
0.80	2.8886	2.8746	2.8610	2.8475	2.8341	2.8207	2.8074	2.7939	2.7803	2.7666
0.82	2.9040	2.8907	2.8776	2.8647	2.8519	2.8392	2.8264	2.8137	2.8007	2.7877
0.84	2.9201	2.9074	2.8949	2.8826	2.8704	2.8583	2.8462	2.8341	2.8219	2.8095
0.86	2.9372	2.9250	2.9131	2.9014	2.8899	2.8784	2.8670	2.8555	2.8440	2.8323
0.88	2.9554	2.9439	2.9326	2.9216	2.9107	2.8998	2.8891	2.8783	2.8674	2.8565
0.90	2.9755	2.9646	2.9539	2.9435	2.9333	2.9231	2.9130	2.9029	2.8928	2.8826
0.92	2.9980	2.9878	2.9779	2.9682	2.9586	2.9492	2.9398	2.9304	2.9210	2.9116
0.94	3.0244	3.0150	3.0059	2.9969	2.9881	2.9795	2.9709	2.9623	2.9537	2.9451
0.96	3.0579	3.0493	3.0411	3.0330	3.0251	3.0173	3.0096	3.0020	2.9944	2.9867
0.98	3.1072	3.0999	3.0928	3.0859	3.0792	3.0726	3.0661	3.0596	3.0532	3.0468

(continued)

TABLE C.3b
(continued)

					a_L					
a_U	*0.42*	*0.44*	*0.46*	*0.48*	*0.50*	*0.52*	*0.54*	*0.56*	*0.58*	*0.60*
0.44	2.3260									
0.46	2.3561	2.3257								
0.48	2.3851	2.3556	2.3255							
0.50	2.4129	2.3845	2.3554	2.3254						
0.52	2.4396	2.4123	2.3842	2.3553	2.3254					
0.54	2.4654	2.4392	2.4121	2.3842	2.3554	2.3255				
0.56	2.4903	2.4651	2.4392	2.4123	2.3845	2.3556	2.3257			
0.58	2.5144	2.4903	2.4654	2.4396	2.4129	2.3851	2.3561	2.3260		
0.60	2.5378	2.5147	2.4908	2.4661	2.4405	2.4138	2.3859	2.3568	2.3264	
0.62	2.5606	2.5385	2.5156	2.4920	2.4674	2.4418	2.4150	2.3871	2.3577	2.3269
0.64	2.5828	2.5617	2.5398	2.5172	2.4937	2.4692	2.4436	2.4167	2.3885	2.3589
0.66	2.6046	2.5844	2.5636	2.5419	2.5195	2.4961	2.4716	2.4459	2.4189	2.3904
0.68	2.6260	2.6068	2.5868	2.5662	2.5448	2.5225	2.4991	2.4746	2.4488	2.4215
0.70	2.6472	2.6288	2.6098	2.5902	2.5697	2.5485	2.5262	2.5029	2.4783	2.4523
0.72	2.6682	2.6506	2.6325	2.6138	2.5944	2.5742	2.5530	2.5308	2.5074	2.4827
0.74	2.6891	2.6724	2.6552	2.6374	2.6189	2.5997	2.5796	2.5586	2.5364	2.5129
0.76	2.7100	2.6941	2.6777	2.6609	2.6433	2.6251	2.6061	2.5862	2.5652	2.5430
0.78	2.7311	2.7160	2.7005	2.6844	2.6679	2.6506	2.6327	2.6138	2.5940	2.5731
0.80	2.7525	2.7382	2.7234	2.7083	2.6926	2.6764	2.6594	2.6417	2.6230	2.6033
0.82	2.7744	2.7608	2.7469	2.7326	2.7178	2.7025	2.6865	2.6699	2.6524	2.6339
0.84	2.7969	2.7841	2.7710	2.7575	2.7436	2.7293	2.7143	2.6987	2.6824	2.6651
0.86	2.8205	2.8084	2.7961	2.7835	2.7705	2.7570	2.7431	2.7285	2.7133	2.6973
0.88	2.8454	2.8341	2.8226	2.8108	2.7987	2.7862	2.7732	2.7597	2.7456	2.7309
0.90	2.8723	2.8618	2.8511	2.8402	2.8290	2.8174	2.8055	2.7931	2.7801	2.7666
0.92	2.9021	2.8924	2.8826	2.8726	2.8623	2.8517	2.8408	2.8295	2.8178	2.8055
0.94	2.9365	2.9277	2.9188	2.9097	2.9005	2.8910	2.8812	2.8710	2.8605	2.8495
0.96	2.9790	2.9713	2.9634	2.9554	2.9473	2.9390	2.9304	2.9216	2.9124	2.9029
0.98	3.0403	3.0338	3.0273	3.0207	3.0140	3.0071	3.0001	2.9928	2.9854	2.9777

TABLE C.3c

Confidence Coefficients $c_{01}(a_L, a_U)$ *for 99% EP Confidence Bands*

	a_L									
a_U	*0.02*	*0.04*	*0.06*	*0.08*	*0.10*	*0.12*	*0.14*	*0.16*	*0.18*	*0.20*
0.10	3.3261	3.1910	3.0740	2.9586						
0.12	3.3563	3.2358	3.1350	3.0386	2.9408					
0.14	3.3802	3.2701	3.1805	3.0968	3.0137	2.9283				
0.16	3.3999	3.2978	3.2163	3.1418	3.0690	2.9953	2.9189			
0.18	3.4167	3.3210	3.2458	3.1780	3.1128	3.0478	2.9811	2.9117		
0.20	3.4314	3.3408	3.2706	3.2082	3.1489	3.0904	3.0311	2.9700	2.9060	
0.22	3.4443	3.3581	3.2921	3.2339	3.1793	3.1260	3.0725	3.0177	2.9610	2.9014
0.24	3.4559	3.3735	3.3109	3.2564	3.2056	3.1564	3.1075	3.0579	3.0068	2.9536
0.26	3.4665	3.3873	3.3278	3.2763	3.2287	3.1829	3.1378	3.0923	3.0458	2.9977
0.28	3.4763	3.3999	3.3430	3.2941	3.2492	3.2064	3.1643	3.1223	3.0796	3.0357
0.30	3.4853	3.4115	3.3569	3.3103	3.2678	3.2274	3.1880	3.1489	3.1094	3.0690
0.32	3.4937	3.4223	3.3698	3.3252	3.2847	3.2464	3.2094	3.1727	3.1359	3.0985
0.34	3.5017	3.4324	3.3817	3.3389	3.3002	3.2639	3.2288	3.1943	3.1598	3.1249
0.36	3.5092	3.4418	3.3929	3.3517	3.3146	3.2800	3.2467	3.2141	3.1816	3.1489
0.38	3.5163	3.4508	3.4034	3.3637	3.3281	3.2950	3.2633	3.2323	3.2016	3.1708
0.40	3.5231	3.4593	3.4133	3.3750	3.3408	3.3090	3.2787	3.2492	3.2201	3.1910
0.42	3.5297	3.4675	3.4228	3.3857	3.3527	3.3222	3.2932	3.2651	3.2374	3.2098
0.44	3.5360	3.4753	3.4319	3.3960	3.3641	3.3347	3.3069	3.2800	3.2536	3.2274
0.46	3.5422	3.4828	3.4406	3.4058	3.3750	3.3467	3.3199	3.2941	3.2689	3.2439
0.48	3.5481	3.4901	3.4490	3.4152	3.3854	3.3581	3.3323	3.3076	3.2834	3.2596
0.50	3.5540	3.4973	3.4572	3.4243	3.3955	3.3691	3.3442	3.3204	3.2973	3.2744
0.52	3.5597	3.5042	3.4651	3.4332	3.4052	3.3797	3.3557	3.3328	3.3105	3.2887
0.54	3.5653	3.5110	3.4729	3.4418	3.4147	3.3899	3.3668	3.3447	3.3233	3.3023
0.56	3.5709	3.5177	3.4805	3.4503	3.4239	3.3999	3.3776	3.3563	3.3357	3.3155
0.58	3.5763	3.5243	3.4880	3.4586	3.4329	3.4097	3.3881	3.3675	3.3477	3.3283
0.60	3.5818	3.5308	3.4954	3.4667	3.4418	3.4193	3.3984	3.3785	3.3594	3.3408
0.62	3.5873	3.5373	3.5027	3.4748	3.4506	3.4288	3.4085	3.3893	3.3709	3.3529
0.64	3.5927	3.5438	3.5100	3.4828	3.4593	3.4381	3.4185	3.3999	3.3821	3.3649
0.66	3.5982	3.5503	3.5173	3.4908	3.4680	3.4474	3.4284	3.4105	3.3933	3.3767
0.68	3.6037	3.5568	3.5247	3.4988	3.4766	3.4567	3.4382	3.4209	3.4043	3.3883
0.70	3.6093	3.5634	3.5320	3.5069	3.4853	3.4659	3.4481	3.4314	3.4154	3.3999
0.72	3.6150	3.5701	3.5395	3.5150	3.4940	3.4753	3.4580	3.4418	3.4264	3.4115
0.74	3.6208	3.5770	3.5471	3.5233	3.5029	3.4847	3.4680	3.4524	3.4375	3.4232
0.76	3.6269	3.5840	3.5549	3.5317	3.5120	3.4944	3.4782	3.4631	3.4488	3.4350
0.78	3.6331	3.5912	3.5629	3.5404	3.5212	3.5042	3.4886	3.4741	3.4602	3.4470
0.80	3.6396	3.5987	3.5712	3.5494	3.5308	3.5144	3.4993	3.4853	3.4720	3.4593
0.82	3.6464	3.6066	3.5799	3.5588	3.5408	3.5249	3.5104	3.4970	3.4842	3.4720
0.84	3.6537	3.6150	3.5891	3.5686	3.5513	3.5360	3.5221	3.5092	3.4970	3.4853
0.86	3.6615	3.6240	3.5989	3.5792	3.5626	3.5478	3.5345	3.5221	3.5104	3.4993
0.88	3.6701	3.6338	3.6096	3.5907	3.5747	3.5606	3.5478	3.5360	3.5249	3.5144
0.90	3.6796	3.6447	3.6215	3.6033	3.5881	3.5747	3.5626	3.5513	3.5408	3.5308
0.92	3.6907	3.6572	3.6350	3.6178	3.6033	3.5907	3.5792	3.5686	3.5588	3.5494
0.94	3.7040	3.6722	3.6513	3.6350	3.6215	3.6096	3.5989	3.5891	3.5799	3.5712
0.96	3.7214	3.6917	3.6722	3.6572	3.6447	3.6338	3.6240	3.6150	3.6066	3.5987
0.98	3.7481	3.7214	3.7040	3.6907	3.6796	3.6701	3.6615	3.6537	3.6464	3.6396

(continued)

TABLE C.3c
(continued)

	a_{L}									
a_{U}	0.22	0.24	0.26	0.28	0.30	0.32	0.34	0.36	0.38	0.40
0.24	2.8977									
0.26	2.9475	2.8946								
0.28	2.9901	2.9424	2.8920							
0.30	3.0273	2.9838	2.9381	2.8899						
0.32	3.0600	3.0201	2.9784	2.9345	2.8880					
0.34	3.0892	3.0524	3.0140	2.9738	2.9314	2.8865				
0.36	3.1156	3.0814	3.0459	3.0089	2.9700	2.9289	2.8852			
0.38	3.1396	3.1076	3.0747	3.0404	3.0045	2.9667	2.9267	2.8841		
0.40	3.1616	3.1317	3.1009	3.0690	3.0357	3.0008	2.9640	2.9249	2.8833	
0.42	3.1821	3.1539	3.1250	3.0952	3.0642	3.0319	2.9978	2.9618	2.9235	2.8826
0.44	3.2011	3.1745	3.1473	3.1193	3.0904	3.0602	3.0286	2.9953	2.9600	2.9223
0.46	3.2189	3.1937	3.1681	3.1418	3.1146	3.0864	3.0570	3.0260	2.9933	2.9586
0.48	3.2358	3.2118	3.1875	3.1627	3.1372	3.1108	3.0832	3.0544	3.0240	2.9918
0.50	3.2517	3.2290	3.2059	3.1824	3.1583	3.1335	3.1077	3.0807	3.0524	3.0225
0.52	3.2670	3.2453	3.2234	3.2011	3.1783	3.1549	3.1306	3.1053	3.0789	3.0510
0.54	3.2816	3.2608	3.2400	3.2188	3.1972	3.1751	3.1522	3.1285	3.1037	3.0777
0.56	3.2956	3.2758	3.2559	3.2358	3.2153	3.1943	3.1727	3.1504	3.1271	3.1027
0.58	3.3092	3.2903	3.2712	3.2521	3.2326	3.2127	3.1923	3.1712	3.1493	3.1264
0.60	3.3224	3.3043	3.2861	3.2678	3.2492	3.2303	3.2110	3.1910	3.1704	3.1489
0.62	3.3353	3.3179	3.3005	3.2830	3.2653	3.2474	3.2290	3.2101	3.1906	3.1704
0.64	3.3480	3.3312	3.3146	3.2978	3.2810	3.2639	3.2464	3.2286	3.2101	3.1910
0.66	3.3604	3.3443	3.3284	3.3124	3.2963	3.2800	3.2634	3.2464	3.2290	3.2110
0.68	3.3727	3.3573	3.3419	3.3267	3.3113	3.2958	3.2800	3.2639	3.2474	3.2303
0.70	3.3849	3.3701	3.3554	3.3408	3.3261	3.3113	3.2963	3.2810	3.2653	3.2492
0.72	3.3971	3.3828	3.3688	3.3548	3.3408	3.3267	3.3124	3.2978	3.2830	3.2678
0.74	3.4093	3.3956	3.3822	3.3688	3.3554	3.3419	3.3284	3.3146	3.3005	3.2861
0.76	3.4216	3.4085	3.3956	3.3828	3.3701	3.3573	3.3443	3.3312	3.3179	3.3043
0.78	3.4342	3.4216	3.4093	3.3971	3.3849	3.3727	3.3604	3.3480	3.3353	3.3224
0.80	3.4470	3.4350	3.4232	3.4115	3.3999	3.3883	3.3767	3.3649	3.3529	3.3408
0.82	3.4602	3.4488	3.4375	3.4264	3.4154	3.4043	3.3933	3.3821	3.3709	3.3594
0.84	3.4741	3.4631	3.4524	3.4418	3.4314	3.4209	3.4105	3.3999	3.3893	3.3785
0.86	3.4886	3.4782	3.4680	3.4580	3.4481	3.4382	3.4284	3.4185	3.4085	3.3984
0.88	3.5042	3.4944	3.4847	3.4753	3.4659	3.4567	3.4474	3.4381	3.4288	3.4193
0.90	3.5212	3.5120	3.5029	3.4940	3.4853	3.4766	3.4680	3.4593	3.4506	3.4418
0.92	3.5404	3.5317	3.5233	3.5150	3.5069	3.4988	3.4908	3.4828	3.4748	3.4667
0.94	3.5629	3.5549	3.5471	3.5395	3.5320	3.5247	3.5173	3.5100	3.5027	3.4954
0.96	3.5912	3.5840	3.5770	3.5701	3.5634	3.5568	3.5503	3.5438	3.5373	3.5308
0.98	3.6331	3.6269	3.6208	3.6150	3.6093	3.6037	3.5982	3.5927	3.5873	3.5818

TABLE C.3c
(continued)

	a_L									
a_U	0.42	0.44	0.46	0.48	0.50	0.52	0.54	0.56	0.58	0.60
0.44	2.8820									
0.46	2.9214	2.8816								
0.48	2.9575	2.9208	2.8813							
0.50	2.9908	2.9569	2.9205	2.8812						
0.52	3.0215	2.9901	2.9565	2.9203	2.8812					
0.54	3.0502	3.0211	2.9899	2.9565	2.9205	2.8813				
0.56	3.0771	3.0499	3.0211	2.9901	2.9569	2.9208	2.8816			
0.58	3.1024	3.0771	3.0502	3.0215	2.9908	2.9575	2.9214	2.8820		
0.60	3.1264	3.1027	3.0777	3.0510	3.0225	2.9918	2.9586	2.9223	2.8826	
0.62	3.1493	3.1271	3.1037	3.0789	3.0524	3.0240	2.9933	2.9600	2.9235	2.8833
0.64	3.1712	3.1504	3.1285	3.1053	3.0807	3.0544	3.0260	2.9953	2.9618	2.9249
0.66	3.1923	3.1727	3.1522	3.1306	3.1077	3.0832	3.0570	3.0286	2.9978	2.9640
0.68	3.2127	3.1943	3.1751	3.1549	3.1335	3.1108	3.0864	3.0602	3.0319	3.0008
0.70	3.2326	3.2153	3.1972	3.1783	3.1583	3.1372	3.1146	3.0904	3.0642	3.0357
0.72	3.2521	3.2358	3.2188	3.2011	3.1824	3.1627	3.1418	3.1193	3.0952	3.0690
0.74	3.2712	3.2559	3.2400	3.2234	3.2059	3.1875	3.1681	3.1473	3.1250	3.1009
0.76	3.2903	3.2758	3.2608	3.2453	3.2290	3.2118	3.1937	3.1745	3.1539	3.1317
0.78	3.3092	3.2956	3.2816	3.2670	3.2517	3.2358	3.2189	3.2011	3.1821	3.1616
0.80	3.3283	3.3155	3.3023	3.2887	3.2744	3.2596	3.2439	3.2274	3.2098	3.1910
0.82	3.3477	3.3357	3.3233	3.3105	3.2973	3.2834	3.2689	3.2536	3.2374	3.2201
0.84	3.3675	3.3563	3.3447	3.3328	3.3204	3.3076	3.2941	3.2800	3.2651	3.2492
0.86	3.3881	3.3776	3.3668	3.3557	3.3442	3.3323	3.3199	3.3069	3.2932	3.2787
0.88	3.4097	3.3999	3.3899	3.3797	3.3691	3.3581	3.3467	3.3347	3.3222	3.3090
0.90	3.4329	3.4239	3.4147	3.4052	3.3955	3.3854	3.3750	3.3641	3.3527	3.3408
0.92	3.4586	3.4503	3.4418	3.4332	3.4243	3.4152	3.4058	3.3960	3.3857	3.3750
0.94	3.4880	3.4805	3.4729	3.4651	3.4572	3.4490	3.4406	3.4319	3.4228	3.4133
0.96	3.5243	3.5177	3.5110	3.5042	3.4973	3.4901	3.4828	3.4753	3.4675	3.4593
0.98	3.5763	3.5709	3.5653	3.5597	3.5540	3.5481	3.5422	3.5360	3.5297	3.5231

TABLE C.4a

Confidence Coefficients $k_{10}(a_L, a_U)$ for 90% Hall–Wellner Confidence Bands

	a_L										
a_U	0.00	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
0.10	0.5985	0.5985	0.5979	0.5930	0.5768						
0.12	0.6509	0.6509	0.6507	0.6484	0.6405	0.6210					
0.14	0.6979	0.6979	0.6978	0.6966	0.6923	0.6819	0.6598				
0.16	0.7406	0.7406	0.7405	0.7399	0.7373	0.7310	0.7184	0.6942			
0.18	0.7796	0.7796	0.7796	0.7792	0.7776	0.7735	0.7653	0.7509	0.7249		
0.20	0.8155	0.8155	0.8155	0.8153	0.8142	0.8114	0.8058	0.7961	0.7801	0.7525	
0.22	0.8487	0.8487	0.8487	0.8485	0.8479	0.8459	0.8419	0.8349	0.8237	0.8063	0.7773
0.24	0.8795	0.8794	0.8794	0.8794	0.8789	0.8775	0.8746	0.8693	0.8611	0.8486	0.8299
0.26	0.9081	0.9081	0.9081	0.9080	0.9077	0.9067	0.9045	0.9005	0.8941	0.8847	0.8711
0.28	0.9348	0.9348	0.9348	0.9347	0.9345	0.9338	0.9320	0.9289	0.9239	0.9165	0.9060
0.30	0.9597	0.9597	0.9597	0.9597	0.9595	0.9589	0.9576	0.9551	0.9510	0.9450	0.9366
0.32	0.9829	0.9829	0.9829	0.9829	0.9828	0.9824	0.9813	0.9793	0.9760	0.9710	0.9641
0.34	1.0047	1.0047	1.0047	1.0046	1.0046	1.0042	1.0034	1.0017	0.9990	0.9948	0.9890
0.36	1.0250	1.0250	1.0250	1.0250	1.0249	1.0246	1.0239	1.0226	1.0202	1.0167	1.0118
0.38	1.0439	1.0439	1.0439	1.0439	1.0439	1.0437	1.0431	1.0419	1.0400	1.0370	1.0327
0.40	1.0616	1.0616	1.0616	1.0616	1.0616	1.0614	1.0610	1.0600	1.0583	1.0557	1.0520
0.42	1.0782	1.0782	1.0782	1.0782	1.0781	1.0780	1.0776	1.0768	1.0753	1.0730	1.0697
0.44	1.0935	1.0935	1.0935	1.0935	1.0935	1.0934	1.0931	1.0923	1.0911	1.0890	1.0862
0.46	1.1078	1.1078	1.1078	1.1078	1.1078	1.1077	1.1074	1.1068	1.1057	1.1039	1.1013
0.48	1.1211	1.1211	1.1211	1.1211	1.1211	1.1210	1.1208	1.1202	1.1192	1.1176	1.1153
0.50	1.1334	1.1334	1.1334	1.1334	1.1334	1.1333	1.1331	1.1326	1.1317	1.1303	1.1281
0.52	1.1447	1.1447	1.1447	1.1447	1.1447	1.1447	1.1445	1.1440	1.1432	1.1419	1.1400
0.54	1.1552	1.1552	1.1552	1.1552	1.1551	1.1551	1.1549	1.1545	1.1538	1.1526	1.1508
0.56	1.1647	1.1647	1.1647	1.1647	1.1647	1.1646	1.1645	1.1641	1.1635	1.1623	1.1607
0.58	1.1734	1.1734	1.1734	1.1734	1.1734	1.1733	1.1732	1.1729	1.1723	1.1712	1.1697
0.60	1.1813	1.1813	1.1813	1.1813	1.1813	1.1812	1.1811	1.1808	1.1802	1.1793	1.1778
0.62	1.1884	1.1884	1.1884	1.1884	1.1883	1.1883	1.1882	1.1879	1.1874	1.1865	1.1851
0.64	1.1947	1.1947	1.1947	1.1947	1.1947	1.1946	1.1945	1.1943	1.1938	1.1929	1.1916
0.66	1.2003	1.2003	1.2003	1.2003	1.2003	1.2002	1.2001	1.1999	1.1994	1.1986	1.1974
0.68	1.2052	1.2052	1.2052	1.2052	1.2052	1.2051	1.2050	1.2048	1.2043	1.2036	1.2024
0.70	1.2094	1.2094	1.2094	1.2094	1.2094	1.2093	1.2093	1.2090	1.2086	1.2078	1.2067
0.72	1.2130	1.2129	1.2129	1.2129	1.2129	1.2129	1.2128	1.2126	1.2122	1.2115	1.2104
0.74	1.2159	1.2159	1.2159	1.2159	1.2159	1.2159	1.2158	1.2156	1.2152	1.2145	1.2134
0.76	1.2183	1.2183	1.2183	1.2183	1.2183	1.2183	1.2182	1.2180	1.2176	1.2169	1.2159
0.78	1.2202	1.2202	1.2202	1.2202	1.2202	1.2201	1.2201	1.2199	1.2195	1.2188	1.2178
0.80	1.2216	1.2216	1.2216	1.2216	1.2216	1.2215	1.2215	1.2213	1.2209	1.2202	1.2192
0.82	1.2226	1.2226	1.2226	1.2226	1.2226	1.2225	1.2225	1.2223	1.2219	1.2212	1.2202
0.84	1.2232	1.2232	1.2232	1.2232	1.2232	1.2232	1.2231	1.2229	1.2225	1.2219	1.2209
0.86	1.2236	1.2236	1.2236	1.2236	1.2236	1.2236	1.2235	1.2233	1.2229	1.2223	1.2213
0.88	1.2238	1.2238	1.2238	1.2238	1.2238	1.2237	1.2237	1.2235	1.2231	1.2225	1.2215
0.90	1.2239	1.2238	1.2238	1.2238	1.2238	1.2238	1.2237	1.2236	1.2232	1.2225	1.2215
0.92	1.2239	1.2238	1.2238	1.2238	1.2238	1.2238	1.2238	1.2236	1.2232	1.2226	1.2216
0.94	1.2239	1.2238	1.2238	1.2238	1.2238	1.2238	1.2238	1.2236	1.2232	1.2226	1.2216
0.96	1.2239	1.2238	1.2238	1.2238	1.2238	1.2238	1.2238	1.2236	1.2232	1.2226	1.2216
0.98	1.2239	1.2238	1.2238	1.2238	1.2238	1.2238	1.2238	1.2236	1.2232	1.2226	1.2216
1.00	1.2239	1.2239	1.2239	1.2239	1.2239	1.2239	1.2238	1.2236	1.2232	1.2226	1.2216

TABLE C.4a
(continued)

	a_{L}									
a_{U}	*0.22*	*0.24*	*0.26*	*0.28*	*0.30*	*0.32*	*0.34*	*0.36*	*0.38*	*0.40*
0.24	0.7996									
0.26	0.8512	0.8198								
0.28	0.8913	0.8703	0.8379							
0.30	0.9251	0.9094	0.8875	0.8540						
0.32	0.9548	0.9423	0.9257	0.9028	0.8685					
0.34	0.9813	0.9710	0.9577	0.9402	0.9165	0.8812				
0.36	1.0052	0.9966	0.9855	0.9713	0.9530	0.9284	0.8924			
0.38	1.0270	1.0196	1.0102	0.9983	0.9833	0.9642	0.9389	0.9020		
0.40	1.0470	1.0406	1.0324	1.0222	1.0095	0.9938	0.9739	0.9478	0.9102	
0.42	1.0654	1.0597	1.0525	1.0436	1.0326	1.0192	1.0027	0.9821	0.9552	0.9169
0.44	1.0822	1.0772	1.0708	1.0629	1.0532	1.0415	1.0274	1.0102	0.9888	0.9613
0.46	1.0978	1.0932	1.0875	1.0804	1.0717	1.0614	1.0490	1.0342	1.0163	0.9942
0.48	1.1121	1.1079	1.1027	1.0963	1.0885	1.0792	1.0682	1.0551	1.0396	1.0210
0.50	1.1252	1.1214	1.1167	1.1108	1.1037	1.0952	1.0853	1.0735	1.0598	1.0436
0.52	1.1373	1.1338	1.1294	1.1240	1.1175	1.1097	1.1006	1.0899	1.0775	1.0631
0.54	1.1483	1.1451	1.1410	1.1360	1.1299	1.1228	1.1143	1.1046	1.0933	1.0802
0.56	1.1584	1.1554	1.1516	1.1469	1.1412	1.1345	1.1267	1.1177	1.1072	1.0952
0.58	1.1675	1.1647	1.1611	1.1567	1.1514	1.1451	1.1378	1.1293	1.1196	1.1085
0.60	1.1758	1.1731	1.1697	1.1656	1.1606	1.1546	1.1477	1.1398	1.1307	1.1203
0.62	1.1832	1.1807	1.1775	1.1735	1.1688	1.1631	1.1566	1.1490	1.1404	1.1307
0.64	1.1898	1.1874	1.1843	1.1806	1.1760	1.1707	1.1644	1.1572	1.1490	1.1398
0.66	1.1956	1.1933	1.1904	1.1868	1.1824	1.1773	1.1713	1.1644	1.1566	1.1477
0.68	1.2007	1.1985	1.1957	1.1922	1.1880	1.1830	1.1773	1.1707	1.1631	1.1546
0.70	1.2051	1.2030	1.2002	1.1969	1.1928	1.1880	1.1824	1.1760	1.1688	1.1606
0.72	1.2088	1.2067	1.2041	1.2008	1.1969	1.1922	1.1868	1.1806	1.1735	1.1656
0.74	1.2119	1.2099	1.2073	1.2041	1.2002	1.1957	1.1904	1.1843	1.1775	1.1697
0.76	1.2144	1.2124	1.2099	1.2067	1.2030	1.1985	1.1933	1.1874	1.1807	1.1731
0.78	1.2163	1.2144	1.2119	1.2088	1.2051	1.2007	1.1956	1.1898	1.1832	1.1758
0.80	1.2178	1.2159	1.2134	1.2104	1.2067	1.2024	1.1974	1.1916	1.1851	1.1778
0.82	1.2188	1.2169	1.2145	1.2115	1.2078	1.2036	1.1986	1.1929	1.1865	1.1793
0.84	1.2195	1.2176	1.2152	1.2122	1.2086	1.2043	1.1994	1.1938	1.1874	1.1802
0.86	1.2199	1.2180	1.2156	1.2126	1.2090	1.2048	1.1999	1.1943	1.1879	1.1808
0.88	1.2201	1.2182	1.2158	1.2128	1.2093	1.2050	1.2001	1.1945	1.1882	1.1811
0.90	1.2201	1.2183	1.2159	1.2129	1.2093	1.2051	1.2002	1.1946	1.1883	1.1812
0.92	1.2202	1.2183	1.2159	1.2129	1.2094	1.2052	1.2003	1.1947	1.1883	1.1813
0.94	1.2202	1.2183	1.2159	1.2129	1.2094	1.2052	1.2003	1.1947	1.1884	1.1813
0.96	1.2202	1.2183	1.2159	1.2129	1.2094	1.2052	1.2003	1.1947	1.1884	1.1813
0.98	1.2202	1.2183	1.2159	1.2129	1.2094	1.2052	1.2003	1.1947	1.1884	1.1813
1.00	1.2202	1.2183	1.2159	1.2130	1.2094	1.2052	1.2003	1.1947	1.1884	1.1813

(continued)

TABLE C.4a
(continued)

	a_L									
a_U	0.42	0.44	0.46	0.48	0.50	0.52	0.54	0.56	0.58	0.60
0.44	0.9223									
0.46	0.9660	0.9263								
0.48	0.9982	0.9693	0.9289							
0.50	1.0243	1.0009	0.9713	0.9302						
0.52	1.0462	1.0263	1.0022	0.9720	0.9302					
0.54	1.0651	1.0476	1.0270	1.0022	0.9713	0.9289				
0.56	1.0815	1.0658	1.0476	1.0263	1.0009	0.9693	0.9263			
0.58	1.0959	1.0815	1.0651	1.0462	1.0243	0.9982	0.9660	0.9223		
0.60	1.1085	1.0952	1.0802	1.0631	1.0436	1.0210	0.9942	0.9613	0.9169	
0.62	1.1196	1.1072	1.0933	1.0775	1.0598	1.0396	1.0163	0.9888	0.9552	0.9102
0.64	1.1293	1.1177	1.1046	1.0899	1.0735	1.0551	1.0342	1.0102	0.9821	0.9478
0.66	1.1378	1.1267	1.1143	1.1006	1.0853	1.0682	1.0490	1.0274	1.0027	0.9739
0.68	1.1451	1.1345	1.1228	1.1097	1.0952	1.0792	1.0614	1.0415	1.0192	0.9938
0.70	1.1514	1.1412	1.1299	1.1175	1.1037	1.0885	1.0717	1.0532	1.0326	1.0095
0.72	1.1567	1.1469	1.1360	1.1240	1.1108	1.0963	1.0804	1.0629	1.0436	1.0222
0.74	1.1611	1.1516	1.1410	1.1294	1.1167	1.1027	1.0875	1.0708	1.0525	1.0324
0.76	1.1647	1.1554	1.1451	1.1338	1.1214	1.1079	1.0932	1.0772	1.0597	1.0406
0.78	1.1675	1.1584	1.1483	1.1373	1.1252	1.1121	1.0978	1.0822	1.0654	1.0470
0.80	1.1697	1.1607	1.1508	1.1400	1.1281	1.1153	1.1013	1.0862	1.0697	1.0520
0.82	1.1712	1.1623	1.1526	1.1419	1.1303	1.1176	1.1039	1.0890	1.0730	1.0557
0.84	1.1723	1.1635	1.1538	1.1432	1.1317	1.1192	1.1057	1.0911	1.0753	1.0583
0.86	1.1729	1.1641	1.1545	1.1440	1.1326	1.1202	1.1068	1.0923	1.0768	1.0600
0.88	1.1732	1.1645	1.1549	1.1445	1.1331	1.1208	1.1074	1.0931	1.0776	1.0610
0.90	1.1733	1.1646	1.1551	1.1447	1.1333	1.1210	1.1077	1.0934	1.0780	1.0614
0.92	1.1734	1.1647	1.1551	1.1447	1.1334	1.1211	1.1078	1.0935	1.0781	1.0616
0.94	1.1734	1.1647	1.1552	1.1447	1.1334	1.1211	1.1078	1.0935	1.0782	1.0616
0.96	1.1734	1.1647	1.1552	1.1447	1.1334	1.1211	1.1078	1.0935	1.0782	1.0616
0.98	1.1734	1.1647	1.1552	1.1447	1.1334	1.1211	1.1078	1.0935	1.0782	1.0616
1.00	1.1734	1.1647	1.1552	1.1447	1.1334	1.1211	1.1078	1.0935	1.0782	1.0616

TABLE C.4b

Confidence Coefficients $k_{05}(a_L, a_U)$ for 95% Hall–Wellner Confidence Bands

	a_L										
a_U	0.00	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
0.10	0.6825	0.6825	0.6822	0.6793	0.6666						
0.12	0.7418	0.7418	0.7417	0.7405	0.7351	0.7191					
0.14	0.7948	0.7948	0.7948	0.7943	0.7916	0.7838	0.7651				
0.16	0.8428	0.8428	0.8428	0.8426	0.8412	0.8369	0.8270	0.8060			
0.18	0.8866	0.8866	0.8866	0.8865	0.8857	0.8832	0.8772	0.8655	0.8425		
0.20	0.9269	0.9268	0.9268	0.9268	0.9263	0.9247	0.9209	0.9134	0.9001	0.8753	
0.22	0.9639	0.9639	0.9639	0.9639	0.9636	0.9626	0.9600	0.9549	0.9460	0.9311	0.9049
0.24	0.9982	0.9982	0.9982	0.9982	0.9980	0.9973	0.9955	0.9919	0.9856	0.9754	0.9592
0.26	1.0299	1.0299	1.0299	1.0299	1.0298	1.0294	1.0281	1.0254	1.0208	1.0134	1.0019
0.28	1.0594	1.0594	1.0594	1.0594	1.0593	1.0590	1.0581	1.0561	1.0526	1.0470	1.0384
0.30	1.0868	1.0868	1.0868	1.0868	1.0868	1.0865	1.0859	1.0843	1.0816	1.0772	1.0706
0.32	1.1123	1.1123	1.1123	1.1123	1.1123	1.1121	1.1116	1.1104	1.1083	1.1047	1.0994
0.34	1.1360	1.1360	1.1360	1.1360	1.1360	1.1359	1.1355	1.1346	1.1328	1.1299	1.1256
0.36	1.1581	1.1581	1.1581	1.1581	1.1580	1.1580	1.1576	1.1569	1.1555	1.1531	1.1495
0.38	1.1785	1.1785	1.1785	1.1785	1.1785	1.1785	1.1782	1.1776	1.1764	1.1745	1.1714
0.40	1.1976	1.1976	1.1976	1.1976	1.1975	1.1975	1.1973	1.1968	1.1958	1.1941	1.1915
0.42	1.2152	1.2152	1.2152	1.2152	1.2152	1.2151	1.2150	1.2146	1.2137	1.2123	1.2100
0.44	1.2315	1.2315	1.2315	1.2315	1.2315	1.2314	1.2313	1.2310	1.2302	1.2290	1.2270
0.46	1.2465	1.2465	1.2465	1.2465	1.2465	1.2465	1.2464	1.2461	1.2455	1.2444	1.2426
0.48	1.2604	1.2604	1.2604	1.2604	1.2604	1.2603	1.2603	1.2600	1.2595	1.2585	1.2570
0.50	1.2731	1.2731	1.2731	1.2731	1.2731	1.2731	1.2730	1.2728	1.2723	1.2714	1.2700
0.52	1.2847	1.2847	1.2847	1.2847	1.2847	1.2847	1.2846	1.2844	1.2840	1.2832	1.2820
0.54	1.2953	1.2952	1.2952	1.2952	1.2952	1.2952	1.2952	1.2950	1.2946	1.2939	1.2928
0.56	1.3048	1.3048	1.3048	1.3048	1.3048	1.3048	1.3047	1.3046	1.3042	1.3036	1.3025
0.58	1.3134	1.3134	1.3134	1.3134	1.3134	1.3134	1.3133	1.3132	1.3129	1.3123	1.3113
0.60	1.3211	1.3211	1.3211	1.3211	1.3211	1.3211	1.3210	1.3209	1.3206	1.3201	1.3191
0.62	1.3279	1.3279	1.3279	1.3279	1.3279	1.3279	1.3278	1.3277	1.3274	1.3269	1.3261
0.64	1.3338	1.3338	1.3338	1.3338	1.3338	1.3338	1.3338	1.3337	1.3334	1.3329	1.3321
0.66	1.3390	1.3390	1.3390	1.3390	1.3390	1.3390	1.3389	1.3388	1.3386	1.3381	1.3374
0.68	1.3434	1.3434	1.3434	1.3434	1.3434	1.3434	1.3433	1.3432	1.3430	1.3426	1.3418
0.70	1.3471	1.3471	1.3471	1.3471	1.3471	1.3471	1.3470	1.3469	1.3467	1.3463	1.3456
0.72	1.3501	1.3501	1.3501	1.3501	1.3501	1.3501	1.3501	1.3500	1.3498	1.3494	1.3487
0.74	1.3525	1.3525	1.3525	1.3525	1.3525	1.3525	1.3525	1.3524	1.3522	1.3518	1.3511
0.76	1.3544	1.3544	1.3544	1.3544	1.3544	1.3544	1.3544	1.3543	1.3541	1.3537	1.3530
0.78	1.3558	1.3558	1.3558	1.3558	1.3558	1.3558	1.3558	1.3557	1.3555	1.3551	1.3544
0.80	1.3568	1.3568	1.3568	1.3568	1.3568	1.3568	1.3567	1.3567	1.3565	1.3561	1.3554
0.82	1.3574	1.3574	1.3574	1.3574	1.3574	1.3574	1.3574	1.3573	1.3571	1.3567	1.3561
0.84	1.3578	1.3578	1.3578	1.3578	1.3578	1.3578	1.3578	1.3577	1.3575	1.3571	1.3565
0.86	1.3580	1.3580	1.3580	1.3580	1.3580	1.3580	1.3580	1.3579	1.3577	1.3573	1.3567
0.88	1.3581	1.3581	1.3581	1.3581	1.3581	1.3581	1.3580	1.3580	1.3578	1.3574	1.3567
0.90	1.3581	1.3581	1.3581	1.3581	1.3581	1.3581	1.3581	1.3580	1.3578	1.3574	1.3568
0.92	1.3581	1.3581	1.3581	1.3581	1.3581	1.3581	1.3581	1.3580	1.3578	1.3574	1.3568
0.94	1.3581	1.3581	1.3581	1.3581	1.3581	1.3581	1.3581	1.3580	1.3578	1.3574	1.3568
0.96	1.3581	1.3581	1.3581	1.3581	1.3581	1.3581	1.3581	1.3580	1.3578	1.3574	1.3568
0.98	1.3581	1.3581	1.3581	1.3581	1.3581	1.3581	1.3581	1.3580	1.3578	1.3574	1.3568
1.00	1.3581	1.3581	1.3581	1.3581	1.3581	1.3581	1.3581	1.3580	1.3578	1.3574	1.3568

(continued)

TABLE C.4b
(continued)

	a_L									
a_U	0.22	0.24	0.26	0.28	0.30	0.32	0.34	0.36	0.38	0.40
0.24	0.9315									
0.26	0.9844	0.9554								
0.28	1.0258	1.0071	0.9769							
0.30	1.0610	1.0473	1.0275	0.9962						
0.32	1.0918	1.0813	1.0666	1.0457	1.0134					
0.34	1.1194	1.1109	1.0994	1.0838	1.0619	1.0286				
0.36	1.1444	1.1374	1.1280	1.1155	1.0989	1.0761	1.0419			
0.38	1.1671	1.1612	1.1533	1.1430	1.1297	1.1122	1.0885	1.0533		
0.40	1.1878	1.1827	1.1760	1.1673	1.1562	1.1420	1.1237	1.0991	1.0631	
0.42	1.2068	1.2024	1.1966	1.1891	1.1796	1.1676	1.1526	1.1334	1.1080	1.0711
0.44	1.2242	1.2203	1.2152	1.2086	1.2003	1.1900	1.1773	1.1615	1.1414	1.1152
0.46	1.2401	1.2367	1.2321	1.2263	1.2189	1.2099	1.1988	1.1853	1.1686	1.1478
0.48	1.2547	1.2516	1.2475	1.2422	1.2357	1.2276	1.2178	1.2060	1.1916	1.1742
0.50	1.2680	1.2652	1.2615	1.2567	1.2508	1.2435	1.2347	1.2241	1.2115	1.1964
0.52	1.2801	1.2775	1.2741	1.2698	1.2643	1.2577	1.2497	1.2401	1.2288	1.2154
0.54	1.2911	1.2887	1.2856	1.2815	1.2765	1.2704	1.2630	1.2543	1.2440	1.2319
0.56	1.3010	1.2988	1.2959	1.2921	1.2874	1.2817	1.2749	1.2668	1.2574	1.2463
0.58	1.3099	1.3078	1.3051	1.3016	1.2972	1.2919	1.2855	1.2779	1.2691	1.2589
0.60	1.3178	1.3158	1.3133	1.3100	1.3058	1.3008	1.2948	1.2877	1.2794	1.2699
0.62	1.3248	1.3229	1.3205	1.3174	1.3134	1.3087	1.3030	1.2963	1.2884	1.2794
0.64	1.3309	1.3291	1.3268	1.3238	1.3201	1.3155	1.3101	1.3037	1.2963	1.2877
0.66	1.3362	1.3345	1.3323	1.3294	1.3258	1.3215	1.3162	1.3101	1.3030	1.2948
0.68	1.3407	1.3391	1.3369	1.3342	1.3307	1.3265	1.3215	1.3155	1.3087	1.3008
0.70	1.3445	1.3429	1.3408	1.3382	1.3348	1.3307	1.3258	1.3201	1.3134	1.3058
0.72	1.3476	1.3461	1.3441	1.3414	1.3382	1.3342	1.3294	1.3238	1.3174	1.3100
0.74	1.3501	1.3486	1.3466	1.3441	1.3408	1.3369	1.3323	1.3268	1.3205	1.3133
0.76	1.3520	1.3506	1.3486	1.3461	1.3429	1.3391	1.3345	1.3291	1.3229	1.3158
0.78	1.3534	1.3520	1.3501	1.3476	1.3445	1.3407	1.3362	1.3309	1.3248	1.3178
0.80	1.3544	1.3530	1.3511	1.3487	1.3456	1.3418	1.3374	1.3321	1.3261	1.3191
0.82	1.3551	1.3537	1.3518	1.3494	1.3463	1.3426	1.3381	1.3329	1.3269	1.3201
0.84	1.3555	1.3541	1.3522	1.3498	1.3467	1.3430	1.3386	1.3334	1.3274	1.3206
0.86	1.3557	1.3543	1.3524	1.3500	1.3469	1.3432	1.3388	1.3337	1.3277	1.3209
0.88	1.3558	1.3544	1.3525	1.3501	1.3470	1.3433	1.3389	1.3338	1.3278	1.3210
0.90	1.3558	1.3544	1.3525	1.3501	1.3471	1.3434	1.3390	1.3338	1.3279	1.3211
0.92	1.3558	1.3544	1.3525	1.3501	1.3471	1.3434	1.3390	1.3338	1.3279	1.3211
0.94	1.3558	1.3544	1.3525	1.3501	1.3471	1.3434	1.3390	1.3338	1.3279	1.3211
0.96	1.3558	1.3544	1.3525	1.3501	1.3471	1.3434	1.3390	1.3338	1.3279	1.3211
0.98	1.3558	1.3544	1.3525	1.3501	1.3471	1.3434	1.3390	1.3338	1.3279	1.3211
1.00	1.3558	1.3544	1.3525	1.3501	1.3471	1.3434	1.3390	1.3338	1.3279	1.3211

TABLE C.4b
(continued)

	a_L									
a_U	*0.42*	*0.44*	*0.46*	*0.48*	*0.50*	*0.52*	*0.54*	*0.56*	*0.58*	*0.60*
0.44	1.0775									
0.46	1.1208	1.0822								
0.48	1.1526	1.1247	1.0854							
0.50	1.1781	1.1557	1.1271	1.0869						
0.52	1.1995	1.1805	1.1573	1.1279	1.0869					
0.54	1.2177	1.2011	1.1813	1.1573	1.1271	1.0854				
0.56	1.2335	1.2185	1.2011	1.1805	1.1557	1.1247	1.0822			
0.58	1.2471	1.2335	1.2177	1.1995	1.1781	1.1526	1.1208	1.0775		
0.60	1.2589	1.2463	1.2319	1.2154	1.1964	1.1742	1.1478	1.1152	1.0711	
0.62	1.2691	1.2574	1.2440	1.2288	1.2115	1.1916	1.1686	1.1414	1.1080	1.0631
0.64	1.2779	1.2668	1.2543	1.2401	1.2241	1.2060	1.1853	1.1615	1.1334	1.0991
0.66	1.2855	1.2749	1.2630	1.2497	1.2347	1.2178	1.1988	1.1773	1.1526	1.1237
0.68	1.2919	1.2817	1.2704	1.2577	1.2435	1.2276	1.2099	1.1900	1.1676	1.1420
0.70	1.2972	1.2874	1.2765	1.2643	1.2508	1.2357	1.2189	1.2003	1.1796	1.1562
0.72	1.3016	1.2921	1.2815	1.2698	1.2567	1.2422	1.2263	1.2086	1.1891	1.1673
0.74	1.3051	1.2959	1.2856	1.2741	1.2615	1.2475	1.2321	1.2152	1.1966	1.1760
0.76	1.3078	1.2988	1.2887	1.2775	1.2652	1.2516	1.2367	1.2203	1.2024	1.1827
0.78	1.3099	1.3010	1.2911	1.2801	1.2680	1.2547	1.2401	1.2242	1.2068	1.1878
0.80	1.3113	1.3025	1.2928	1.2820	1.2700	1.2570	1.2426	1.2270	1.2100	1.1915
0.82	1.3123	1.3036	1.2939	1.2832	1.2714	1.2585	1.2444	1.2290	1.2123	1.1941
0.84	1.3129	1.3042	1.2946	1.2840	1.2723	1.2595	1.2455	1.2302	1.2137	1.1958
0.86	1.3132	1.3046	1.2950	1.2844	1.2728	1.2600	1.2461	1.2310	1.2146	1.1968
0.88	1.3133	1.3047	1.2952	1.2846	1.2730	1.2603	1.2464	1.2313	1.2150	1.1973
0.90	1.3134	1.3048	1.2952	1.2847	1.2731	1.2603	1.2465	1.2314	1.2151	1.1975
0.92	1.3134	1.3048	1.2952	1.2847	1.2731	1.2604	1.2465	1.2315	1.2152	1.1975
0.94	1.3134	1.3048	1.2952	1.2847	1.2731	1.2604	1.2465	1.2315	1.2152	1.1976
0.96	1.3134	1.3048	1.2952	1.2847	1.2731	1.2604	1.2465	1.2315	1.2152	1.1976
0.98	1.3134	1.3048	1.2952	1.2847	1.2731	1.2604	1.2465	1.2315	1.2152	1.1976
1.00	1.3134	1.3048	1.2952	1.2847	1.2731	1.2604	1.2465	1.2315	1.2152	1.1976

TABLE C.4c

Confidence Coefficients $k_{01}(a_{\mathrm{L}}, a_{\mathrm{U}})$ for 99% Hall–Wellner Confidence Bands

	a_{L}										
a_{U}	0.00	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
0.10	0.8512	0.8512	0.8512	0.8502	0.8428						
0.12	0.9243	0.9243	0.9243	0.9240	0.9217	0.9113					
0.14	0.9895	0.9895	0.9895	0.9894	0.9886	0.9845	0.9715				
0.16	1.0483	1.0483	1.0483	1.0483	1.0479	1.0461	1.0404	1.0249			
0.18	1.1017	1.1017	1.1017	1.1017	1.1016	1.1007	1.0978	1.0903	1.0727		
0.20	1.1505	1.1505	1.1505	1.1505	1.1505	1.1500	1.1484	1.1443	1.1352	1.1157	
0.22	1.1953	1.1953	1.1953	1.1953	1.1953	1.1950	1.1941	1.1917	1.1863	1.1757	1.1544
0.24	1.2365	1.2365	1.2365	1.2365	1.2365	1.2364	1.2358	1.2343	1.2309	1.2243	1.2122
0.26	1.2745	1.2745	1.2745	1.2745	1.2745	1.2744	1.2740	1.2730	1.2708	1.2664	1.2586
0.28	1.3095	1.3095	1.3095	1.3095	1.3095	1.3095	1.3092	1.3086	1.3070	1.3039	1.2985
0.30	1.3419	1.3419	1.3419	1.3419	1.3419	1.3418	1.3417	1.3412	1.3401	1.3379	1.3340
0.32	1.3717	1.3717	1.3717	1.3717	1.3717	1.3717	1.3716	1.3713	1.3705	1.3688	1.3659
0.34	1.3993	1.3993	1.3993	1.3993	1.3993	1.3993	1.3992	1.3990	1.3984	1.3971	1.3949
0.36	1.4247	1.4247	1.4247	1.4247	1.4247	1.4247	1.4247	1.4245	1.4240	1.4231	1.4213
0.38	1.4481	1.4481	1.4481	1.4481	1.4481	1.4481	1.4481	1.4479	1.4476	1.4468	1.4454
0.40	1.4696	1.4696	1.4696	1.4696	1.4696	1.4696	1.4696	1.4695	1.4692	1.4686	1.4674
0.42	1.4893	1.4893	1.4893	1.4893	1.4893	1.4893	1.4893	1.4892	1.4890	1.4885	1.4875
0.44	1.5073	1.5073	1.5073	1.5073	1.5073	1.5073	1.5073	1.5072	1.5071	1.5066	1.5058
0.46	1.5237	1.5237	1.5237	1.5237	1.5237	1.5237	1.5237	1.5236	1.5235	1.5231	1.5225
0.48	1.5386	1.5386	1.5386	1.5386	1.5386	1.5386	1.5385	1.5385	1.5384	1.5381	1.5375
0.50	1.5520	1.5520	1.5520	1.5520	1.5520	1.5520	1.5519	1.5519	1.5518	1.5515	1.5510
0.52	1.5640	1.5640	1.5640	1.5640	1.5640	1.5640	1.5640	1.5639	1.5638	1.5636	1.5631
0.54	1.5747	1.5747	1.5747	1.5747	1.5747	1.5747	1.5747	1.5746	1.5746	1.5744	1.5739
0.56	1.5841	1.5841	1.5841	1.5841	1.5841	1.5841	1.5841	1.5841	1.5840	1.5838	1.5835
0.58	1.5924	1.5924	1.5924	1.5924	1.5924	1.5924	1.5924	1.5924	1.5923	1.5921	1.5918
0.60	1.5996	1.5996	1.5996	1.5996	1.5996	1.5996	1.5996	1.5996	1.5995	1.5993	1.5990
0.62	1.6057	1.6057	1.6057	1.6057	1.6057	1.6057	1.6057	1.6057	1.6056	1.6055	1.6052
0.64	1.6109	1.6109	1.6109	1.6109	1.6109	1.6109	1.6109	1.6109	1.6108	1.6107	1.6104
0.66	1.6152	1.6152	1.6152	1.6152	1.6152	1.6152	1.6152	1.6151	1.6151	1.6150	1.6147
0.68	1.6186	1.6186	1.6186	1.6186	1.6186	1.6186	1.6186	1.6186	1.6186	1.6184	1.6182
0.70	1.6214	1.6214	1.6214	1.6214	1.6214	1.6214	1.6214	1.6214	1.6213	1.6212	1.6209
0.72	1.6235	1.6235	1.6235	1.6235	1.6235	1.6235	1.6235	1.6235	1.6234	1.6233	1.6230
0.74	1.6250	1.6250	1.6250	1.6250	1.6250	1.6250	1.6250	1.6250	1.6249	1.6248	1.6246
0.76	1.6261	1.6261	1.6261	1.6261	1.6261	1.6261	1.6261	1.6261	1.6260	1.6259	1.6257
0.78	1.6268	1.6268	1.6268	1.6268	1.6268	1.6268	1.6268	1.6268	1.6267	1.6266	1.6264
0.80	1.6272	1.6272	1.6272	1.6272	1.6272	1.6272	1.6272	1.6272	1.6272	1.6271	1.6268
0.82	1.6275	1.6275	1.6275	1.6275	1.6275	1.6275	1.6275	1.6274	1.6274	1.6273	1.6271
0.84	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6275	1.6274	1.6272
0.86	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6274	1.6272
0.88	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6275	1.6272
0.90	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6275	1.6272
0.92	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6275	1.6272
0.94	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6275	1.6272
0.96	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6275	1.6272
0.98	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6275	1.6272
1.00	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6276	1.6275	1.6272

TABLE C.4c
(continued)

	a_{L}									
a_{U}	*0.22*	*0.24*	*0.26*	*0.28*	*0.30*	*0.32*	*0.34*	*0.36*	*0.38*	*0.40*
0.24	1.1893									
0.26	1.2452	1.2207								
0.28	1.2896	1.2748	1.2489							
0.30	1.3276	1.3175	1.3014	1.2742						
0.32	1.3611	1.3537	1.3425	1.3253	1.2967					
0.34	1.3912	1.3855	1.3771	1.3648	1.3464	1.3166				
0.36	1.4184	1.4139	1.4073	1.3979	1.3845	1.3650	1.3341			
0.38	1.4431	1.4395	1.4342	1.4267	1.4162	1.4018	1.3812	1.3491		
0.40	1.4655	1.4625	1.4582	1.4520	1.4436	1.4322	1.4167	1.3950	1.3619	
0.42	1.4859	1.4834	1.4797	1.4746	1.4676	1.4583	1.4459	1.4294	1.4067	1.3724
0.44	1.5045	1.5023	1.4992	1.4948	1.4888	1.4810	1.4707	1.4574	1.4399	1.4161
0.46	1.5213	1.5194	1.5167	1.5129	1.5077	1.5009	1.4922	1.4810	1.4667	1.4482
0.48	1.5365	1.5348	1.5324	1.5291	1.5245	1.5186	1.5109	1.5013	1.4892	1.4739
0.50	1.5501	1.5487	1.5465	1.5435	1.5395	1.5342	1.5274	1.5189	1.5083	1.4953
0.52	1.5623	1.5610	1.5591	1.5564	1.5527	1.5480	1.5419	1.5343	1.5249	1.5134
0.54	1.5732	1.5720	1.5703	1.5678	1.5644	1.5601	1.5545	1.5476	1.5391	1.5288
0.56	1.5828	1.5817	1.5801	1.5778	1.5747	1.5707	1.5656	1.5592	1.5514	1.5420
0.58	1.5912	1.5902	1.5887	1.5865	1.5837	1.5799	1.5751	1.5692	1.5620	1.5533
0.60	1.5984	1.5975	1.5961	1.5941	1.5914	1.5878	1.5834	1.5778	1.5710	1.5629
0.62	1.6047	1.6038	1.6024	1.6005	1.5980	1.5946	1.5904	1.5851	1.5787	1.5710
0.64	1.6099	1.6090	1.6078	1.6060	1.6035	1.6003	1.5962	1.5912	1.5851	1.5778
0.66	1.6142	1.6134	1.6122	1.6104	1.6081	1.6050	1.6011	1.5962	1.5904	1.5834
0.68	1.6177	1.6169	1.6157	1.6141	1.6118	1.6088	1.6050	1.6003	1.5946	1.5878
0.70	1.6205	1.6197	1.6186	1.6169	1.6147	1.6118	1.6081	1.6035	1.5980	1.5914
0.72	1.6226	1.6218	1.6207	1.6191	1.6169	1.6141	1.6104	1.6060	1.6005	1.5941
0.74	1.6241	1.6234	1.6223	1.6207	1.6186	1.6157	1.6122	1.6078	1.6024	1.5961
0.76	1.6252	1.6245	1.6234	1.6218	1.6197	1.6169	1.6134	1.6090	1.6038	1.5975
0.78	1.6260	1.6252	1.6241	1.6226	1.6205	1.6177	1.6142	1.6099	1.6047	1.5984
0.80	1.6264	1.6257	1.6246	1.6230	1.6209	1.6182	1.6147	1.6104	1.6052	1.5990
0.82	1.6266	1.6259	1.6248	1.6233	1.6212	1.6184	1.6150	1.6107	1.6055	1.5993
0.84	1.6267	1.6260	1.6249	1.6234	1.6213	1.6186	1.6151	1.6108	1.6056	1.5995
0.86	1.6268	1.6261	1.6250	1.6235	1.6214	1.6186	1.6151	1.6109	1.6057	1.5996
0.88	1.6268	1.6261	1.6250	1.6235	1.6214	1.6186	1.6152	1.6109	1.6057	1.5996
0.90	1.6268	1.6261	1.6250	1.6235	1.6214	1.6186	1.6152	1.6109	1.6057	1.5996
0.92	1.6268	1.6261	1.6250	1.6235	1.6214	1.6186	1.6152	1.6109	1.6057	1.5996
0.94	1.6268	1.6261	1.6250	1.6235	1.6214	1.6186	1.6152	1.6109	1.6057	1.5996
0.96	1.6268	1.6261	1.6250	1.6235	1.6214	1.6186	1.6152	1.6109	1.6057	1.5996
0.98	1.6268	1.6261	1.6250	1.6235	1.6214	1.6186	1.6152	1.6109	1.6057	1.5996
1.00	1.6268	1.6261	1.6250	1.6235	1.6214	1.6186	1.6152	1.6109	1.6057	1.5996

(continued)

TABLE C.4c
(continued)

	a_L									
a_U	*0.42*	*0.44*	*0.46*	*0.48*	*0.50*	*0.52*	*0.54*	*0.56*	*0.58*	*0.60*
0.44	1.3808									
0.46	1.4234	1.3870								
0.48	1.4544	1.4286	1.3912							
0.50	1.4790	1.4585	1.4316	1.3932						
0.52	1.4993	1.4821	1.4605	1.4327	1.3932					
0.54	1.5164	1.5013	1.4831	1.4605	1.4316	1.3912				
0.56	1.5308	1.5174	1.5013	1.4821	1.4585	1.4286	1.3870			
0.58	1.5430	1.5308	1.5164	1.4993	1.4790	1.4544	1.4234	1.3808		
0.60	1.5533	1.5420	1.5288	1.5134	1.4953	1.4739	1.4482	1.4161	1.3724	
0.62	1.5620	1.5514	1.5391	1.5249	1.5083	1.4892	1.4667	1.4399	1.4067	1.3619
0.64	1.5692	1.5592	1.5476	1.5343	1.5189	1.5013	1.4810	1.4574	1.4294	1.3950
0.66	1.5751	1.5656	1.5545	1.5419	1.5274	1.5109	1.4922	1.4707	1.4459	1.4167
0.68	1.5799	1.5707	1.5601	1.5480	1.5342	1.5186	1.5009	1.4810	1.4583	1.4322
0.70	1.5837	1.5747	1.5644	1.5527	1.5395	1.5245	1.5077	1.4888	1.4676	1.4436
0.72	1.5865	1.5778	1.5678	1.5564	1.5435	1.5291	1.5129	1.4948	1.4746	1.4520
0.74	1.5887	1.5801	1.5703	1.5591	1.5465	1.5324	1.5167	1.4992	1.4797	1.4582
0.76	1.5902	1.5817	1.5720	1.5610	1.5487	1.5348	1.5194	1.5023	1.4834	1.4625
0.78	1.5912	1.5828	1.5732	1.5623	1.5501	1.5365	1.5213	1.5045	1.4859	1.4655
0.80	1.5918	1.5835	1.5739	1.5631	1.5510	1.5375	1.5225	1.5058	1.4875	1.4674
0.82	1.5921	1.5838	1.5744	1.5636	1.5515	1.5381	1.5231	1.5066	1.4885	1.4686
0.84	1.5923	1.5840	1.5746	1.5638	1.5518	1.5384	1.5235	1.5071	1.4890	1.4692
0.86	1.5924	1.5841	1.5746	1.5639	1.5519	1.5385	1.5236	1.5072	1.4892	1.4695
0.88	1.5924	1.5841	1.5747	1.5640	1.5519	1.5385	1.5237	1.5073	1.4893	1.4696
0.90	1.5924	1.5841	1.5747	1.5640	1.5520	1.5386	1.5237	1.5073	1.4893	1.4696
0.92	1.5924	1.5841	1.5747	1.5640	1.5520	1.5386	1.5237	1.5073	1.4893	1.4696
0.94	1.5924	1.5841	1.5747	1.5640	1.5520	1.5386	1.5237	1.5073	1.4893	1.4696
0.96	1.5924	1.5841	1.5747	1.5640	1.5520	1.5386	1.5237	1.5073	1.4893	1.4696
0.98	1.5924	1.5841	1.5747	1.5640	1.5520	1.5386	1.5237	1.5073	1.4893	1.4696
1.00	1.5924	1.5841	1.5747	1.5640	1.5520	1.5386	1.5237	1.5073	1.4893	1.4696

TABLE C.5

Survival Function of the Supremum of the Absolute Value of a Standard Brownian Motion Process over the Range 0 to 1

$\Pr[\sup \lvert B(t) \rvert > x]$	x	$\Pr[\sup \lvert B(t) \rvert > x]$	x	$\Pr[\sup \lvert B(t) \rvert > x]$	x
0.01	2.8070	0.34	1.3721	0.67	0.9559
0.02	2.5758	0.35	1.3562	0.68	0.9452
0.03	2.4324	0.36	1.3406	0.69	0.9345
0.04	2.3263	0.37	1.3253	0.70	0.9238
0.05	2.2414	0.38	1.3103	0.71	0.9132
0.06	2.1701	0.39	1.2956	0.72	0.9025
0.07	2.1084	0.40	1.2812	0.73	0.8919
0.08	2.0537	0.41	1.2670	0.74	0.8812
0.09	2.0047	0.42	1.2531	0.75	0.8706
0.10	1.9600	0.43	1.2394	0.76	0.8598
0.11	1.9189	0.44	1.2259	0.77	0.8491
0.12	1.8808	0.45	1.2126	0.78	0.8383
0.13	1.8453	0.46	1.1995	0.79	0.8274
0.14	1.8119	0.47	1.1866	0.80	0.8164
0.15	1.7805	0.48	1.1739	0.81	0.8053
0.16	1.7507	0.49	1.1614	0.82	0.7941
0.17	1.7224	0.50	1.1490	0.83	0.7828
0.18	1.6954	0.51	1.1367	0.84	0.7712
0.19	1.6696	0.52	1.1246	0.85	0.7595
0.20	1.6448	0.53	1.1127	0.86	0.7475
0.21	1.6211	0.54	1.1009	0.87	0.7353
0.22	1.5982	0.55	1.0892	0.88	0.7227
0.23	1.5761	0.56	1.0776	0.89	0.7098
0.24	1.5548	0.57	1.0661	0.90	0.6964
0.25	1.5341	0.58	1.0547	0.91	0.6824
0.26	1.5141	0.59	1.0434	0.92	0.6677
0.27	1.4946	0.60	1.0322	0.93	0.6521
0.28	1.4758	0.61	1.0211	0.94	0.6355
0.29	1.4574	0.62	1.0101	0.95	0.6173
0.30	1.4395	0.63	0.9992	0.96	0.5971
0.31	1.4220	0.64	0.9883	0.97	0.5737
0.32	1.4050	0.65	0.9774	0.98	0.5450
0.33	1.3883	0.66	0.9666	0.99	0.5045

TABLE C.6

Survival Function of $W = \int_0^1 [B(t)]^2 dt$*, where* $B(t)$ *is a Standard Brownian Motion*

W	*0.00*	*0.01*	*0.02*	*0.03*	*0.04*	*0.05*	*0.06*	*0.07*	*0.08*	*0.09*
0.00	1.0000	1.0000	.9994	.9945	.9824	.9642	.9417	.9169	.8910	.8648
0.10	.8390	.8138	.7894	.7659	.7434	.7218	.7012	.6814	.6626	.6445
0.20	.6273	.6108	.5949	.5798	.5652	.5513	.5378	.5249	.5125	.5006
0.30	.4890	.4779	.4672	.4568	.4468	.4371	.4278	.4187	.4099	.4014
0.40	.3931	.3851	.3773	.3697	.3623	.3552	.3482	.3414	.3348	.3284
0.50	.3222	.3161	.3101	.3043	.2987	.2932	.2878	.2825	.2774	.2724
0.60	.2675	.2627	.2580	.2534	.2489	.2446	.2403	.2361	.2320	.2280
0.70	.2240	.2202	.2164	.2127	.2091	.2056	.2021	.1987	.1953	.1921
0.80	.1889	.1857	.1826	.1796	.1767	.1738	.1709	.1681	.1654	.1627
0.90	.1600	.1574	.1549	.1524	.1499	.1475	.1451	.1428	.1405	.1383
1.00	.1361	.1339	.1318	.1297	.1277	.1257	.1237	.1218	.1198	.1180
1.10	.1161	.1143	.1125	.1108	.1091	.1074	.1057	.1041	.1025	.1009
1.20	.0994	.0978	.0963	.0949	.0934	.0920	.0906	.0892	.0878	.0865
1.30	.0852	.0839	.0826	.0814	.0802	.0789	.0778	.0766	.0754	.0743
1.40	.0732	.0721	.0710	.0700	.0689	.0679	.0669	.0659	.0649	.0639
1.50	.0630	.0621	.0611	.0602	.0593	.0585	.0576	.0568	.0559	.0551
1.60	.0543	.0535	.0527	.0519	.0512	.0504	.0497	.0490	.0482	.0475
1.70	.0469	.0462	.0455	.0448	.0442	.0435	.0429	.0423	.0417	.0411
1.80	.0405	.0399	.0393	.0388	.0382	.0376	.0371	.0366	.0360	.0355
1.90	.0350	.0345	.0340	.0335	.0330	.0326	.0321	.0317	.0312	.0308
2.00	.0303	.0299	.0295	.0290	.0286	.0282	.0278	.0274	.0270	.0266
2.10	.0263	.0259	.0255	.0252	.0248	.0245	.0241	.0238	.0234	.0231
2.20	.0228	.0225	.0221	.0218	.0215	.0212	.0209	.0206	.0203	.0201
2.30	.0198	.0195	.0192	.0190	.0187	.0184	.0182	.0179	.0177	.0174
2.40	.0172	.0169	.0167	.0165	.0162	.0160	.0158	.0156	.0153	.0151
2.50	.0149	.0147	.0145	.0143	.0141	.0139	.0137	.0135	.0133	.0132
2.60	.0130	.0128	.0126	.0124	.0123	.0121	.0119	.0118	.0116	.0114
2.70	.0113	.0111	.0110	.0108	.0107	.0105	.0104	.0102	.0101	.0100
2.80	.0098	.0097	.0096	.0094	.0093	.0092	.0090	.0089	.0088	.0087
2.90	.0086	.0084	.0083	.0082	.0081	.0080	.0079	.0078	.0077	.0076
3.00	.0075	.0074	.0073	.0072	.0071	.0070	.0069	.0068	.0067	.0066
3.10	.0065	.0064	.0063	.0062	.0062	.0061	.0060	.0059	.0058	.0057
3.20	.0057	.0056	.0055	.0054	.0054	.0053	.0052	.0052	.0051	.0050
3.30	.0049	.0049	.0048	.0047	.0047	.0046	.0046	.0045	.0044	.0044
3.40	.0043	.0043	.0042	.0041	.0041	.0040	.0040	.0039	.0039	.0038
3.50	.0038	.0037	.0037	.0036	.0036	.0035	.0035	.0034	.0034	.0033
3.60	.0033	.0032	.0032	.0032	.0031	.0031	.0030	.0030	.0030	.0029
3.70	.0029	.0028	.0028	.0028	.0027	.0027	.0026	.0026	.0026	.0025
3.80	.0025	.0025	.0024	.0024	.0024	.0023	.0023	.0023	.0023	.0022
3.90	.0022	.0022	.0021	.0021	.0021	.0021	.0020	.0020	.0020	.0019

Selected upper percentage points: $W_{0.01} = 2.787$, $W_{0.025} = 2.135$, $W_{0.05} = 1.656$, $W_{0.10} = 1.196$.

TABLE C.7

Upper Percentiles of $R = \int_0^k |B^o(u)|du$, *where* $B^o(u)$ *is a Brownian Bridge*

$P(R > r)$	$k = 0.1$	$k = 0.2$	$k = 0.3$	$k = 0.4$	$k = 0.5$	$k = 0.6$	$k = 0.7$	$k = 0.8$	$k = 0.9$	$k = 1.0$
0.99	0.0003	0.0011	0.0026	0.0044	0.0068	0.0093	0.0124	0.0155	0.0184	0.0200
0.98	0.0004	0.0014	0.0032	0.0054	0.0083	0.0115	0.0151	0.0189	0.0220	0.0237
0.97	0.0004	0.0016	0.0036	0.0062	0.0094	0.0130	0.0172	0.0213	0.0249	0.0267
0.96	0.0005	0.0018	0.0040	0.0069	0.0105	0.0144	0.0189	0.0235	0.0274	0.0292
0.95	0.0005	0.0020	0.0044	0.0075	0.0114	0.0157	0.0205	0.0253	0.0295	0.0314
0.94	0.0005	0.0021	0.0047	0.0081	0.0123	0.0168	0.0220	0.0271	0.0315	0.0334
0.93	0.0006	0.0023	0.0051	0.0086	0.0131	0.0180	0.0234	0.0288	0.0334	0.0354
0.92	0.0006	0.0024	0.0054	0.0092	0.0138	0.0191	0.0248	0.0303	0.0352	0.0372
0.91	0.0007	0.0026	0.0057	0.0097	0.0147	0.0201	0.0261	0.0319	0.0369	0.0390
0.90	0.0007	0.0027	0.0060	0.0102	0.0154	0.0212	0.0273	0.0335	0.0388	0.0408
0.89	0.0007	0.0029	0.0063	0.0107	0.0162	0.0221	0.0286	0.0349	0.0404	0.0425
0.88	0.0008	0.0030	0.0066	0.0112	0.0169	0.0231	0.0299	0.0364	0.0420	0.0442
0.87	0.0008	0.0031	0.0068	0.0117	0.0176	0.0241	0.0311	0.0379	0.0437	0.0459
0.86	0.0008	0.0033	0.0071	0.0122	0.0184	0.0251	0.0322	0.0394	0.0452	0.0475
0.85	0.0009	0.0034	0.0074	0.0127	0.0191	0.0261	0.0334	0.0408	0.0468	0.0491
0.84	0.0009	0.0036	0.0077	0.0132	0.0198	0.0271	0.0346	0.0422	0.0484	0.0508
0.83	0.0010	0.0037	0.0080	0.0137	0.0205	0.0280	0.0358	0.0436	0.0500	0.0524
0.82	0.0010	0.0039	0.0083	0.0142	0.0213	0.0290	0.0370	0.0449	0.0515	0.0539
0.81	0.0010	0.0040	0.0086	0.0147	0.0220	0.0299	0.0382	0.0463	0.0532	0.0556
0.80	0.0011	0.0041	0.0089	0.0152	0.0228	0.0309	0.0394	0.0477	0.0546	0.0572
0.79	0.0011	0.0043	0.0092	0.0157	0.0236	0.0319	0.0407	0.0491	0.0562	0.0588
0.78	0.0011	0.0044	0.0096	0.0162	0.0243	0.0330	0.0419	0.0506	0.0578	0.0604
0.77	0.0012	0.0046	0.0099	0.0168	0.0250	0.0340	0.0431	0.0520	0.0594	0.0621
0.76	0.0012	0.0048	0.0102	0.0173	0.0258	0.0350	0.0443	0.0534	0.0610	0.0636
0.75	0.0013	0.0049	0.0105	0.0179	0.0266	0.0360	0.0456	0.0549	0.0626	0.0652
0.74	0.0013	0.0051	0.0109	0.0185	0.0274	0.0370	0.0468	0.0563	0.0642	0.0668
0.73	0.0014	0.0052	0.0112	0.0190	0.0283	0.0380	0.0481	0.0578	0.0657	0.0684
0.72	0.0014	0.0054	0.0116	0.0196	0.0291	0.0391	0.0494	0.0593	0.0673	0.0701
0.71	0.0014	0.0056	0.0119	0.0202	0.0299	0.0402	0.0508	0.0607	0.0690	0.0718
0.70	0.0015	0.0057	0.0123	0.0208	0.0307	0.0413	0.0521	0.0623	0.0707	0.0735
0.69	0.0015	0.0059	0.0127	0.0213	0.0316	0.0424	0.0535	0.0637	0.0723	0.0752
0.68	0.0016	0.0061	0.0131	0.0219	0.0324	0.0435	0.0549	0.0653	0.0740	0.0770
0.67	0.0016	0.0063	0.0135	0.0226	0.0333	0.0446	0.0562	0.0669	0.0757	0.0787
0.66	0.0017	0.0065	0.0138	0.0232	0.0342	0.0458	0.0576	0.0686	0.0775	0.0804
0.65	0.0017	0.0067	0.0143	0.0238	0.0351	0.0470	0.0591	0.0703	0.0793	0.0822
0.64	0.0018	0.0069	0.0147	0.0245	0.0360	0.0482	0.0605	0.0719	0.0812	0.0840
0.63	0.0018	0.0071	0.0151	0.0252	0.0370	0.0495	0.0620	0.0736	0.0831	0.0859
0.62	0.0019	0.0073	0.0155	0.0259	0.0380	0.0507	0.0636	0.0753	0.0850	0.0876
0.61	0.0019	0.0075	0.0159	0.0266	0.0390	0.0520	0.0651	0.0770	0.0869	0.0895
0.60	0.0020	0.0077	0.0164	0.0273	0.0401	0.0533	0.0668	0.0789	0.0889	0.0913
0.59	0.0021	0.0080	0.0168	0.0281	0.0412	0.0547	0.0684	0.0807	0.0909	0.0934

(continued)

TABLE C.7
(continued)

$P(R > r)$	$k = 0.1$	$k = 0.2$	$k = 0.3$	$k = 0.4$	$k = 0.5$	$k = 0.6$	$k = 0.7$	$k = 0.8$	$k = 0.9$	$k = 1.0$
0.58	0.0021	0.0082	0.0173	0.0288	0.0423	0.0561	0.0700	0.0825	0.0929	0.0955
0.57	0.0022	0.0084	0.0178	0.0296	0.0434	0.0575	0.0718	0.0845	0.0950	0.0974
0.56	0.0023	0.0087	0.0182	0.0304	0.0445	0.0590	0.0735	0.0865	0.0973	0.0997
0.55	0.0023	0.0089	0.0188	0.0312	0.0456	0.0605	0.0752	0.0885	0.0994	0.1018
0.54	0.0024	0.0092	0.0193	0.0321	0.0468	0.0620	0.0771	0.0905	0.1016	0.1040
0.53	0.0025	0.0094	0.0198	0.0330	0.0480	0.0636	0.0789	0.0925	0.1038	0.1063
0.52	0.0025	0.0097	0.0204	0.0338	0.0493	0.0652	0.0808	0.0947	0.1061	0.1087
0.51	0.0026	0.0100	0.0209	0.0348	0.0506	0.0668	0.0828	0.0969	0.1086	0.1111
0.50	0.0027	0.0102	0.0215	0.0357	0.0519	0.0684	0.0847	0.0991	0.1110	0.1135
0.49	0.0028	0.0105	0.0221	0.0367	0.0533	0.0702	0.0868	0.1013	0.1134	0.1159
0.48	0.0028	0.0108	0.0228	0.0377	0.0547	0.0719	0.0888	0.1035	0.1159	0.1185
0.47	0.0029	0.0111	0.0234	0.0387	0.0561	0.0737	0.0911	0.1060	0.1185	0.1210
0.46	0.0030	0.0115	0.0241	0.0398	0.0575	0.0756	0.0934	0.1084	0.1213	0.1237
0.45	0.0031	0.0118	0.0247	0.0409	0.0590	0.0776	0.0957	0.1110	0.1239	0.1263
0.44	0.0032	0.0121	0.0254	0.0421	0.0605	0.0795	0.0980	0.1136	0.1265	0.1291
0.43	0.0033	0.0125	0.0262	0.0432	0.0621	0.0816	0.1004	0.1163	0.1292	0.1320
0.42	0.0034	0.0128	0.0269	0.0444	0.0639	0.0838	0.1031	0.1191	0.1321	0.1348
0.41	0.0035	0.0132	0.0277	0.0456	0.0656	0.0860	0.1055	0.1219	0.1350	0.1380
0.40	0.0036	0.0136	0.0285	0.0468	0.0674	0.0883	0.1081	0.1248	0.1383	0.1413
0.39	0.0037	0.0140	0.0293	0.0482	0.0692	0.0907	0.1108	0.1279	0.1416	0.1445
0.38	0.0038	0.0144	0.0301	0.0496	0.0710	0.0932	0.1138	0.1310	0.1449	0.1481
0.37	0.0039	0.0148	0.0310	0.0510	0.0731	0.0958	0.1166	0.1343	0.1484	0.1516
0.36	0.0041	0.0153	0.0320	0.0525	0.0752	0.0983	0.1196	0.1378	0.1519	0.1552
0.35	0.0042	0.0158	0.0330	0.0541	0.0772	0.1012	0.1228	0.1412	0.1555	0.1587
0.34	0.0043	0.0163	0.0340	0.0556	0.0795	0.1040	0.1259	0.1446	0.1593	0.1623
0.33	0.0045	0.0168	0.0351	0.0573	0.0817	0.1068	0.1291	0.1482	0.1632	0.1663
0.32	0.0046	0.0173	0.0362	0.0590	0.0841	0.1097	0.1327	0.1523	0.1671	0.1705
0.31	0.0048	0.0179	0.0373	0.0608	0.0867	0.1129	0.1364	0.1565	0.1713	0.1746
0.30	0.0049	0.0185	0.0384	0.0627	0.0892	0.1160	0.1404	0.1606	0.1759	0.1790
0.29	0.0051	0.0191	0.0396	0.0646	0.0918	0.1195	0.1444	0.1650	0.1803	0.1837
0.28	0.0052	0.0198	0.0409	0.0666	0.0947	0.1231	0.1487	0.1698	0.1849	0.1886
0.27	0.0054	0.0204	0.0423	0.0687	0.0977	0.1267	0.1527	0.1747	0.1900	0.1934
0.26	0.0056	0.0211	0.0437	0.0709	0.1008	0.1305	0.1572	0.1796	0.1949	0.1984
0.25	0.0058	0.0218	0.0451	0.0731	0.1042	0.1346	0.1620	0.1846	0.2006	0.2037
0.24	0.0060	0.0226	0.0467	0.0756	0.1076	0.1388	0.1672	0.1900	0.2066	0.2091
0.23	0.0063	0.0234	0.0482	0.0781	0.1112	0.1434	0.1726	0.1956	0.2127	0.2150
0.22	0.0065	0.0242	0.0500	0.0808	0.1152	0.1483	0.1779	0.2015	0.2192	0.2216
0.21	0.0067	0.0251	0.0518	0.0837	0.1193	0.1532	0.1839	0.2082	0.2260	0.2285
0.20	0.0070	0.0260	0.0539	0.0868	0.1235	0.1585	0.1901	0.2146	0.2332	0.2356
0.19	0.0073	0.0271	0.0560	0.0900	0.1278	0.1645	0.1967	0.2219	0.2408	0.2427
0.18	0.0076	0.0281	0.0582	0.0936	0.1326	0.1702	0.2039	0.2295	0.2488	0.2505
0.17	0.0079	0.0293	0.0605	0.0974	0.1379	0.1764	0.2113	0.2378	0.2572	0.2589
0.16	0.0082	0.0306	0.0629	0.1013	0.1433	0.1834	0.2194	0.2462	0.2660	0.2675
0.15	0.0086	0.0318	0.0655	0.1055	0.1489	0.1909	0.2280	0.2553	0.2756	0.2776
0.14	0.0090	0.0333	0.0685	0.1104	0.1547	0.1989	0.2372	0.2652	0.2861	0.2883

TABLE C.7
(continued)

$P(R > r)$	$k = 0.1$	$k = 0.2$	$k = 0.3$	$k = 0.4$	$k = 0.5$	$k = 0.6$	$k = 0.7$	$k = 0.8$	$k = 0.9$	$k = 1.0$
0.13	0.0094	0.0348	0.0717	0.1155	0.1618	0.2073	0.2469	0.2759	0.2975	0.3002
0.12	0.0099	0.0365	0.0750	0.1209	0.1694	0.2169	0.2580	0.2881	0.3100	0.3122
0.11	0.0104	0.0383	0.0787	0.1272	0.1779	0.2282	0.2697	0.3015	0.3238	0.3253
0.10	0.0110	0.0404	0.0830	0.1339	0.1873	0.2403	0.2837	0.3158	0.3388	0.3406
0.09	0.0116	0.0427	0.0874	0.1414	0.1975	0.2531	0.2979	0.3323	0.3559	0.3573
0.08	0.0123	0.0454	0.0929	0.1497	0.2092	0.2679	0.3150	0.3514	0.3743	0.3774
0.07	0.0131	0.0485	0.0986	0.1589	0.2224	0.2840	0.3359	0.3723	0.3948	0.3994
0.06	0.0140	0.0520	0.1056	0.1699	0.2373	0.3038	0.3579	0.3963	0.4201	0.4239
0.05	0.0151	0.0561	0.1142	0.1823	0.2555	0.3273	0.3848	0.4259	0.4506	0.4543
0.04	0.0165	0.0612	0.1245	0.1990	0.2766	0.3567	0.4157	0.4609	0.4859	0.4941
0.03	0.0184	0.0676	0.1377	0.2206	0.3063	0.3917	0.4613	0.5074	0.5381	0.5419
0.02	0.0210	0.0768	0.1577	0.2517	0.3487	0.4473	0.5247	0.5753	0.6099	0.6106
0.01	0.0253	0.0935	0.1938	0.3098	0.4241	0.5390	0.6386	0.6848	0.7339	0.7379

D

Data on 137 Bone Marrow Transplant Patients

TABLE D.1

Data on 137 Bone Marrow Transplant Patients

g—Disease group
- 1-ALL
- 2-AML low-risk
- 3-AML high-risk

T_1—Time (in days) to death or on study time

T_2—Disease-Free survival time (time to relapse, death or end of study)

δ_1—Death indicator
- 1-Dead 0-Alive

δ_2—Relapse indicator
- 1-Relapsed 0-Disease-Free

δ_3—Disease-Free survival indicator
- 1-Dead or relapsed 0-Alive disease-free

T_A—Time (in days) to acute graft-versus-host disease

δ_A—Acute graft-versus-host disease indicator
- 1-Developed acute graft-versus-host disease
- 0-Never developed acute graft-versus-host disease

T_C—Time (in days) to chronic graft-versus-host disease

δ_C—Chronic graft-versus-host disease indicator
- 1-Developed chronic graft-versus-host disease
- 0-Never developed chronic graft-versus-host disease

T_P—Time (in days) to return of platelets to normal levels

δ_P—Platelet recovery indicator
- 1-Platelets returned to normal levels
- 0-Platelets never returned to normal levels

Z_1—Patient age in years

Z_2—Donor age in years

Z_3—Patient sex
- 1-Male 0-Female

Z_4—Donor Sex
- 1-Male 0-Female

Z_5—Patient CMV status
- 1-CMV positive 0-CMV negative

Z_6—Donor CMV status
- 1-CMV positive 0-CMV negative

Z_7—Waiting time to transplant in days

Z_8—FAB
- 1-FAB Grade 4 Or 5 and AML 0-Otherwise

Z_9—Hospital
- 1-The Ohio State University 2-Alfred
- 3-St. Vincent 4-Hahnemann

Z_{10}—MTX used as a graft-versus-host-prophylactic
- 1-Yes 0-No

TABLE D.1
(continued)

g	T_1	T_2	δ_1	δ_2	δ_3	T_A	δ_A	T_C	δ_C	T_P	δ_P	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}
1	2081	2081	0	0	0	67	1	121	1	13	1	26	33	1	0	1	1	98	0	1	0
1	1602	1602	0	0	0	1602	0	139	1	18	1	21	37	1	1	0	0	1720	0	1	0
1	1496	1496	0	0	0	1496	0	307	1	12	1	26	35	1	1	1	0	127	0	1	0
1	1462	1462	0	0	0	70	1	95	1	13	1	17	21	0	1	0	0	168	0	1	0
1	1433	1433	0	0	0	1433	0	236	1	12	1	32	36	1	1	1	1	93	0	1	0
1	1377	1377	0	0	0	1377	0	123	1	12	1	22	31	1	1	1	1	2187	0	1	0
1	1330	1330	0	0	0	1330	0	96	1	17	1	20	17	1	0	1	1	1006	0	1	0
1	996	996	0	0	0	72	1	121	1	12	1	22	24	1	0	0	0	1319	0	1	0
1	226	226	0	0	0	226	0	226	0	10	1	18	21	0	1	0	0	208	0	1	0
1	1199	1199	0	0	0	1199	0	91	1	29	1	24	40	1	1	0	1	174	0	3	1
1	1111	1111	0	0	0	1111	0	1111	0	22	1	19	28	1	1	0	1	236	0	3	1
1	530	530	0	0	0	38	1	84	1	34	1	17	28	1	1	0	0	151	0	3	1
1	1182	1182	0	0	0	1182	0	112	1	22	1	24	23	0	0	0	1	203	0	2	1
1	1167	1167	0	0	0	39	1	487	1	1167	0	27	22	0	1	1	1	191	0	2	1
1	418	418	1	0	1	418	0	220	1	21	1	18	14	1	1	0	0	110	0	1	0
1	417	383	1	1	1	417	0	417	0	16	1	15	20	1	1	0	0	824	0	1	0
1	276	276	1	0	1	276	0	81	1	21	1	18	5	0	0	0	0	146	0	1	0
1	156	104	1	1	1	28	1	156	0	20	1	20	33	1	1	0	1	85	0	1	0
1	781	609	1	1	1	781	0	781	0	26	1	27	27	1	0	1	1	187	0	1	0
1	172	172	1	0	1	22	1	172	0	37	1	40	37	0	0	0	1	129	0	1	0
1	487	487	1	0	1	487	0	76	1	22	1	22	20	1	1	0	0	128	0	1	0
1	716	662	1	1	1	716	0	716	0	17	1	28	32	1	1	0	0	84	0	1	0
1	194	194	1	0	1	194	0	94	1	25	1	26	32	0	1	0	0	329	0	1	0
1	371	230	1	1	1	371	0	184	1	9	1	39	31	0	1	0	1	147	0	1	0
1	526	526	1	0	1	526	0	121	1	11	1	15	20	1	1	0	0	943	0	1	0
1	122	122	1	0	1	88	1	122	0	13	1	20	26	1	0	0	1	2616	0	1	0
1	1279	129	1	1	1	1279	0	1279	0	22	1	17	20	0	0	0	0	937	0	3	1
1	110	74	1	1	1	110	0	110	0	49	1	28	25	1	0	1	0	303	0	3	1
1	243	122	1	1	1	243	0	243	0	23	1	37	38	0	1	1	1	170	0	3	1
1	86	86	1	0	1	86	0	86	0	86	0	17	26	1	0	1	0	239	0	3	1
1	466	466	1	0	1	466	0	119	1	100	1	15	18	1	1	0	0	508	0	3	1
1	262	192	1	1	1	10	1	84	1	59	1	29	32	1	1	1	0	74	0	3	1
1	162	109	1	1	1	162	0	162	0	40	1	36	43	1	1	1	0	393	0	2	1
1	262	55	1	1	1	262	0	262	0	24	1	23	16	0	1	1	1	331	0	2	1
1	1	1	1	0	1	1	0	1	0	1	0	42	48	1	1	0	0	196	0	2	1
1	107	107	1	0	1	107	0	107	0	107	0	30	19	1	1	1	1	178	0	2	1
1	269	110	1	1	1	269	0	120	1	27	1	29	20	0	1	1	1	361	0	2	1
1	350	332	0	1	1	350	0	350	0	33	1	22	20	1	0	0	0	834	0	2	1
2	2569	2569	0	0	0	2569	0	2569	0	21	1	19	13	1	1	1	0	270	1	1	0
2	2506	2506	0	0	0	2506	0	2506	0	17	1	31	34	1	1	0	0	60	0	1	0
2	2409	2409	0	0	0	2409	0	2409	0	16	1	35	31	1	1	1	1	120	0	1	0
2	2218	2218	0	0	0	2218	0	2218	0	11	1	16	16	1	1	1	0	60	1	1	0
2	1857	1857	0	0	0	1857	0	260	1	15	1	29	35	0	0	1	0	90	0	1	0
2	1829	1829	0	0	0	1829	0	1829	0	19	1	19	18	1	1	1	0	210	0	1	0
2	1562	1562	0	0	0	1562	0	1562	0	18	1	26	30	1	1	1	1	90	0	1	0
2	1470	1470	0	0	0	1470	0	180	1	14	1	27	34	1	1	0	1	240	0	1	0

(continued)

TABLE D.1
(continued)

g	T_1	T_2	δ_1	δ_2	δ_3	T_A	δ_A	T_C	δ_C	T_P	δ_P	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}
2	1363	1363	0	0	0	1363	0	200	1	12	1	13	24	1	1	1	0	90	0	1	0
2	1030	1030	0	0	0	1030	0	210	1	14	1	25	29	0	0	0	0	210	0	1	0
2	860	860	0	0	0	860	0	860	0	15	1	25	31	0	1	0	1	180	0	1	0
2	1258	1258	0	0	0	1258	0	120	1	66	1	30	16	0	1	1	0	180	0	2	1
2	2246	2246	0	0	0	52	1	380	1	15	1	45	39	0	0	0	0	105	0	4	0
2	1870	1870	0	0	0	1870	0	230	1	16	1	33	30	0	0	1	1	225	0	4	0
2	1799	1799	0	0	0	1799	0	140	1	12	1	32	23	1	0	0	0	120	0	4	0
2	1709	1709	0	0	0	20	1	348	1	19	1	23	28	0	1	1	0	90	1	4	0
2	1674	1674	0	0	0	1674	0	1674	0	24	1	37	34	1	1	0	0	60	1	4	0
2	1568	1568	0	0	0	1568	0	1568	0	14	1	15	19	1	0	0	0	90	0	4	0
2	1527	1527	0	0	0	1527	0	1527	0	13	1	22	12	0	1	0	1	450	1	4	0
2	1324	1324	0	0	0	25	1	1324	0	15	1	46	31	1	1	1	1	75	0	4	0
2	957	957	0	0	0	957	0	957	0	69	1	18	17	1	1	0	0	90	0	4	0
2	932	932	0	0	0	29	1	932	0	7	1	27	30	0	0	0	0	60	1	4	0
2	847	847	0	0	0	847	0	847	0	16	1	28	29	1	1	0	0	75	0	4	0
2	848	848	0	0	0	848	0	155	1	16	1	23	26	1	1	0	0	180	0	4	0
2	1850	1850	0	0	0	1850	0	1850	0	9	1	37	36	0	0	0	1	180	0	3	1
2	1843	1843	0	0	0	1843	0	1843	0	19	1	34	32	0	0	1	1	270	0	3	1
2	1535	1535	0	0	0	1535	0	1535	0	21	1	35	32	0	1	0	0	180	1	3	1
2	1447	1447	0	0	0	1447	0	220	1	24	1	33	28	0	1	1	1	150	0	3	1
2	1384	1384	0	0	0	1384	0	200	1	19	1	21	18	0	0	0	0	120	0	3	1
2	414	414	1	0	1	414	0	414	0	27	1	21	15	1	1	0	1	120	1	1	0
2	2204	2204	1	0	1	2204	0	2204	0	12	1	25	19	0	0	0	1	60	0	1	0
2	1063	1063	1	0	1	1063	0	240	1	16	1	50	38	1	0	1	0	270	1	1	0
2	481	481	1	0	1	30	1	120	1	24	1	35	36	1	0	1	1	90	1	1	0
2	105	105	1	0	1	21	1	105	0	15	1	37	34	1	0	1	1	120	0	1	0
2	641	641	1	0	1	641	0	641	0	11	1	26	24	1	1	0	0	90	0	1	0
2	390	390	1	0	1	390	0	390	0	11	1	50	48	1	1	0	0	120	0	1	0
2	288	288	1	0	1	18	1	100	1	288	0	45	43	1	1	1	1	90	0	1	0
2	522	421	1	1	1	25	1	140	1	20	1	28	30	1	1	0	1	90	1	1	0
2	79	79	1	0	1	16	1	79	0	79	0	43	43	0	0	0	0	90	0	1	0
2	1156	748	1	1	1	1156	0	180	1	18	1	14	19	1	0	0	0	60	0	1	0
2	583	486	1	1	1	583	0	583	0	11	1	17	14	0	1	0	0	120	0	1	0
2	48	48	1	0	1	48	0	48	0	14	1	32	33	0	1	1	0	150	1	1	0
2	431	272	1	1	1	431	0	431	0	12	1	30	23	0	1	1	0	120	1	1	0
2	1074	1074	1	0	1	1074	0	120	1	19	1	30	32	1	1	1	0	150	1	1	0
2	393	381	1	1	1	393	0	100	1	16	1	33	28	0	0	0	0	120	1	1	0
2	10	10	1	0	1	10	0	10	0	10	0	34	54	1	0	1	1	240	0	2	1
2	53	53	1	0	1	53	0	53	0	53	0	33	41	0	1	1	1	180	0	2	1
2	80	80	1	0	1	10	1	80	0	80	0	30	35	0	0	0	1	150	0	2	1
2	35	35	1	0	1	35	0	35	0	35	0	23	25	0	1	1	1	150	0	2	1
2	1499	248	0	1	1	1499	0	1499	0	9	1	35	18	1	1	0	1	30	0	4	0
2	704	704	1	0	1	36	1	155	1	18	1	29	21	0	1	1	0	105	0	4	0
2	653	211	1	1	1	653	0	653	0	23	1	23	16	1	0	0	0	90	1	4	0
2	222	219	1	1	1	222	0	123	1	52	1	28	30	1	1	1	1	120	1	3	1
2	1356	606	0	1	1	1356	0	1356	0	14	1	33	22	1	1	1	0	210	1	3	1

TABLE D.1
(continued)

g	T_1	T_2	δ_1	δ_2	δ_3	T_A	δ_A	T_C	δ_C	T_P	δ_P	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}
3	2640	2640	0	0	0	2640	0	2640	0	22	1	18	23	1	1	0	0	750	0	1	0
3	2430	2430	0	0	0	2430	0	2430	0	14	1	29	26	1	1	0	1	24	0	1	0
3	2252	2252	0	0	0	2252	0	150	1	17	1	35	31	1	0	0	0	120	0	1	0
3	2140	2140	0	0	0	2140	0	220	1	18	1	27	17	1	1	1	1	210	0	1	0
3	2133	2133	0	0	0	2133	0	250	1	17	1	36	39	0	1	0	0	240	0	1	0
3	1238	1238	0	0	0	1238	0	250	1	18	1	24	28	1	0	1	1	240	0	1	0
3	1631	1631	0	0	0	1631	0	150	1	40	1	27	21	1	0	1	0	690	1	2	1
3	2024	2024	0	0	0	2024	0	180	1	16	1	35	41	0	1	0	0	105	1	4	0
3	1345	1345	0	0	0	32	1	360	1	14	1	50	36	1	1	1	1	120	0	4	0
3	1136	1136	0	0	0	1136	0	140	1	15	1	47	27	1	0	1	0	900	0	3	1
3	845	845	0	0	0	845	0	845	0	20	1	40	39	0	0	1	1	210	1	3	1
3	491	422	1	1	1	491	0	180	1	491	0	22	21	0	0	0	0	210	1	1	0
3	162	162	1	0	1	162	0	162	0	13	1	22	23	1	0	0	1	300	0	1	0
3	1298	84	1	1	1	1298	0	1298	0	1298	0	8	2	0	0	1	0	105	1	1	0
3	121	100	1	1	1	28	1	121	0	65	1	39	48	1	1	1	1	210	1	1	0
3	2	2	1	0	1	2	0	2	0	2	0	20	19	1	1	0	0	75	1	1	0
3	62	47	1	1	1	62	0	62	0	11	1	27	25	1	1	0	0	90	1	1	0
3	265	242	1	1	1	265	0	210	1	14	1	32	32	1	0	0	0	180	1	1	0
3	547	456	1	1	1	547	0	130	1	24	1	31	28	1	0	1	1	630	1	1	0
3	341	268	1	1	1	21	1	100	1	17	1	20	23	0	1	1	1	180	1	1	0
3	318	318	1	0	1	318	0	140	1	12	1	35	40	0	1	1	1	300	0	1	0
3	195	32	1	1	1	195	0	195	0	16	1	36	39	1	1	0	0	90	1	1	0
3	469	467	1	1	1	469	0	90	1	20	1	35	33	0	0	1	0	120	0	1	0
3	93	47	1	1	1	93	0	93	0	28	1	7	2	1	1	0	0	135	1	1	0
3	515	390	1	1	1	515	0	515	0	31	1	23	25	1	1	1	0	210	1	1	0
3	183	183	1	0	1	183	0	130	1	21	1	11	7	0	1	0	0	120	1	1	0
3	105	105	1	0	1	105	0	105	0	105	0	14	18	1	0	0	0	150	1	1	0
3	128	115	1	1	1	128	0	128	0	12	1	37	35	0	0	1	1	270	0	1	0
3	164	164	1	0	1	164	0	164	0	164	0	19	32	0	0	0	1	285	1	1	0
3	129	93	1	1	1	129	0	129	0	51	1	37	34	0	1	1	0	240	1	1	0
3	122	120	1	1	1	122	0	122	0	12	1	25	29	0	1	1	1	510	1	1	0
3	80	80	1	0	1	21	1	80	0	0	1	35	28	1	0	0	0	780	1	1	0
3	677	677	1	0	1	677	0	150	1	8	1	15	14	1	1	1	0	150	1	1	0
3	73	64	1	1	1	73	0	73	0	38	1	45	42	0	1	1	0	180	1	2	1
3	168	168	1	0	1	168	0	200	1	48	1	32	43	0	1	1	1	150	1	2	1
3	74	74	1	0	1	29	1	74	0	24	1	41	29	0	1	1	1	750	0	2	1
3	16	16	1	0	1	16	0	16	0	16	0	27	36	0	0	1	0	180	0	4	0
3	248	157	1	1	1	248	0	100	1	52	1	33	39	0	0	1	1	180	1	4	0
3	732	625	1	1	1	732	0	732	0	18	1	39	43	0	1	1	1	150	1	4	0
3	105	48	1	1	1	105	0	105	0	30	1	17	14	0	1	0	0	210	1	4	0
3	392	273	1	1	1	392	0	122	1	24	1	43	50	1	1	1	0	240	0	3	1
3	63	63	1	0	1	38	1	63	0	16	1	44	37	1	1	0	0	360	1	3	1
3	97	76	1	1	1	97	0	97	0	97	0	48	56	1	1	1	1	330	0	3	1
3	153	113	1	1	1	153	0	153	0	59	1	31	25	0	1	1	1	240	0	3	1
3	363	363	1	0	1	363	0	363	0	19	1	52	48	1	1	1	0	180	0	3	1

E
Selected Solutions to Exercises

Solutions to Chapter 2

2.1
- (a) 1000
- (b) 693.15
- (c) 0.1353

2.3
- (a) 50 days: 0.2205
  100 days: 0.0909
  150 days: 0.0516
- (b) 21.5 days
- (c) The inflection point is at 13.572
- (d) 52.1 days

2.5
- (a) Mean = 210.2985 days
  Median = 23.9747 days
- (b) 100 days: 0.2466
  200 days: 0.1544
  300 days: 0.1127

2.7
- (a) 0.3027
- (b) 0.4303
- (c) 15 months

2.9 (a)

	Treatment A	Treatment B
1 year	0.5030	0.4042
2 years	0.3673	0.2779
5 years	0.2127	0.1475

(b)

	Treatment A	Treatment B
1 year	0.5019	0.4397
2 years	0.4161	0.3569
5 years	0.3106	0.2598

2.11 (a)
$$\begin{array}{rll} f(x) = & 0, & x < \phi \\ & \lambda\alpha(x-\phi)^{(\alpha-1)}\exp\{-\lambda(x-\phi)^{\alpha}\}, & x \geq \phi \\ h(x) = & 0, & x < \phi \\ & \lambda\alpha(x-\phi)^{(\alpha-1)}, & x \geq \phi \end{array}$$

(b) Mean lifetime = 233.3333

Median lifetime = 192.4196

2.13 (a)
$$\begin{array}{rll} S(x) = & 1, & x < 1 \\ & 1-p, & 1 \leq x < 2 \\ & \vdots & \\ & (1-p)^{i}, & i \leq x < i+1, i = 1, 2, \ldots \end{array}$$

(b)
$$\begin{array}{ll} h(x) = p, & x = 1.2, \ldots \\ 0, & \text{elsewhere} \end{array}$$

For both the exponential and the geometric distributions the hazard rate is constant and is equal to the reciprocal of the mean.

2.15 (a) $S(x) = \exp[-x(\alpha + \beta x/2)]$

(b) $f(x) = (\alpha + \beta x)\exp[-x(\alpha + \beta x/2)]$

2.17 (a) $E(X) = 10$

(b) $h(x) = 2/(x + 10)$

(c) $S(x) = 100/(x + 10)^2$

2.19 (a) $S_X(x) = 1 - x, 0 < x < 1$

(b) $\text{CI}(x) = x - .75x^2 + .5x^3 - .25x^4, 0 < x < 1$

Solutions to Chapter 3

3.1 (a) Generalized Type I right censoring

(b) Generalized Type I right censoring

3.3 (a) Left censoring at 42 days

(b) Type I right censoring at 140 days

(c) Interval censoring in 84–91 days

(d) Random right censoring at 37 days

(e) $L \propto [1 - S(42)]S(140)[S(84) - S(91)]S(37)$

3.5 From (3.5.1) the likelihood has the form

$$L \propto \left[\frac{\alpha\lambda(0.5)^{\alpha-1}}{(1+\lambda(0.5)^{\alpha})^2}\right]\left[\frac{\alpha\lambda(1)^{\alpha-1}}{(1+\lambda)^2}\right]\left[\frac{\alpha\lambda(0.75)^{\alpha-1}}{(1+\lambda(0.75)^{\alpha})^2}\right]$$
$$\left[1-\frac{1}{1+\lambda(0.25)^{\alpha}}\right]\left[1-\frac{1}{1+\lambda(1.25)^{\alpha}}\right]$$

3.7 (a) First 4 observations are interval-censored
Last 4 observations are Type I right-censored

(b) $L \propto [\exp(-\lambda 55^{\alpha}) - \exp(-\lambda 56^{\alpha})][\exp(-\lambda 58^{\alpha}) - \exp(-\lambda 59^{\alpha})]$
$[\exp(-\lambda 52^{\alpha}) - \exp(-\lambda 53^{\alpha})][\exp(-\lambda 59^{\alpha}) - \exp(-\lambda 60^{\alpha})]$
$[\exp(-\lambda 60^{\alpha})]^4$

Solutions to Chapter 4

4.1 (a) $\hat{S}(12) = 0.9038$, $SE(\hat{S}(12)) = 0.0409$, $\hat{S}(60) = 0.6538$, $SE(\hat{S}(60)) = 0.0660$

(b) $\hat{H}(60) = 0.4178 \quad SE(\hat{H}(60)) = 0.0992$
$\exp\{-\hat{H}(60)\} = 0.6585$

(c) (0.5245, 0.7832)

(d) (0.5083, 0.7659)

(e) (0.5205, 0.7759)

(f, g)

Time	*EP Band (f)*	*Hall–Wellner Band (g)*
36–41	(0.5060–0.8200)	(0.5014–0.8221)
41–51	(0.4866–0.8045)	(0.4855–0.8050)
51–65	(0.4674–0.7887)	(0.4692–0.7878)
65–67	(0.4281–0.7724)	(0.4520–0.7699)
67–70	(0.4089–0.7558)	(0.4345–0.7520)
70–72	(0.3900–0.7389)	(0.4169–0.7340)

(h) $\hat{\mu} = 146.6$ Confidence interval: (92.4, 200.9)

(i) Median = 93 Confidence interval: (67–157)

4.3 Solution to parts a and c

	Part a		*Part b*	
Time	*Estimated Survival*	*Standard Error*	*Estimated Survival*	*Standard Error*
0–22	1.0000	0.0000	1.0000	0.0000
22–27	0.9600	0.0392	0.9600	0.0392
27–50	0.9200	0.0543	0.9200	0.0543
50–68	0.8800	0.0650	0.8800	0.0650
68–99	0.8400	0.0733	0.8400	0.0733
99–101	0.8400	0.0733	0.8400	0.0733
101–108	0.8400	0.0733	0.8000	0.0800
108–121	0.8400	0.0733	0.8000	0.0800
121–131	0.8400	0.0733	0.8000	0.0800
131–134	0.8400	0.0840	0.8000	0.0800
134–136	0.7906	0.0922	0.7600	0.0854
136–139	0.7412	0.0984	0.7200	0.0898
139–144	0.6918	0.1030	0.6800	0.0933
144–186	0.6424	0.1030	0.6400	0.0960
186–191	0.6424	0.1030	0.6400	0.0960
191–198	0.6424	0.1030	0.6000	0.0980
198–203	0.6424	0.1030	0.6000	0.0980
203–210	0.5840	0.1090	0.5600	0.0993
210–217	0.5256	0.1126	0.5200	0.0999
217–224	0.5256	0.1126	0.5200	0.0999
224–231	0.5256	0.1126	0.4800	0.0999
231–248	0.5256	0.1126	0.4800	0.0999
248–256	0.4505	0.1190	0.4400	0.0993
256–290	0.4505	0.1190	0.4000	0.0980
290–306	0.4505	0.1190	0.4000	0.0980
306–308	0.4505	0.1190	0.4000	0.0980
308–320	0.4505	0.1190	0.3600	0.0960
320–363	0.4505	0.1190	0.3600	0.0960
363–410	0.2252	0.1700	0.3200	0.0933
410–441	0.2252	0.1700	0.3200	0.0933
441–482	0.2252	0.1700	0.2800	0.0898
482–511	0.2252	0.1700	0.2400	0.0854
511–559	0.2252	0.1700	0.2000	0.0800
559–561	0.2252	0.1700	0.1600	0.0733
561–580	0.2252	0.1700	0.1200	0.0650
580–683	0.2252	0.1700	0.0800	0.0543
683–724	0.2252	0.1700	0.0400	0.0392
724 infinity	0.2252	0.1700	0.0000	0.0000

(b) For $t > 363$, estimate $S(t)$ by $\exp\{-0.004t\}$

(d) $\hat{u} = 312.3$, $SE = 70.9$ days

(e) $\hat{\mu} = 294.6$, $SE = 42.7$

(f) $\bar{x} = 294.6, s = 213.4, SE = 42.7$

4.5 (a)

Time in Days	*Estimated Survival*	*Standard Error*
0–40	1.0000	0.0000
40–45	0.9828	0.0171
45–106	0.9655	0.0240
106–121	0.9480	0.0293
121–229	0.9297	0.0339
229–344	0.9115	0.0378
344–864	0.8912	0.0421
864–929	0.8672	0.0473
929–943	0.8383	0.0539
943–1016	0.8093	0.0592
1,016–1,196	0.7782	0.0646
1,196–2,171	0.7458	0.0696
2,171–2,276	0.6884	0.0846
2,276–2,650	0.6258	0.0974
>2,650	0.5364	0.1176

(b) $(0.8087, 0.9737)$

(c) $(0.7731, 0.9497)$

(d) $(0.7960, 0.9497)$

4.7

		Left Truncated			*No Truncation*	
Time	*Y*	$S(t \mid \text{Alive at } 60)$	$S(t \mid \text{Alive at } 70)$	*Y*	$S(t \mid \text{Alive at } 60)$	$S(t \mid \text{Alive at } 70)$
58	2	1.0000	1.0000	30	1.0000	1.0000
59	3	1.0000	1.0000	30	1.0000	1.0000
60	5	0.8000	1.0000	30	0.9667	1.0000
61	6	0.8000	1.0000	30	0.9667	1.0000
62	9	0.7111	1.0000	29	0.9333	1.0000
63	10	0.6400	1.0000	28	0.9000	1.0000
64	10	0.6400	1.0000	28	0.9000	1.0000
65	10	0.5120	0.8000	27	0.8333	0.9259
66	10	0.4608	0.7200	25	0.8000	0.8889
67	12	0.4608	0.7200	25	0.8000	0.8889
68	13	0.3899	0.6092	24	0.7333	0.8148
69	14	0.3342	0.5222	22	0.6667	0.7407
70	13	0.2828	0.4419	18	0.5926	0.6584
71	12	0.2357	0.3682	16	0.5185	0.5761
72	12	0.1964	0.3068	14	0.4444	0.4938
73	11	0.1785	0.2790	11	0.4040	0.4489
74	9	0.1587	0.2480	9	0.3591	0.3991
76	7	0.1360	0.2125	7	0.3078	0.3420
77	5	0.1088	0.1700	5	0.2463	0.2736
78	4	0.1088	0.1700	4	0.2463	0.2736
79	3	0.1088	0.1700	3	0.2463	0.2736
80	1	0.1088	0.1700	1	0.2463	0.2736

4.9 Parts a,b

	Thymic Lymphoma		*Reticulum Cell Sarcoma*		*Other Causes*		*Overall Survival:*
Time	*CI*	*KME*	*CI*	*KME*	*CI*	*KME*	*KME*
200	0.0633	0.0641	0.0000	0.0000	0.0127	0.0127	0.9241
300	0.2025	0.2061	0.0000	0.0000	0.0380	0.0433	0.7595
400	0.2278	0.2326	0.0000	0.0000	0.0506	0.0598	0.7215
500	0.2785	0.2880	0.0127	0.0182	0.0633	0.0766	0.6456
600	0.3038	0.3159	0.0253	0.0386	0.0759	0.0954	0.5949
700	0.3165	0.3314	0.1139	0.2082	0.1899	0.2827	0.3797
800	0.3418	0.3877	0.1646	0.3358	0.2658	0.4397	0.2278
900	0.3418	0.3877	0.1772	0.3773	0.4051	0.8008	0.0759
1,000	0.3418	0.3877	0.1899	0.5849	0.4430	0.9004	0.0253

(c) The Kaplan–Meier estimator (KME) for thymic lymphoma estimates the probability of having died from this cause in a hypothetical world where no other cause of death is possible.

(d) At 500 days we have 0.0759, at 800 days we have 0.6000. These are estimates of the conditional probability of dying from thymic lymphoma among survivors who have not died from one of the other two causes of death.

Solutions to Chapter 5

5.1

Age in Years	*Estimated Survival Function*
0–14	1.000
14–15	0.984
15–16	0.961
16–17	0.882
17–18	0.780
18–19	0.741
19–20	0.717
20–21	0.702
21–22	0.693
22–23	0.667
23–26	0.644
>26	0.515

5.3

Time	*Estimated Survival Function*
0–9	1.000
9–24	0.889
24–36	0.694
36–42	0.580
42–60	0.435
>60	0.000

5.5

T_i	X_i	R_i	d_i	Y_i	$P[X \le x_i \mid X \le 42]$
2	30	12	1	10	0.9000
4	27	15	1	13	0.8308
7	25	17	1	14	0.7714
14	19	23	1	17	0.7261
20	18	24	1	16	0.6807
18	17	25	1	16	0.6381
8	16	26	2	16	0.5584
13	16	26			
17	15	27	3	14	0.4387
26	15	27			
20	15	27			
15	13	29	2	11	0.3589
23	13	29			
5	12	30	1	9	0.3191
16	11	31	1	8	0.2792
15	9	33	1	8	0.2443
11	8	34	3	7	0.1396
6	8	34			
33	8	34			
4	7	35	1	5	0.1117
8	6	36	3	5	0.0447
35	6	36			
10	6	36			
36	4	38	2	2	0.0000
25	4	38			

5.7

1 *Time*	*2* Y'_j	*3* W_j	*4* Y_j	*5* d_j	*6* $\hat{S}(a_j)$	*7* $\hat{f}(a_{mj})$	*8* $\hat{h}(a_{mj})$	*9* $SE(\hat{S})$	*10* $SE(\hat{f}(a_{mj}))$	*11* $SE(\hat{h}(a_{mj}))$
45–50	1571	29	1556.5	29	1.0000	0.0022	0.0022	0	0.0005	0.0005
50–55	1525	60	1495.0	60	0.9891	0.0048	0.0049	0.0026	0.0008	0.0008
55–60	1429	83	1387.5	83	0.9653	0.0085	0.0091	0.0047	0.0011	0.0012
60–65	1284	441	1063.5	441	0.9221	0.0132	0.0148	0.0070	0.0015	0.0017
65–70	767	439	547.5	439	0.8562	0.0156	0.0192	0.0098	0.0021	0.0027
70–75	278	262	147.0	262	0.7780	0.0031	0.0126	0.0138	0.0031	0.0042
75–80	7	7	3.5	7	0.7304	0	0	0.0201		

5.9

t	$\hat{S}(t)$
0–46	1.000
46–49	0.877
49–54	0.754
54–61	0.631
61–62	0.508
62–64	0.385
64–68	0.308
68–120	0.231
120–150	0.154
150–160	0.077
> 160	0.000

Solutions to Chapter 6

6.1 (a) $\hat{h} = 0.0369, SE = 0.0144$

(b) $\hat{h} = 0.0261, SE = 0.0142$

(c) $\hat{h} = 0.0258, SE = 0.0154$

(d) Uniform Kernel $\hat{h} = 0.0223, SE = 0.0100$

Epanechnikov Kernel $\hat{h} = 0.0263, SE = 0.0122$

Biweight Kernel $\hat{h} = 0.0293, SE = 0.0142$

6.3 (a) $\hat{h} = 0.0161$ for Surgical Placement, $\hat{h} = 0.0229$ for Percutaneous Placement

(b) $\hat{h} = 0.0326$ for Surgical Placement, $\hat{h} = 0.0014$ for Percutaneous Placement

6.5 (a, b)

time	$B(t)$	Se$[B(t)]$
28	182	182
32	382	270
49	604	350
84	854	430
357	1140	516

(c)

time	Nelson-Aalen	$\Theta(t)$	$A(t)$
28	0.091	0.014	0.077
32	0.191	0.016	0.175
49	0.302	0.023	0.279
84	0.427	0.036	0.392
357	0.570	0.122	0.448
933	0.570	0.279	0.290
1078	0.570	0.312	0.257
1183	0.570	0.332	0.238
1560	0.570	0.383	0.187
2114	0.570	0.433	0.137
2144	0.570	0.435	0.135

6.7

Time	Estimate Using Dirichlet Prior	Estimate Using Beta Prior	Kaplan-Mejer Estimate	Prior
20	0.868	0.852	1.000	0.670
40	0.580	0.634	0.667	0.449
60	0.454	0.448	0.556	0.301

Solutions to Chapter 7

7.1 Log rank $\chi^2 = 6.03$, p-value $= 0.03$

7.3 (a) Log rank $\chi^2 = 3.793$, p-value $= 0.052$

(b) Gehan $\chi^2 = 2.864$, p-value $= 0.09$

(c) Tarone–Ware $\chi^2 = 3.150$, p-value $= 0.08$

7.5 Log rank $\chi^2 = 22.763$, p-value < 0.0001

7.7 (a) Log rank $\chi^2 = 33.380$, p-value $=< 0.0001$

(b) Untreated vs. Radiation

Log rank $\chi^2 = 11.412$, p-value $= 0.0007$

Untreated vs. Radiation + BPA

Log rank $\chi^2 = 21.671$, p-value < 0.0001

Radiation vs. Radiation + BPA

Log rank $\chi^2 = 10.148$, p-value $= 0.0014$

The Bonferroni multiple comparison procedure would test at the $0.05/3 = 0.0167$ level. All pairwise comparisons are significant.

(c) Log rank trend test $\chi^2 = 30.051$, p-value < 0.0001

7.9 (a) Log rank $\chi^2 = 4.736$, p-value $= 0.19$
Gehan $\chi^2 = 3.037$, p-value $= 0.39$
(b) For males,
Log rank $\chi^2 = 0.097$, p-value $= 0.76$
Gehan $\chi^2 = 0.366$, p-value $= 0.55$
For females,
Log rank $\chi^2 = 4.847$, p-value $= 0.03$
Gehan $\chi^2 = 2.518$, p-value $= 0.11$

To test the hypothesis that blacks have a higher mortality rate than whites, after adjusting by stratification for sex, we get

$$Z = 1.064, \quad p\text{-value} = 0.30$$

7.11 Stratified $\chi^2 = 23.25$, p-value < 0.0001

Prior to 1975 log rank $\chi^2 = 12.00$, p-value $= 0.007$

1975 or later log rank $\chi^2 = 11.59$, p-value $= 0.009$

7.13 (a) Log rank $\chi^2 = 5.4943$, p-value $= 0.0195$
(b) $Q = 2.34$, p-value $= 0.04$
(c) $Q_1 = 1.10$, p-value $= 0.27$
(d) $W_{KM} = 116.22$, $Z = 1.90$, p-value $= 0.06$

7.15 Let treatment 1, 2, and 3 = ALL, AML-Low, and AML-High, respectively. Then

$Z_{12}(365) = -0.83,$ two-sided p-value $= 0.41$,

$Z_{13}(365) = 0.94,$ two-sided p-value $= 0.35$,

$Z_{23}(365) = 1.88,$ two-sided p-value $= 0.06$.

Solutions to Chapter 8

8.1 (a) $Z_1 = 1$ if HOD Allo patient, 0 otherwise
$Z_2 = 1$ if NHL Auto patient, 0 otherwise
$Z_3 = 1$ if HOD Auto patient, 0 otherwise
(b) Let
$Z_1 = 1$ if Auto patient, 0 otherwise
$Z_2 = 1$ if HOD patient, 0 otherwise
$Z_3 = Z_1 X Z_2$
(c) $\beta_1 = 1.5, \beta_2 = 2, \beta_3 = -3$

8.3 (a) Score test using Breslow method gives a p-value of 0.098.

(b) $b = -0.461$, $se(b) = 0.281$, $RR = 0.63$, 95% CI is $(0.36, 1.09)$

(c) p-value = 0.106

(d) p-value = 0.100

8.5 (a)

Testing Global Null Hypothesis

Test	*Chi-Square*	*DF*	*p-value*
Likelihood Ratio	7.89	3	0.048
Score	11.08	3	0.011
Wald	9.26	3	0.026

ANOVA Table

Variable	*DF*	*Parameter Estimate*	*Standard Error*	*Chi-Square*	*p-value*
HOD Allo	1	1.830	0.675	7.34	0.007
NHL Auto	1	0.664	0.564	1.38	0.24
HOD Auto	1	0.154	0.589	0.07	0.79

(b) The global tests are the same as in 8.5(a) above.

ANOVA Table

Variable	*DF*	*Parameter Estimate*	*Standard Error*	*Chi-Square*	*p-value*
Auto	1	0.664	0.564	1.38	0.24
HOD	1	1.830	0.675	7.34	0.007
Auto & HOD	1	−2.340	0.852	7.55	0.006

Likelihood ratio p-value = 0.007 and Wald p-value = 0.006. Thus we conclude that there is a significant interaction between disease type and transplant type.

(c) Either model gives a relative risk for an NHL Auto transplant patient to an NHL Allo transplant patient of 1.94 with a 95% confidence interval of $(0.64, 5.87)$.

(d) Comparing Allo patients gives a p-value of 0.007. Comparing Auto patients gives a p-value of 0.31.

(e) The test statistic is 8.50, which has a chi-square distribution with 2 degrees of freedom and the p-value is 0.014.

Note that the inferential conclusions in this entire problem do not depend upon the coding scheme employed.

8.7 (a) For the routine bathing care, the cut point is 25% of total surface area burned. For the chlorhexidine gluconate method, the cut point is 22% of total surface area burned.

(b) For routine bathing care method, $Q = .8080$, p-value > 0.30, $RR = 1.59$.

For chlorhexidine gluconate method, $Q = 1.3404$, p-value $= 0.055$, $RR = 2.31$.

(c) For routine bathing care method, $b = 0.007$, $SE = 0.009$, p-value $= 0.44$, $RR = 1.01$.

For chlorhexidine gluconate method, $b = 0.007$, $SE = 0.012$, p-value $= 0.54$, and $RR = 1.01$.

8.9 (a) Let Z_1 = type of disinfectant, Z_4 = % of surface area burned, p-value $= 0.056$.

(b) p-value $= 0.077$.

(c) Tests of hypothesis that the times to staphylococcus infection are the same for the two disinfectant groups adjusting for each of the listed factors in a separate model are shown below.

Tests for Z_1 adjusted for gender, p-value $= 0.040$; race, p-value $= 0.045$; area burned, p-value $= 0.077$; type of burn, p-value $= 0.045$.

(d) The final model along with the parameter estimates is given below. Although we have used Wald tests in this exercise, similar conclusions are obtained if the likelihood ratio statistic is used throughout.

Variable	*DF*	*Parameter Estimate*	*Standard Error*	*Chi-Square*	*p-value*	*Hazard Ratio*
Disinfectant	1	−0.601	0.298	4.07	0.044	0.55
Type of Burn*	3			7.94	0.047	
Scald	1	1.557	1.087	2.05	0.15	4.7
Electric	1	2.151	1.086	3.92	0.048	8.6
Flame	1	0.999	1.016	0.97	0.33	2.7
Race	1	2.269	1.025	4.90	0.027	9.7

*Chemical burn is referent group.

The local Wald test of the primary hypothesis of no difference between the times to staphylococcus infection for the two disinfectant groups has a p-value of 0.044, which suggests that the times to staphylococcus infection are different for the two disinfectant groups after adjustment for the type of the burn and race of the patient. Note that the test was insignificant when no cofounders were adjusted for in part a.

8.11 (a)

Test	*Chi-Square*	*DF*	*p-value*
Likelihood Ratio	16.58	1	$<.001$
Score	15.04	1	$<.001$
Wald	13.62	1	$<.001$

Variable	*DF*	*Parameter Estimate*	*Standard Error*	*Chi-Square*	*p-value*	*Hazard Ratio*
Z	1	-1.095	0.297	13.62	$<.001$	0.33

(b) Tests for Z adjusted for: mother's age p-value $< .001$; urban p-value $< .001$; alcohol p-value $< .001$; smoking status p-value $< .001$; region p-value $< .001$; birthweight p-value $< .001$; poverty p-value $< .001$; race p-value $< .001$; siblings p-value $< .001$.

(c) The final model along with the parameter estimates is given below.

Variable	*DF*	*Parameter Estimate*	*Standard Error*	*Chi-Square*	*p-value*	*Hazard Ratio*
Z	1	-0.882	0.303	8.49	0.004	0.4
Smoking status*	2			9.39	0.009	
$<$ 1 pack/day	1	0.751	0.256	8.60	0.003	2.1
$>$ 1 pack/day	1	0.632	0.349	3.28	0.070	1.9
siblings	1	0.387	0.124	9.77	0.002	1.5
mother's age	1	-0.121	0.050	5.88	0.015	0.9

*Referent group is nonsmokers.

8.13 At 20 days for a patient with 25% of the total body area burned, 95% confidence intervals for the survival functions based on the log transformation (4.3.2), for the two bathing solutions, are $(0.52, 0.75)$ and $(0.67, 0.85)$, respectively.

Solutions to Chapter 9

9.1 Using $g(t) = \log(t)$ the Wald p-value is 0.47. No evidence of a departure from proportional hazards.

9.3 (a) Relative Risk = 0.90, Confidence Interval = $(0.58, 1.39)$

(b) Using $g(t) = \log(t)$ the Wald p-value is 0.009.

(c) The best cut point is at 254 days.

(d) Up to 254 days the relative risk of Chemo only to Chemo+Radiation is 0.24 95% Confidence Interval (0.10, 0.56). Among 254-day survivors the relative risk is 1.89 (104, 3.44).

9.5 (a)

Variable	*DF*	*Parameter Estimate*	*Standard Error*	*Chi Square*	*Pr > Chi Square*
Stage II	1	0.112	0.464	0.06	0.81
Stage III	1	0.619	0.356	3.03	0.082
Stage IV	1	1.697	0.443	14.86	<0.001
Age	1	0.017	0.015	1.30	0.25

(b) From a model with common covariate values in each strata -2 LOG $L = 323.869$ is greater than a model with different covariates values in each strata -2 LOG $L = 320.806$, Chi Square = 3.06, $p = .55$.

(c) Chi Square = 2.82, $p = 0.59$

9.7 Step 1: Waiting time: df = 1, $p = 0.075$, FAB Class: df = 1, $p = 0.883$.

MTX: df = 1, $p = 0.091$. Sex: df = 3, $p = 0.752$. CMV: df = 3, $p = 0.049$.

Age: df = 3, $p = 0.17$.

⇒ Add CMV.

Step 2: Waiting time: $p = 0.26$. FAB Class: $p = 0.67$.

MTX: $p = 0.062$. Sex: $p = 0.46$. Age: $p = 0.29$.

Final Model:

Variable	*DF*	*Parameter Estimate*	*Standard Error*	*Chi-Square*	*Pr > Chi Square*	*Hazard Ratio*
AML low-risk ($Z1$)	1	0.343	0.658	0.27	0.60	1.41
AML high-risk ($Z2$)	1	2.136	0.884	5.84	0.016	8.47
Donor CMV positive ($Z9$)	1	1.764	0.778	5.14	0.023	5.83
Patient CMV positive ($Z10$)	1	0.401	0.955	0.18	0.67	1.49
Both CMV positive ($Z11$)	1	−2.530	1.346	3.53	0.060	0.08

Solutions to Chapter 10

10.1 (a) Let $Z = 1$ if Allo, 0 if Auto, and model $h(t \mid Z) = \beta_0(t) + \beta_1(t)$. Estimation restricted to 1,345 days.

Time	$B_1(t)$	*Standard Error*
0–2	0	0
2–4	0.0625	0.0625
4–28	0.1292	0.0914
28–30	0.2006	0.1160
30–32	0.1636	0.1218
32–36	0.2405	0.1440
36–41	0.2020	0.1491
41–42	0.1620	0.1543
42–49	0.1204	0.1599
49–52	0.2037	0.1803
52–53	0.1602	0.1855
53–57	0.1148	0.1909
57–62	0.0671	0.1968
62–63	0.0171	0.2030
63–72	−0.0355	0.2098
72–77	0.0554	0.2286
77–79	0.1554	0.2495
79–81	0.2665	0.2731
81–84	0.1554	0.2842
84–108	0.2804	0.3105
108–132	0.2179	0.3167
132–140	0.1512	0.3237
140–252	0.0798	0.3314
252–357	−0.0111	0.3437
357–524	0.1318	0.3722
524–1345	−0.0682	0.4225

(b) $U = 3.37$, $V(U) = 117.74$, chi-square $= 0.96$, $p = 0.7560$.

(c) Inference is restricted to 0–79 days.

	Disease		*Type of Transplant*		*Interaction*	
time	$B_1(t)$	*SE*	$B_2(t)$	*SE*	$B_3(t)$	*SE*
0–2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2–4	0.0000	0.0000	0.2000	0.2000	−0.2000	0.2000
4–28	0.0000	0.0000	0.4500	0.3000	−0.4500	0.3202
28–30	0.0000	0.0000	0.4500	0.3202	−0.3591	0.3328
30–32	−0.0667	0.0667	0.3833	0.3202	−0.2924	0.3394
32–36	−0.0667	0.0067	0.3833	0.3270	−0.1924	0.3538
36–41	−0.1381	0.0977	0.3119	0.3270	−0.1210	0.3610
41–42	−0.2150	0.1244	0.2350	0.3347	−0.0441	0.3691
42–49	−0.1317	0.1497	0.2350	0.3435	−0.1274	0.3784
49–52	−0.1317	0.1497	0.2350	0.3435	−0.1063	0.3944
52–53	−0.2150	0.1713	0.1516	0.3435	0.0670	0.4031
53–57	−0.1241	0.1940	0.1516	0.3534	−0.0239	0.4132
57–62	−0.0241	0.1282	0.1516	0.3534	−0.1239	0.4251
62–63	−0.1150	0.2364	0.0607	0.3649	−0.0330	0.4347
63–72	−0.0039	0.2612	0.0607	0.3649	−0.1441	0.4487
72–77	−0.0039	0.2612	0.3941	0.4943	−0.4774	0.5590
77–79	−0.0039	0.2612	0.8941	0.7031	−0.9744	0.7500
79	−0.0039	0.2612	1.8941	1.2224	−1.9774	1.2500

Analysis of Variance (Using number at risk as weights)

Effect	*Chi-Square*	*df*	*p-value*
Disease	0.6587	1	0.4170
Type of transplant	0.0113	1	0.9153
Interaction	6.6123	1	0.0101

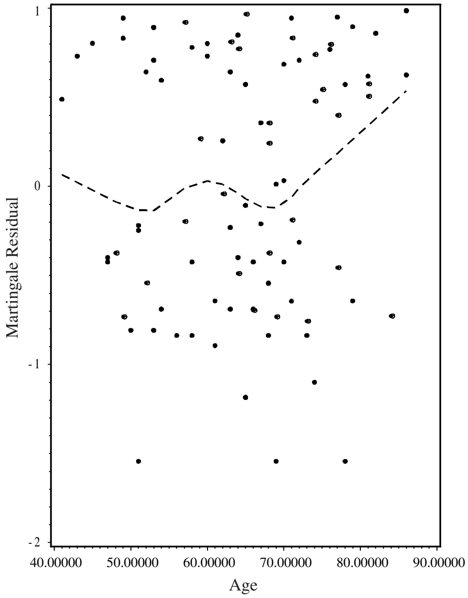
10.3 (a) $b = -0.00023$, SE $= 0.00064$, $\chi^2 = 0.129$, $p = 0.7193$.

(b)

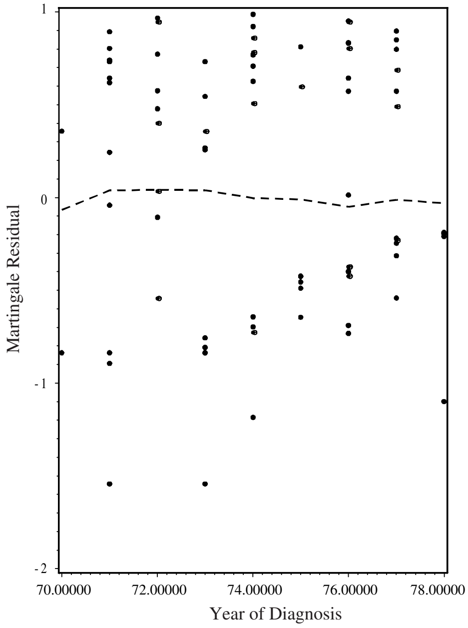
Effect	α	*Standard Error*	*Chi-Square*	*df*	*p-value*
Type of Transplant	0.0012	0.0010	1.38	1	0.2390
Disease	0.0170	0.0097	3.09	1	0.0786
Interaction	−0.0182	0.0098	3.45	1	0.0632

Solutions to Chapter 11

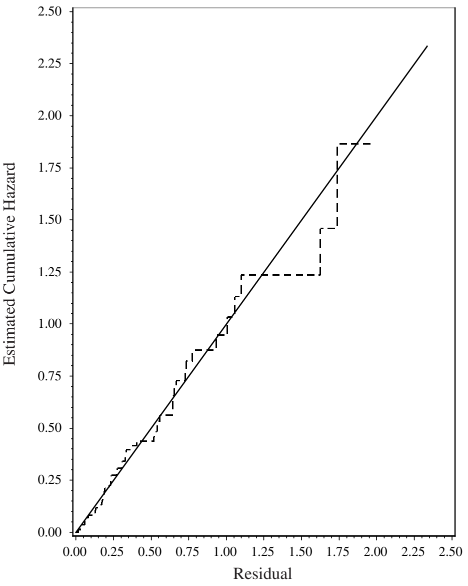
11.1 (a) From the Martingale residual plot a quadratic or a threshold model (see section 8.6) is suggested.



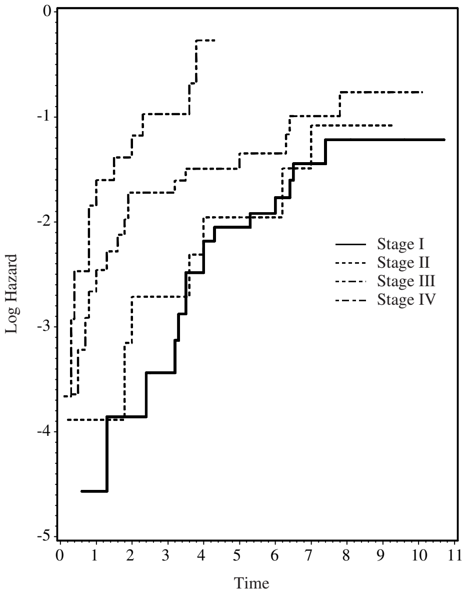
(b) The covariate year of transplant seems to enter the model as a linear term. The plot suggests its regression coefficient is not significantly different from zero.

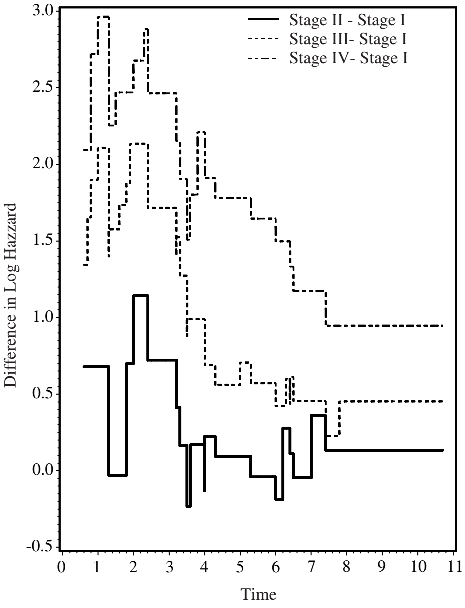


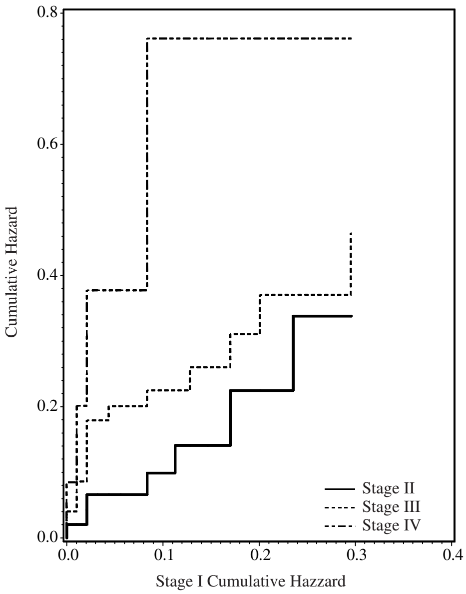
(c) The model seems to fit the data well.

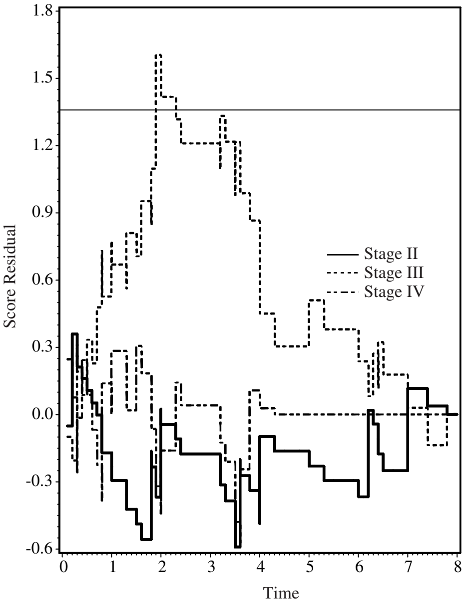


11.3 All four plots seem to suggest that proportional hazards is suspect for Stage IV as compared to Stage I.



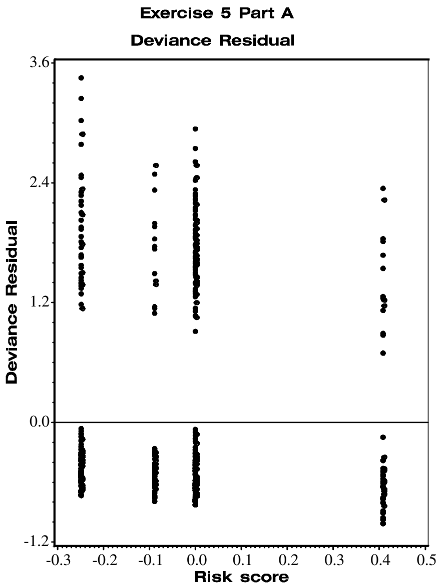




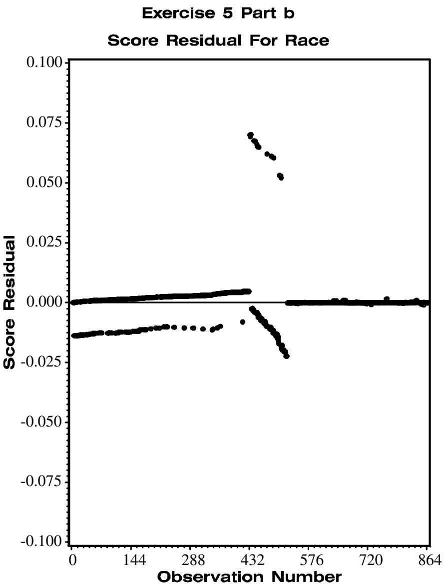


11.5 (a) The deviance residual suggests that there are a number of data points for which the model does not fit well. These are points with a deviance residual greater than 2. The worst three points are observations 529, 527, 526, which all have a risk score of -0.248. These patients, who the model predicts are good-risk patients, die very early at 2, 3, and 7 days after transplant. Most of the "outliers" are of this nature, patients who died too soon after transplant. A possible remedy is to add additional covariates which could help to explain these early deaths.

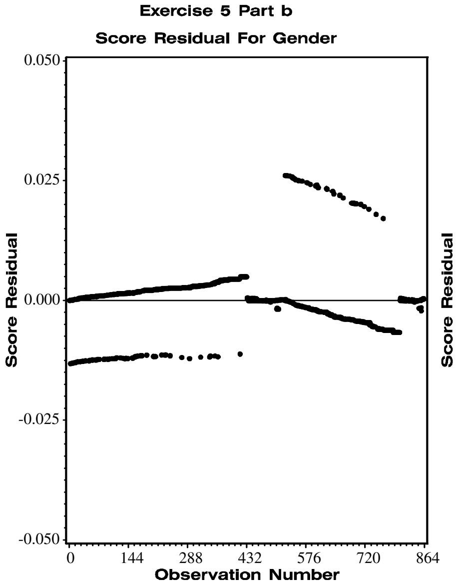
(b) For race these are observations 444, 435, 434, and 433. All are the four black males with the shortest survival time. For gender these are observations 532, 529, 527, and 526, the four white females who die the soonest after transplant. For the interaction term these are observations 812, 809, 807, and 806, the four black females with the shortest survival time.


Exercise 5 Part A

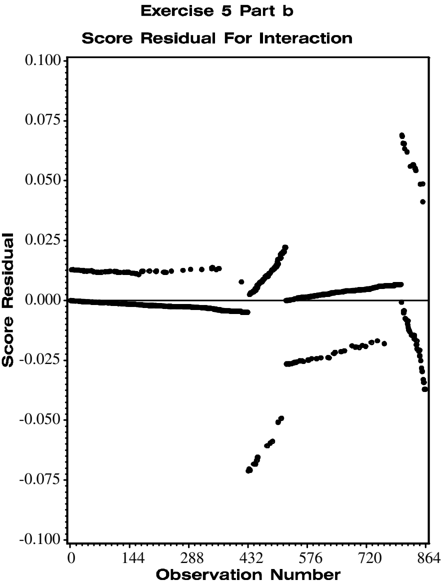
Deviance Residual


Exercise 5 Part b

Score Residual For Race


Exercise 5 Part b

Score Residual For Gender


Exercise 5 Part b

Score Residual For Interaction

Solutions to Chapter 12

12.1 (a) For aneuploid group, $\hat{\lambda} = 0.016$, $\hat{\alpha} = 0.832$ with standard errors 0.010 and 0.128, respectively, and for diploid group, $\hat{\lambda} = 0.036$, $\hat{\alpha} = 0.775$ with standard errors 0.023 and 0.136, respectively.

(b) For aneuploid group, L.R. p-value is 0.21 and Wald p-value is 0.27; for diploid group, L.R. p-value is 0.12 and Wald p-value is 0.19.

(c) MLE of median of aneuploid and diploid groups, respectively, are 91.8 and 45.8. SE of median of aneuploid and diploid groups, respectively, are 19.8 and 17.6.

(d) LR and Wald p-values are 0.059 and 0.057, respectively. Estimate of RR is 0.58 and the 95% confidence interval is $(0.34, 1.01)$. Estimate of the acceleration factor is 0.51 and the 95% confidence interval is $(0.26, 1.02)$. This means that the median lifetime for diploid patients is between 0.26 and 1.02 times that of aneuploid patients with 95% confidence.

12.3 (a) LR global test p-value $= 0.004$.

ANOVA Table for $\hat{\mu}$, $\hat{\sigma}$, and $\hat{\gamma}_i$

Variable	*df*	*Parameter Estimate*	*Standard Error*	*Chi-Square*	*p-Value*
Intercept ($\hat{\mu}$)	1	7.831	0.753		
Scale ($\hat{\sigma}$)	1	1.653	0.277		
Auto ($\hat{\gamma}_i$)	1	−2.039	0.930	4.81	0.028
Hod ($\hat{\gamma}_2$)	1	−4.198	1.067	15.48	<0.001
Auto by hod ($\hat{\gamma}_3$)	1	5.358	1.377	15.14	<0.001

ANOVA Table for $\hat{\lambda}$, $\hat{\alpha}$, and $\hat{\beta}_i$

Variable	*df*	*Parameter Estimate*	*Standard Error*	*Chi-Square*	*p-Value*
Intercept ($\hat{\lambda}$)	1	0.009	0.007		
Scale ($\hat{\alpha}$)	1	0.605	0.101		
Auto ($\hat{\beta}_1$)	1	1.233	0.574	4.61	0.032
Hod ($\hat{\beta}_2$)	1	2.539	0.699	13.2	<0.001
Auto by hod ($\hat{\beta}_3$)	1	−3.241	0.878	13.6	<0.001

(b) p-value < 0.001.

(c) $RR = 3.4$ and 95% confidence interval for RR is $(1.1, 10.6)$.

(d) p-value < 0.001 and 0.17, respectively.

(e) Using contrast matrix $C = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, p-value < 0.001.

12.5 (a) For aneuploid group, $\hat{\lambda} = 0.009$, $\hat{\alpha} = 1.048$ with standard errors 0.007 and 0.163, respectively, and for diploid group, $\hat{\lambda} = 0.022$, $\hat{\alpha} = 1.035$ with standard errors 0.016 and 0.181, respectively.

(b) p-value > 0.6 for both aneuploid and diploid groups.

(c) MLE of median of aneuploid and diploid groups, respectively, are 87.2 and 39.6. SE of median of aneuploid and diploid groups, respectively, are 21.2 and 12.9.

(d) LR and Wald p-values are both 0.051. Estimate of relative odds is 0.44 and the 95% confidence interval is $(0.19, 1.01)$. Estimate of the acceleration factor is 0.45 and the 95% confidence interval is $(0.21, 1.002)$.

12.7 (a) For aneuploid group $\hat{\mu} = 4.46$, $\hat{\sigma} = 1.72$, and var-cov matrix for $\hat{\mu}$ and $\hat{\sigma}$ is

$$\begin{array}{rr} 0.07 & 0.02 \\ 0.02 & 0.06 \end{array}$$

For diploid group $\hat{\mu} = 3.64$, $\hat{\sigma} = 1.634$ and var-cov matrix for $\hat{\mu}$ and $\hat{\sigma}$ is

$$\begin{array}{rr} 0.10 & 0.01 \\ 0.01 & 0.07 \end{array}$$

(b) For aneuploid group $\hat{\mu} = 112.21$, $\hat{\sigma} = 96.71$ and var-cov matrix for $\hat{\mu}$ and $\hat{\sigma}$ is

$$\begin{array}{rr} 233.22 & 61.47 \\ 61.47 & 169.44 \end{array}$$

For diploid group $\hat{\mu} = 71.32$, $\hat{\sigma} = 70.67$ and var-cov matrix for $\hat{\mu}$ and $\hat{\sigma}$ is

$$\begin{array}{rr} 194.27 & 19.89 \\ 19.89 & 121.03 \end{array}$$

(c) For aneuploid group $\hat{\mu} = 4.75$, $\hat{\sigma} = 1.44$, $\hat{\theta} = 0.53$ and var-cov matrix for $\hat{\mu}$, $\hat{\sigma}$, and $\hat{\theta}$ is

$$\begin{array}{rrr} 0.15 & -0.09 & 0.19 \\ -0.09 & 0.15 & -0.20 \\ 0.19 & -0.20 & 0.38 \end{array}$$

For diploid group $\hat{\mu} = 3.86$, $\hat{\sigma} = 1.55$, $\hat{\theta} = 0.32$ and var-cov matrix for $\hat{\mu}$, $\hat{\sigma}$, and $\hat{\theta}$ is

$$\begin{array}{rrr} 0.40 & -0.14 & 0.45 \\ -0.14 & 0.13 & -0.20 \\ 0.45 & -0.20 & 0.65 \end{array}$$

(d) For aneuploid and diploid group the Wald p-values for testing $\hat{\theta} = 0$ is 0.39 and 0.69, respectively. Cannot reject log normal fit for both groups.

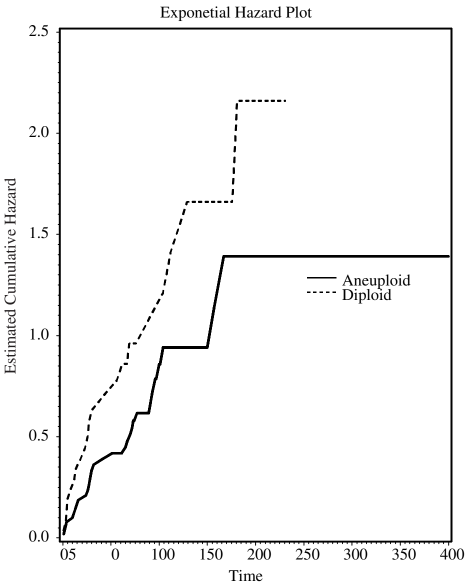
(e) For aneuploid and diploid group the Wald p-values for testing $\hat{\theta} = 1$ are 0.75 and 0.70, respectively. Cannot reject Weibull fit for both groups.

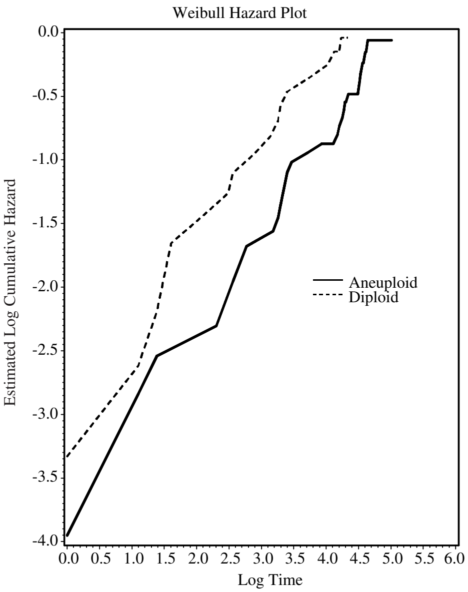
(f)

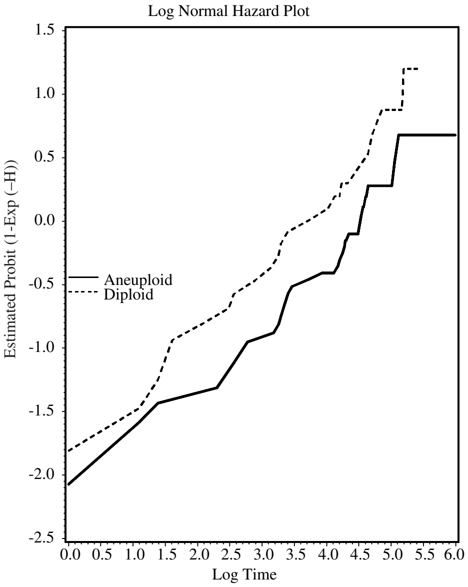
		Aneuploid	*Diploid*
Exponential	Log Likelihood	−77.14	−48.59
	AIC	156.28	99.18
Weibull	Log Likelihood	−76.36	−47.40
	AIC	156.72	98.80
Log logistic	Log likelihood	−76.09	−47.51
	AIC	156.18	99.02
Log normal	Log Likelihood	−76.42	−47.15
	AIC	156.84	98.30
Generalized	Log Likelihood	−76.08	−47.08
gamma	AIC	158.16	100.16

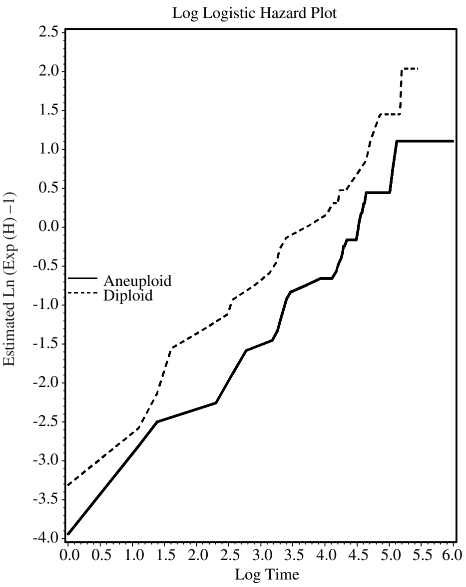
For aneuploid group log logistic is slightly better fit and for diploid group lognormal is slightly better fit.

12.9 The hazard plots, all of which should be linear, suggest that any of the models would be reasonable.

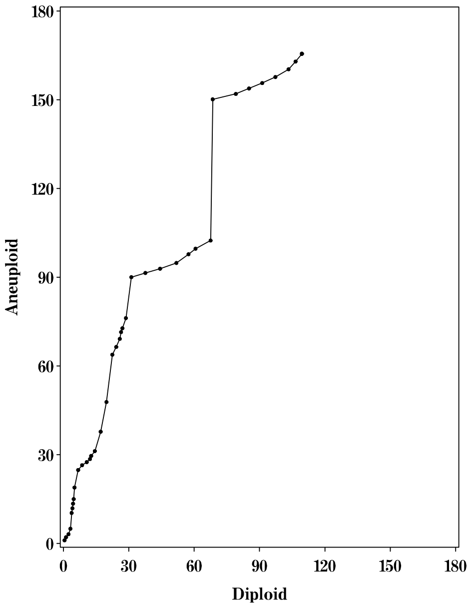




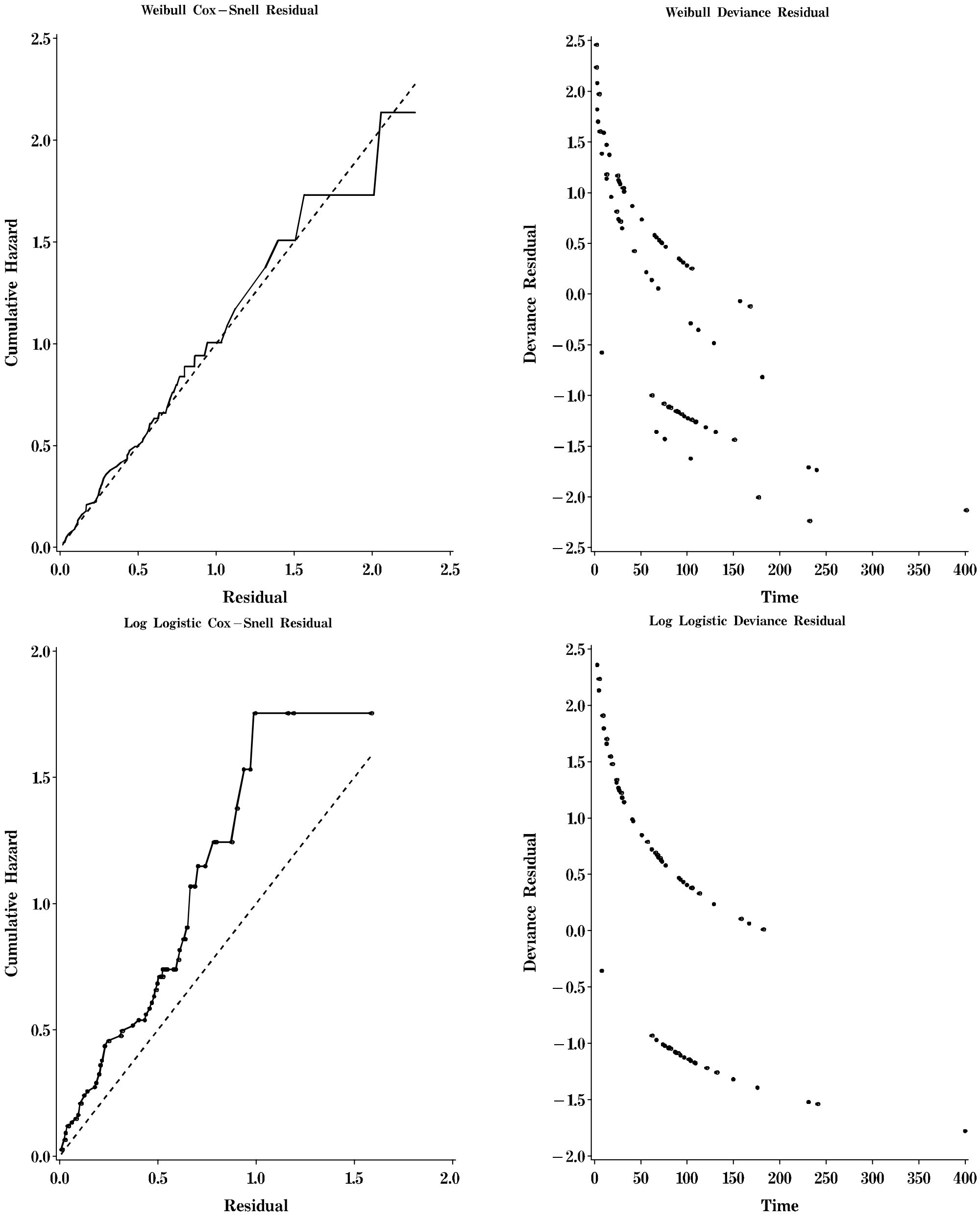




12.11 The plot appears to be roughly linear with a slope of about 3.5. The slope is a crude estimate of the acceleration function.



12.13 The model seems to fit well to the Weibull but is suspect for the log logistic model. Both Deviance residual plots suggest the models do not fit well to early events.



Solutions to Chapter 13

13.1 $T = 14.8$, $V = 107.5$, $Z = 1.4$, $p = 0.1539$. No evidence of random effect.

13.3 (a) Standard Cox model $b = -1.035$, SE $(b) = 0.44$, $p = 0.0187$.

(b) Gamma Frailty Model $b = -1.305$, SE $(b) = 0.528$, $p = 0.0133$.

Estimate of $\theta = 0.713$, SE $= 0.622$, Wald p-value of test of $\theta = 0 = 0.2517$, likelihood ratio p-value $= 0.1286$.

13.5 (a) See 13.1.

(b) Adjusted SE $= 0.3852$, test statistic $= -1.035/0.3852$, $p = .0072$.

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