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# Nonlinear elliptic systems with variable exponents and measure data

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Abstract. In this paper we prove existence results for distributional solutions of nonlinear elliptic systems with a measure data. The functional setting involves Lebesgue-Sobolev spaces as well as weak Lebesgue (Marcinkiewicz) spaces with variable exponents $W_0^{1,p(\cdot)}(\Omega)$ and $M^{p(\cdot)}(\Omega)$ respectively.

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## 1. Introduction

Let $\Omega$ be a bounded open set in $\mathbb{R}^N$ $(N \geq 2)$ with Lipchitz boundary $\partial\Omega$. Our aim is to prove the existence of at least one distributional solution $u = (u_1, \ldots, u_m)^\top$ $(m \geq 1)$ to the nonlinear elliptic system

$$
\begin{aligned}
-\sum_{l=1}^{N} \frac{\partial}{\partial x_l}\sigma_l\left(x, \frac{\partial u}{\partial x_l}\right) &= \mu, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial\Omega,
\end{aligned}
\tag{1.1}
$$

where the right-hand side $\mu = (\mu_1, \ldots, \mu_m)^\top$ is a given vector-valued Radon measure on $\Omega$ of finite mass.

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We assume that the vector fields $\sigma_l : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$, $l = 1, \dots, N$, satisfy the following conditions concerning continuity, coercivity, growth, and strict monotonicity:

$$\begin{aligned} &\sigma_l(x,\xi) \text{ is measurable in } x \in \Omega \text{ for every } \xi \in \mathbb{R}^m \text{ and} \\ &\sigma_l(x,\xi) \text{ is continuous in } \xi \in \mathbb{R}^m \text{ for a.e. } x \in \Omega; \\ &\sigma_l(x,\xi) \cdot \xi \geq c_1 |\xi|^{p(x)} - c_2, \quad \forall (x,\xi) \in \Omega \times \mathbb{R}^m; \\ &|\sigma_l(x,\xi)| \leq c_1' |\xi|^{p(x)-1} + |h|, \quad \forall (x,\xi) \in \Omega \times \mathbb{R}^m,\ h \in L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Omega, \mathbb{R}^m); \end{aligned} \tag{1.2}$$

and for all $x \in \Omega$, and all $\xi, \xi' \in \mathbb{R}^m$,

$$(\sigma_l(x,\xi) - \sigma_l(x,\xi')) \cdot (\xi - \xi') \geq \begin{cases} c_3 |\xi - \xi'|^{p(x)}, & \text{if } p(x) \geq 2, \\ c_4 \frac{|\xi - \xi'|^2}{(|\xi| + |\xi'|)^{2-p(x)}}, & \text{if } 1 < p(x) < 2, \end{cases} \tag{1.3}$$

for some positive constants $c_1, c_2, c_1', c_3, c_4$. We assume that the variable exponents $p(\cdot)$ is continuous function on $\overline{\Omega}$ and

$$2 - \frac{1}{N} < p(x) < N, \quad \forall x \in \overline{\Omega}. \tag{1.4}$$

Note that this condition is classical (this can be found in [5] and [8] for single and system of elliptic equation, respectively, with constant exponents ($p(\cdot) = p$ constant)). Under the assumption (1.4), this work proves existence and regularity solutions for distributional solutions. Inspired by this work, we extend the result on the previous work of Dolzmann et al [8] to the nonlinear elliptic systems with a measure data and with variable exponents. The existence result and the method rely heavily on the paper [8]. To our knowledge, the system (1.1) with variable exponents is new and has never been studied before.

One of our motivations for studying (1.1) comes from applications to electro-rheological fluids as an important class of non-Newtonian fluids (sometimes referred to as smart fluids). The electro-rheological fluids are characterized by their ability to drastically change the mechanical properties under the influence of an external electromagnetic field. A mathematical model of electro-rheological fluids was proposed in [12, 13]. Other important application is related to image processing [6] where this kind of the diffusion operator is used to underline the borders of the distorted image and to eliminate the noise. We mention also that our space appears in the study of the elasticity [17] and of the calculus of variations with variable exponents [1]. The study of (1.1) is a new and interesting topic when the data is measure data. The scalar case (m=1) and $L^1$ or measure data, can be found in [4, 11]. If $m = 1$, $\mu \in L^1(\Omega)$, and under the additional hypothesis that the variable exponent $p(\cdot) > 1$ is log-Hölder continuous (2.1), similar results are established in [14] and references therein. We cite the papers ([2],[15],[16]) and references therein ($m = 1$), where other types of elliptic problems were also considered. The classical case $p(x) = p$ (a constant), was treated in [8] for the isotropic case (see also [3] for the anisotropic case). In this paper we will use the so-called (right-)angle condition:

$$\begin{aligned} &\forall x \in \Omega,\ \forall \xi \in \mathbb{R}^m, \text{ and } \forall a \in \mathbb{R}^m \text{ with } |a| \le 1, \\ &\sigma_l(x,\xi) \cdot [(I - a \otimes a)\,\xi] \ge 0, \quad l = 1, \dots, N, \end{aligned} \tag{1.5}$$

where $(I - a \otimes a)$ is the rank $m-1$ orthogonal projector onto the space orthogonal to the unit vector $a \in \mathbb{R}^m$. If $\sigma_{i,l}$, $i = 1, \dots, m$, denotes the components of the vector $\sigma_l$, then the angle condition can be stated more explicitly as

$$\sum_{i,j=1}^{m} \sigma_{i,l}(x,\xi)\xi_j(\delta_{i,j} - a_i a_j) \ge 0.$$

A prototype example that is covered by our assumptions is the $p(x)$-harmonic, system

$$-\sum_{l=1}^{N} \frac{\partial}{\partial x_l}\Big( \Big|\frac{\partial u}{\partial x_l}\Big|^{p(x)-2} \frac{\partial u}{\partial x_l}\Big) = \mu. \tag{1.6}$$

We would like to stress that the method used in the constant case [3, 8] cannot be applied here because the operator and nonlinearity are not homogeneous. The essentially difficulties introduced in extending the results of single equation to systems, is to obtain truncation estimate, since truncation behaves quite differently for scalar and vector functions. As the exponent which appear in (1.6) depends on the variable $x$, the functional setting involves Lebesgue and Sobolev spaces with variable exponent $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$. The existence of the solutions to the non-Newtonian fluids model with regular data (in $L^\infty$) will be the subject of a forthcoming paper (recall that there is no result for Navier-stokes equations with source and initial measure data).

In this paper, we prove the existence of a solution to (1.1) where the variable exponents $p(\cdot)$ is assumed to be merely continuous function. The proof is based on the usual strategy of deriving a priori estimates for a sequence of suitable approximate solutions $(u_\varepsilon)_{0<\varepsilon\le 1}$ (for which existence is straightforward to prove) and then to pass to the limit as $\varepsilon \to 0$.

The remaining part of this paper is organized as follows: Section 2 is devoted to mathematical preliminaries, including a brief discussion of Sobolev spaces with variable exponents. We also prove a weak Lebesgue space estimate that will be used later to obtain a priori estimates for our approximate solutions. The main existence result is stated and proved in Section 3. Finally, in Section 4 we discuss some extensions.

## 2. Mathematical preliminaries

We recall in what follows some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, where $\Omega$ is an open subset of $\mathbb{R}^N$. We refer to Fan and Zhao [9] for further properties of variable exponent Lebesgue-Sobolev spaces.

The continuous real-valued function $p : \overline{\Omega} \to [1, +\infty)$ satisfies the log-continuity if

$$\forall x, y \in \overline{\Omega}, \quad |x - y| < 1, \quad |p(x) - p(y)| < w(|x - y|), \tag{2.1}$$

where $\limsup_{\alpha\to 0^+} w(\alpha)\ln\left(\frac{1}{\alpha}\right) < +\infty$. Let $p^- = \min_{x\in\overline{\Omega}} p(x)$ and $p^+ = \max_{x\in\overline{\Omega}} p(x)$. We define the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable } \mid \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} = \inf\left\{ \mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1 \right\}.$$

The following inequality will be used later

$$\min\left\{ \|u\|_{p(\cdot)}^{p_-}, \|u\|_{p(\cdot)}^{p_+} \right\} \le \int_\Omega |u(x)|^{p(x)}\ dx \le \max\left\{ \|u\|_{p(\cdot)}^{p_-}, \|u\|_{p(\cdot)}^{p_+} \right\}. \tag{2.2}$$

If $p^- > 1$, then $L^{p(\cdot)}(\Omega)$ is reflexive and the dual space of $L^{p(\cdot)}(\Omega)$ can be identified with $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the Hölder type inequality

$$\left| \int_\Omega uv\, dx \right| \le \left( \frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)} \tag{2.3}$$

holds true.

We define also the variable Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) :\ |\nabla u| \in L^{p(\cdot)}(\Omega) \right\},$$

which is a Banach space under the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$$

We define also $W_0^{1,p(\cdot)}(\Omega) := \overline{C_c^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)}$. Assuming $p^- > 1$ the spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces.

**Remark 2.1.** Log-continuity condition (2.1) is used to obtain several regularity results for Sobolev spaces with variable exponents; in particular, $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega)$. Moreover, if $p$ satisfies the log-continuity (2.1) and $1 < p^- \le p^+ < N$, then the Sobolev embedding holds also (see e.g. [7] for more details) $W^{1,p(\cdot)}(\Omega) \subset L^{p^\star(\cdot)}(\Omega)$.

**Definition 2.1.** *Let $q(\cdot)$ be a measurable function such that $q_- > 0$. We say that a measurable function $u$ belongs to the Marcinkiewicz space $\mathcal{M}^{q(\cdot)}(\Omega)$ if there exists a positive constant $M$ such that*

$$\int_{\{|u|>t\}} t^{q(x)}\, dx \le M, \quad \text{for all}\, t > 0.$$

We remark that for $q(\cdot) = q$ constant this definition coincides with the classical definition of the Marcinkiewicz space $\mathcal{M}^q(\Omega)$.

**Lemma 2.1.** *Let $p(\cdot)$ be a continuous function on $\overline{\Omega}$ and $g$ a nonnegative function in $W_0^{1,p(\cdot)}(\Omega)$. Suppose $p(\cdot) < N$, and that there exists a constant $c$ such that*

$$\int_{\{g\leq\gamma\}} |\nabla g|^{p(x)}\, dx \leq c(\gamma+1), \quad \forall \gamma > 0. \tag{2.4}$$

*Then there exists a constant $C$, depending on $c$, such that*

$$\int_{\{|g|>t\}} t^{q(x)}\, dx \leq C, \quad \forall\, t > 0.$$

*for all continuous functions $q(\cdot)$ satisfying*

$$1 \leq q(x) < \frac{N(p(x)-1)}{N-p(x)}, \quad \forall\, x \in \overline{\Omega} \tag{2.5}$$

*Proof.* First, let $q^+$ be a constant satisfying

$$1 \leq q^+ < \min_{x\in\overline{\Omega}} \frac{N(p(x)-1)}{N-p(x)} = \frac{N(p^- - 1)}{N-p^-}. \tag{2.6}$$

Using the techniques of proof of in [8] for the constant case, we have

$$g \in \mathcal{M}^{\frac{N(p^- -1)}{N-p^-}}(\Omega), \tag{2.7}$$

and we can therefore conclude that

$$\|g\|_{\mathcal{M}^{q^+}(\Omega)} \leq C \quad \text{for all} \quad 1 \leq q^+ < \frac{N(p^- - 1)}{N-p^-},$$

for some $C > 0$. In particular, there exists a constant $C' > 0$ such that

$$\|g\|_{L^1(\Omega)} \leq C'. \tag{2.8}$$

Now let us consider a continuous variable exponent $q(\cdot)$ on $\overline{\Omega}$ satisfying the pointwise estimate (2.5). By the continuity of $p(\cdot)$ and $q(\cdot)$ on $\overline{\Omega}$ there exists a constant $\delta > 0$ such that

$$\max_{y\in\overline{B(x,\delta)\cap\Omega}} q(y) < \min_{y\in\overline{B(x,\delta)\cap\Omega}} \frac{N(p(y)-1)}{N-p(y)} \quad \text{for all} \quad x \in \overline{\Omega}. \tag{2.9}$$

Observe that $\overline{\Omega}$ is compact and therefore we can cover it with a finite number of balls $(B_i)_{i=1,\dots,k}$. Moreover, there exists a constant $\alpha > 0$ such that

$$\delta > |\Omega_i| > \alpha,\ \Omega_i = B_i \cap \Omega\ \forall\, i = 1,\dots,k. \tag{2.10}$$

We denote by $q_i^+$ (respectively $p_i^-$) the local maximum of $q$ on $\overline{\Omega}_i$ (respectively the local minimum of $p$ on $\overline{\Omega}_i$). By (2.4) and the fact that $p_i^- \leq p(x)$ on $\Omega_i$, we have for $\gamma > 0$

$$\int_{\Omega_i} |\nabla T_\gamma(g)|^{p_i^-}\, dx \leq C_4(\gamma+1), \quad i = 1,\dots,k. \tag{2.11}$$

In view of (2.8) and (2.10), we deduce

$$|\overline{T_\gamma(g)}_i| \leq \frac{C'}{\alpha}, \quad \overline{T_\gamma(g)}_i = \frac{1}{|\Omega_i|}\int_{\Omega_i} T_\gamma(g(x))dx.. \tag{2.12}$$

By Poincar-Wirtinger inequality and (2.12), we obtain

$$\|T_\gamma(g)\|_{L^{p_i^{-*}}(\Omega_i)} \;\leq\; C_5 + C_5\|\nabla T_\gamma(g)\|_{L^{p_i^-}(\Omega_i)}$$

so that, by (2.11), we get for all $\gamma \geq 1$:

$$\int_{\Omega_i} |T_\gamma(g)|^{p_i^{-*}} \;\leq\; C_6 + C_7\gamma^{\frac{p_i^{-*}}{p_i^-}} \leq C_8\gamma^{\frac{p_i^{-*}}{p_i^-}}.$$

Put

$$\lambda_g^i(\gamma) = |\{x \in \Omega_i \,:\, |g(x)| > \gamma\}|\,, \quad \gamma \geq 0, \quad i = 1, \ldots, k.$$

Taking into account that $|\lambda_g^i(\gamma)| \leq |\Omega|\gamma^{-\frac{N(p_i^- -1)}{N-p_i^-}}$ for all $\gamma \leq 1$, we get

$$\lambda_g^i(\gamma) \leq \gamma^{-p_i^{-\star}} \int_{\Omega_i} |T_\gamma(g)|^{p_i^{-\star}}\, dx \leq C_8\gamma^{-\frac{N(p_i^- -1)}{N-p_i^-}}, \quad \forall\gamma > 0,\ \forall\, i = 1, \ldots, k.$$

Finally, since $q_i^+ \geq q(x)$ for all $x \in \Omega_i$ and all $i = 1, \ldots, k$, the proof of Lemma 2.1 is completed. □

**2.1. Truncation function.** For any $\gamma > 0$, define the spherial (radially symmetric) truncation function $T_\gamma : \mathbb{R}^m \to \mathbb{R}^m$ by

$$T_\gamma(r) := \begin{cases} r, & \text{if } |r| \leq \gamma, \\ \frac{r}{|r|}\gamma, & \text{if } |r| > \gamma. \end{cases} \tag{2.13}$$

This function will be used repeatedly to derive a priori estimates for our approximate solutions. Observe that

$$DT_\gamma(r) = \begin{cases} I, & \text{if } |r| < \gamma, \\ \frac{\gamma}{|r|}\big(I - \frac{r\otimes r}{|r|^2}\big), & \text{if } |r| > \gamma. \end{cases}$$

In particular, (1.5) implies for all $\xi, r \in \mathbb{R}^m$ the crucial property

$$\sigma_l(x,\xi)\cdot DT_\gamma(r)\xi \geq \sigma_l(x,\xi)\cdot \xi\, \chi_{|r|<\gamma}, \quad l = 1, \ldots, N. \tag{2.14}$$

We refer to Landes [10] for a discussion of $T_\gamma$ and other test functions for elliptic systems, which indeed is a delicate issue.

## 3. Existence of a solution

### 3.1. Statement of main theorem.

**Definition 3.1.** A distributional solution of (1.1) is a vector-valued function $u : \Omega \to \mathbb{R}^m$ satisfying

$$u \in W_0^{1,1}(\Omega;\mathbb{R}^m), \quad \sigma_l\big(x, \frac{\partial u}{\partial x_l}\big) \in L^1(\Omega;\mathbb{R}^m), \quad l = 1, \ldots, N, \tag{3.1}$$

and for all $\varphi \in C_c^\infty(\Omega;\mathbb{R}^m)$,

$$\int_\Omega \sum_{l=1}^N \sigma_l\big(x, \frac{\partial u}{\partial x_l}\big) \cdot \frac{\partial \varphi}{\partial x_l}\, dx = \int_\Omega \varphi\, d\mu.$$

**Theorem 3.1.** *Suppose* (1.2)-(1.5) *hold. Let* $\mu = (\mu_1, \dots, \mu_m)^\top$ *be a Radon measure on* $\Omega$ *of finite mass. Then there exists at least one distributional solution* $u = (u_1, \dots, u_m)^\top$ *of* (1.1)*. Moreover,*

$$u \in \mathcal{M}^{q(\cdot)}(\Omega;\mathbb{R}^m), \quad \frac{\partial u}{\partial x_i} \in \mathcal{M}^{\widetilde{q}(\cdot)}(\Omega;\mathbb{R}^m),\ \forall i = 1, ..., N, \tag{3.2}$$

*for all continuous functions* $q(\cdot)$ *and* $\widetilde{q}(\cdot)$ *such that*

$$1 \le q(x) < \frac{N(p(x)-1)}{N-p(x)} \quad \text{and} \quad 1 \le \widetilde{q}(x) < \frac{N(p(x)-1)}{N-1},\ \forall x \in \overline{\Omega}. \tag{3.3}$$

**3.2. Approximate solutions.** Let $(f_\varepsilon)_{0<\varepsilon\le 1} \subset C_c^\infty(\Omega;\mathbb{R}^m)$ be a sequence defined by $f_\varepsilon = \mu\star\omega_\varepsilon$, where $\omega_\varepsilon(x) = \frac{1}{\varepsilon^N}\omega_0\big(\frac{x}{\varepsilon}\big) \ge 0$ and $\omega_0$ is a nonnegative function in $C_c^\infty(B(0,1))$ with $\int \omega_0\, dx = 1$. It is always understood that $\varepsilon$ takes values in a sequence in $(0,\infty)$ tending to zero. Clearly,

$$\begin{gathered} |f_\varepsilon| \le C(\varepsilon) \quad \text{and} \quad \int_\Omega |f_\varepsilon|\, dx \le |\mu|, \\ f_\varepsilon \overset{\star}{\rightharpoonup} \mu \text{ in the sense of measures as } \varepsilon \to 0. \end{gathered} \tag{3.4}$$

Then the result in [4], provide us with the existence of a sequence of functions

$$(u_\varepsilon)_{0<\varepsilon\le 1} \subset W_0^{1,p(\cdot)}(\Omega;\mathbb{R}^m),$$

each of them satisfying the weak formulation

$$\int_\Omega \sum_{l=1}^N \sigma_l\big(x, \frac{\partial u_\varepsilon}{\partial x_l}\big) \cdot \frac{\partial \varphi}{\partial x_l}\, dx = \int_\Omega f_\varepsilon \cdot \varphi\, dx, \quad \forall \varphi \in W_0^{1,p(\cdot)}(\Omega;\mathbb{R}^m). \tag{3.5}$$

Now the proof of Theorem 3.1 consists of two main steps. First, we prove $\varepsilon$-uniform estimates in weak Lebesgue spaces for $u_\varepsilon$ and $\nabla u_\varepsilon$. Second, we pass to the limit in (3.5) as $\varepsilon \to 0$.

**3.3. Uniform estimates.**

**Lemma 3.1.** *There exists a constant* $c$*, not depending on* $\varepsilon$*, such that*

$$\int_{\{|u_\varepsilon|\le\gamma\}} |\nabla u_\varepsilon|^{p(x)}\, dx \le c(\gamma+1), \quad \forall \gamma > 0. \tag{3.6}$$

*Proof.* Inserting $\varphi = T_\gamma(u_\varepsilon)$ into (3.5) gives

$$\int_\Omega \sum_{l=1}^N \sigma_l\big(x, \frac{\partial u_\varepsilon}{\partial x_l}\big) \cdot DT_\gamma(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x_l}\, dx = \int_\Omega f_\varepsilon \cdot T_\gamma(u_\varepsilon)\, dx.$$

Using (2.14) and the coercivity condition in (1.2), we obtain (3.6). □

**Lemma 3.2.** *There exists a constant $C$, not depending on $\varepsilon$, such that*

$$\int_{\{|u_\varepsilon|>t\}} t^{q(x)}\,dx \le C, \quad \forall\, t>0, \tag{3.7}$$

*and*

$$\int_{\{|\nabla u_\varepsilon|>t\}} t^{\widetilde{q}(x)}\,dx \le C, \quad \forall\, t>0. \tag{3.8}$$

*where the variable exponents $q(\cdot)$ and $\widetilde{q}(\cdot)$ are defined in* (3.3).

*Proof.* By Lemma 3.1 and $|\nabla|u_\varepsilon|| \le |\nabla u_\varepsilon|$ yield

$$\int_{\{|u_\varepsilon|\le\gamma\}} |\nabla|u_\varepsilon||^{p(x)}\,dx \le C_9(\gamma+1).$$

Applying Lemma 2.1 to $|u_\varepsilon|$ we obtain (3.7). For the proof of estimate (3.8), we start by the case

$$\widetilde{q}^{+} < \frac{N(p^- - 1)}{N-1}. \tag{3.9}$$

Following [8], we obtain

$$\lambda_{|\nabla u_\varepsilon|}(\alpha) = |\{x\in\Omega \,:\, |\nabla u_\varepsilon|>\alpha\}| \le C_{10}\alpha^{-\frac{N(p^- -1)}{N-1}}, \quad \alpha>0. \tag{3.10}$$

This proves that

$$\left\|\frac{\partial u_\varepsilon}{\partial x_i}\right\|_{\mathcal{M}^{\widetilde{q}^+}(\Omega;\mathbb{R}^m)} \le C_{11}, \quad i=1,\dots,N.$$

Now let us consider a continuous variable exponent $\widetilde{q}(\cdot)$ on $\overline{\Omega}$ satisfying only the pointwise estimate

$$1 \le \widetilde{q}(x) < \frac{N(p(x)-1)}{N-1}, \quad x\in\overline{\Omega}.$$

By the continuity of $p(\cdot)$ and $\widetilde{q}(\cdot)$ on $\overline{\Omega}$ there exists a constant $\delta'>0$ such that

$$\max_{y\in B(x,\delta')\cap\Omega} \widetilde{q}(y) < \min_{y\in B(x,\delta')\cap\Omega} \frac{N(p(y)-1)}{N-1} \quad \text{for all} \quad x\in\overline{\Omega}.$$

We can then cover $\overline{\Omega}$ with a finite number of balls still denoted by $(B_i)_{i=1,\dots,k}$ such that

$$\widetilde{q}_i^{+} = \max_{x\in\overline{\Omega_i}} \widetilde{q}(x) < \min_{x\in\overline{\Omega_i}} \frac{N(p(x)-1)}{N-1} = \frac{N(p_i^- - 1)}{N-1},\ \Omega_i = B_i\cap\Omega.$$

Arguing locally in (3.10), we obtain

$$|\{x\in\Omega_i : |\nabla u_\varepsilon|>\alpha\}| \le C_{12}\alpha^{-\frac{N(p_i^- -1)}{N-1}}, \quad \alpha>0.$$

Finally, we get

$$\int_{\{|\nabla u_\varepsilon|>\alpha,\ x\in\Omega\}} \alpha^{\widetilde{q}(x)}dx \le k|\Omega| + \sum_{i=1}^{k}\int_{\{|\nabla u_\varepsilon|>\alpha,\ x\in\Omega_i\}} \alpha^{\widetilde{q}_i^{+}}\,dx \le C_{13}.$$

This ends the proof of lemma. □

**Remark 3.1.** Note that the result obtained in Lemma 3.2 also holds for any measurable function $q:\Omega\to[1,+\infty)$ (resp. $\widetilde{q}:\Omega\to[1,+\infty)$) such that

$$b:=\operatorname*{ess\,inf}_{x\in\Omega}\Big(\frac{N(p(x)-1)}{N-p(x)}-q(x)\Big)>0,\ \Big(\text{resp.}\ b:=\operatorname*{ess\,inf}_{x\in\Omega}\Big(\frac{N(p(x)-1)}{N-1}-\widetilde{q}(x)\Big)>0\Big)$$

In fact, in this case there exists a continuous function $s(\cdot)$ (resp. $s'(\cdot)$) such that

$$s(x)\geq q(x)\ (\text{resp.}\ s'(x)\geq\widetilde{q}(x))\ \text{for almost every}\ x\in\Omega,$$

and

$$\min_{x\in\Omega}\Big(\frac{N(p(x)-1)}{N-p(x)}-s(x)\Big)(>b/2)>0,\ \Big(\text{resp.}\ \min_{x\in\Omega}\Big(\frac{N(p(x)-1)}{N-1}-s'(x)\Big)(>b/2)>0\Big).$$

From Lemma 3.2, we deduce the bound of $u_\varepsilon$ in $\mathcal{M}^{s(\cdot)}(\Omega;\mathbb{R}^m)$ and the bound of $\nabla u_\varepsilon\in\mathcal{M}^{s'(\cdot)}(\Omega;\mathbb{R}^m)$. Finally the result follows from to the continuous embedding $\mathcal{M}^{s(\cdot)}(\Omega)$ into $\mathcal{M}^{q(\cdot)}(\Omega)$ (resp. $\mathcal{M}^{s'(\cdot)}(\Omega)$ into $\mathcal{M}^{\widetilde{q}(\cdot)}(\Omega)$).

**3.4. Strong convergence.** From Lemma 3.2, $u_\varepsilon$ is uniformly bounded in $L^{s_0(\cdot)}(\Omega;\mathbb{R}^m)$ for some continuous functions $s_0(\cdot)<\frac{N(p(\cdot)-1)}{N-p(\cdot)}$ with $s_0(\cdot)>\frac{N(p(\cdot)-1)}{N-1}$, and $\frac{\partial u_\varepsilon}{\partial x_i}$ is uniformly bounded in $L^{\widetilde{q}(\cdot)}(\Omega;\mathbb{R}^m)$ for all $i=1,\dots,N$ and some continuous functions $\widetilde{q}(\cdot)$ such that

$$\max\{p(x)-1,1\}<\widetilde{q}(x)<\frac{N(p(x)-1)}{N-1},\ x\in\overline{\Omega}.$$

From this we get that $u_\varepsilon$ is uniformly bounded in the Sobolev space

$$W_0^{1,s}(\Omega;\mathbb{R}^m),\quad s=\min_{x\in\overline{\Omega}}\widetilde{q}(x).$$

Consequently, we can assume (without loss of generality) that as $\varepsilon\to0$

$$\begin{aligned}
&u_\varepsilon\to u\quad\text{a.e. in }\Omega\text{ and in }L^s(\Omega;\mathbb{R}^m),\\
&u_\varepsilon\rightharpoonup u\quad\text{in }W_0^{1,s}(\Omega;\mathbb{R}^m),\\
&\left|\frac{\partial u_\varepsilon}{\partial x_l}-\frac{\partial u}{\partial x_l}\right|\rightharpoonup h_l\quad\text{in }L^s(\Omega),\quad l=1,\dots,N,\\
&\sigma_l\big(x,\frac{\partial u_\varepsilon}{\partial x_l}\big)\rightharpoonup\beta_l\quad\text{in }L^{\frac{\widetilde{q}(\cdot)}{p(\cdot)-1}}(\Omega;\mathbb{R}^m),\quad l=1,\dots,N,\\
&f_\varepsilon\overset{\star}{\rightharpoonup}\mu\quad\text{in the sense of measures on }\Omega.
\end{aligned}\tag{3.11}$$

Herein, the convergences obtained in (3.11) are not strong enough if we want to pass to the limit $\varepsilon\to0$ in the nonlinear system (3.5), and the proof of Theorem 3.1 will be completed by Lemma 3.3 below. To prove this lemma we follow closely the argument used in [8] for the $p$-harmonic system, which is based on using a regularized test function and a localization procedure to handle the problem that $u$ does not in general belong to the Sobolev space $W_0^{1,p(\cdot)}(\Omega,\mathbb{R}^m)$.

**Lemma 3.3.** *For $l = 1, \ldots, N$, as $\varepsilon \to 0$ we have*

$$\sigma_l\big(x, \frac{\partial u_\varepsilon}{\partial x_l}\big) \to \sigma_l\big(x, \frac{\partial u}{\partial x_l}\big) \quad \text{a.e. in } \Omega \text{ and in } L^1(\Omega; \mathbb{R}^m). \tag{3.12}$$

*Proof.* The main part of the proof consists in showing that

$$h_l(x) = 0 \quad \text{for a.e. } x \in \Omega, \quad l = 1, \ldots, N, \tag{3.13}$$

where $h_l$ is defined in (3.11). Suppose for the moment the validity of (3.13), and fix any one of the directions $l = 1, \ldots, N$. Then, by Vitali's theorem,

$$\frac{\partial u_\varepsilon}{\partial x_l} \to \frac{\partial u}{\partial x_l} \quad \text{in } L^1(\Omega; \mathbb{R}^m),$$

and, after extracting a subsequence if necessary, $\frac{\partial u_\varepsilon}{\partial x_l} \to \frac{\partial u}{\partial x_l}$ a.e. in $\Omega$. From this we also have $\sigma_l\big(x, \frac{\partial u_\varepsilon}{\partial x_l}\big) \to \sigma_l\big(x, \frac{\partial u}{\partial x_l}\big)$ a.e. in $\Omega$. As $\sigma_l\big(x, \frac{\partial u_\varepsilon}{\partial x_l}\big)$ is uniformly bounded in $L^{\frac{\widetilde{q}(\cdot)}{p(\cdot)-1}}(\Omega; \mathbb{R}^m)$, by (2.2), Vitali's theorem gives

$$\sigma_l\big(x, \frac{\partial u_\varepsilon}{\partial x_l}\big) \to \sigma_l\big(x, \frac{\partial u}{\partial x_l}\big) \quad \text{in } L^{t(\cdot)}(\Omega; \mathbb{R}^m),$$

for any continuous function $t$ such that $1 \le t(\cdot) < \frac{\widetilde{q}(\cdot)}{p(\cdot)-1}$, which proves (3.12).

We now set out to prove (3.13). Choose a nonnegative function $\alpha \in C^\infty([0, \infty) \cap L^\infty([0, \infty))$ such that $\alpha(t) = t$ for $t \in [0, \delta]$ for some $\delta > 0$, $\alpha' \ge 0$, and $\alpha'(t)t \le \alpha(t)$ for all $t \ge 0$ (see [8] for an explicit example of such a function). Then define the function $\psi : \mathbb{R}^m \to \mathbb{R}^m$ by

$$\psi(r) = \frac{r}{|r|}\alpha(|r|),$$

and note that $\psi(r) = r$ when $|r| \le \delta$. We also need two scalar functions $\eta$, $\phi$ of the following type:

$$\eta \in C_c^\infty(\mathbb{R}^m), \quad 0 \le \eta \le 1, \quad \text{supp}(\eta) \subset [0, \delta),$$
$$\phi \in C_c^\infty(\mathbb{R}^n), \quad 0 \le \phi \le 1, \quad \int \phi \, dx = 1.$$

In what follows, let us fix any one of the directions $l = 1, \ldots, N$. Denoting by $v$ a comparison function in $C^1(\Omega; \mathbb{R}^m)$ (to be chosen later), we proceed by using the triangle and Hölder inequalities:

$$\begin{aligned}
&\int_\Omega \sum_{l=1}^N \left|\frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial u}{\partial x_l}\right| \eta(u_\varepsilon - v)\phi\,dx \\
&\le \int_\Omega \sum_{l=1}^N \left|\frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial v}{\partial x_l}\right| \eta(u_\varepsilon - v)\phi\,dx + \int_\Omega \sum_{l=1}^N \left|\frac{\partial v}{\partial x_l} - \frac{\partial u}{\partial x_l}\right| \eta(u_\varepsilon - v)\phi\,dx \\
&\le 2\sum_{l=1}^N \left\| \left|\frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial v}{\partial x_l}\right| (\eta(u_\varepsilon - v)\phi)^{\frac{1}{p(x)}} \right\|_{L^{p(\cdot)}(\Omega)} \left\| (\eta(u_\varepsilon - v)\phi)^{\frac{1}{p'(x)}} \right\|_{L^{p'(\cdot)}(\Omega)} \\
&\quad + \int_\Omega \sum_{l=1}^N \left|\frac{\partial v}{\partial x_l} - \frac{\partial u}{\partial x_l}\right| \eta(u_\varepsilon - v)\phi\,dx \\
&\le 2\sum_{i=1}^N \max\left\{A_\varepsilon^{\frac{1}{p^+}}; A_\varepsilon^{\frac{1}{p^-}}\right\} \max\left\{B_\varepsilon^{1-\frac{1}{p^+}}; B_\varepsilon^{1-\frac{1}{p^-}}\right\} + \int_\Omega \sum_{l=1}^N \left|\frac{\partial v}{\partial x_l} - \frac{\partial u}{\partial x_l}\right| \eta(u_\varepsilon - v)\phi\,dx
\end{aligned}$$

where

$$A_\varepsilon = \int_\Omega \left|\frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial v}{\partial x_l}\right|^{p(x)} \eta(u_\varepsilon - v)\phi\,dx \quad \text{and} \quad B_\varepsilon = \int_\Omega \eta(u_\varepsilon - v)\phi\,dx.$$

Equipped with this and (3.11), using in particular that $u_\varepsilon \to u$ a.e. and the fact that $\eta$, $\psi$, $D\psi$ are continuous and bounded functions, we deduce

$$\begin{aligned}
&\int_\Omega \sum_{l=1}^N h_l(x)\eta(u - v)\phi\,dx \\
&\le 2\sum_{l=1}^N \max\left\{L_l^{\frac{1}{p^+}}; L_l^{\frac{1}{p^-}}\right\} \max\left\{B^{1-\frac{1}{p^+}}; B^{1-\frac{1}{p^-}}\right\} + \int_\Omega \sum_{l=1}^N \left|\frac{\partial v}{\partial x_l} - \frac{\partial u}{\partial x_l}\right| \eta(u - v)\phi\,dx,
\end{aligned} \tag{3.14}$$

where

$$L_l = L_l(\eta, \phi, \psi) := \limsup_{\varepsilon\to 0} A_\varepsilon \quad \text{and} \quad B = \int_\Omega \eta(u - v)\phi\,dx.$$

Put

$$\Omega_1 = \{x \in \Omega \mid p(x) \ge 2\} \quad \text{and} \quad \Omega_2 = \{x \in \Omega \mid 1 < p(x) < 2\}.$$

We must analyze $L_l$, and start with the case $p(x) \geq 2$. By (1.3),

$$\begin{aligned}
&\int_{\Omega_1} c_3 \left|\frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial v}{\partial x_l}\right|^{p(x)} \eta(u_\varepsilon - v)\phi\,dx \\
&\leq \int_{\Omega_1} \sum_{l=1}^{N} \Big(\sigma_l\big(x, \frac{\partial u_\varepsilon}{\partial x_l}\big) - \sigma_l\big(x, \frac{\partial v}{\partial x_l}\big)\Big) \cdot \Big(\frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial v}{\partial x_l}\Big)\eta(u_\varepsilon - v)\phi\,dx \\
&\leq \int_{\Omega} \sum_{l=1}^{N} \Big(\sigma_l\big(x, \frac{\partial u_\varepsilon}{\partial x_l}\big) - \sigma_l\big(x, \frac{\partial v}{\partial x_l}\big)\Big) \cdot \frac{\partial \psi(u_\varepsilon - v)}{\partial x_l}\eta(u_\varepsilon - v)\phi\,dx \\
&= \int_{\Omega} \sum_{l=1}^{N} \sigma_l\big(x, \frac{\partial u_\varepsilon}{\partial x_l}\big) \cdot \frac{\partial \psi(u_\varepsilon - v)}{\partial x_l}\phi\,dx \\
&\quad - \int_{\Omega} \sum_{l=1}^{N} \sigma_l\big(x, \frac{\partial u_\varepsilon}{\partial x_l}\big) \cdot \frac{\partial \psi(u_\varepsilon - v)}{\partial x_l}(1 - \eta(u_\varepsilon - v))\phi\,dx \\
&\quad - \int_{\Omega} \sum_{l=1}^{N} \sigma_l\Big(x, \frac{\partial v}{\partial x_l}\Big) \cdot \Big(\frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial v}{\partial x_l}\Big)\eta(u_\varepsilon - v)\phi\,dx \\
&=: E_{1\varepsilon} + E_{2\varepsilon} + E_{3\varepsilon} = E(\varepsilon),
\end{aligned} \tag{3.15}$$

On the set where $1 < p(x) < 2$, we write

$$\begin{aligned}
&\int_{\Omega_2} \left|\frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial v}{\partial x_l}\right|^{p(x)} \eta(u_\varepsilon - v)\phi\,dx \\
&= \int_{\Omega_2} \frac{\Big(\Big|\frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial v}{\partial x_l}\Big|\Big)^{p(x)}}{\Big(\Big|\frac{\partial u_\varepsilon}{\partial x_l}\Big| + \Big|\frac{\partial v}{\partial x_l}\Big|\Big)^{\frac{p(x)(2-p(x))}{2}}} (\eta(u_\varepsilon - v)\phi)^{\frac{p(x)}{2} + \frac{2-p(x)}{2}} \left(\left|\frac{\partial u_\varepsilon}{\partial x_l}\right| + \left|\frac{\partial v}{\partial x_l}\right|\right)^{\frac{p(x)(2-p(x))}{2}} dx \\
&\leq 2 \left\| \frac{\Big(\Big|\frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial v}{\partial x_l}\Big|\Big)^{p(x)}}{\Big(\Big|\frac{\partial u_\varepsilon}{\partial x_l}\Big| + \Big|\frac{\partial v}{\partial x_l}\Big|\Big)^{\frac{p(x)(2-p(x))}{2}}} (\eta(u_\varepsilon - v)\phi)^{\frac{p(x)}{2}} \right\|_{L^{2/p(\cdot)}(\Omega_2)} \times \\
&\quad \left\| \left(\left|\frac{\partial u_\varepsilon}{\partial x_l}\right| + \left|\frac{\partial v}{\partial x_l}\right|\right)^{\frac{p(x)(2-p(x))}{2}} (\eta(u_\varepsilon - v)\phi)^{\frac{2-p(x)}{2}} \right\|_{L^{2/(2-p(\cdot))}(\Omega_2)} \\
&\leq 2 \max\left\{w_{1\varepsilon}^{p^-/2}; w_{1\varepsilon}^{p^+/2}\right\} \max\left\{w_{2\varepsilon}^{(2-p^-)/2}; w_{2\varepsilon}^{(2-p^+)/2}\right\}
\end{aligned} \tag{3.16}$$

where

$$w_{1\varepsilon} = \int_{\Omega_2} \frac{\Big|\frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial v}{\partial x_l}\Big|^2}{\Big(\Big|\frac{\partial u_\varepsilon}{\partial x_l}\Big| + \Big|\frac{\partial v}{\partial x_l}\Big|\Big)^{2-p(x)}} \eta(u_\varepsilon - v)\phi\,dx,$$

and

$$w_{2\varepsilon} = \int_{\Omega_2} \left( \left| \frac{\partial u_\varepsilon}{\partial x_l} \right| + \left| \frac{\partial v}{\partial x_l} \right| \right)^{p(x)} \eta(u_\varepsilon - v)\phi \, dx.$$

We employ (1.3) instead as follows:

$$\begin{aligned} \int_{\Omega_2} \left| \frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial v}{\partial x_l} \right|^{p(x)} & \eta(u_\varepsilon - v)\phi \, dx \\ & \leq 2 \max \left\{ c_4^{-p^-/2} E(\varepsilon)^{p^-/2}; c_4^{-p^+/2} E(\varepsilon)^{p^+/2} \right\} \\ & \qquad \times \max \left\{ w_{2\varepsilon}^{(2-p^-)/2}; w_{2\varepsilon}^{(2-p^+)/2} \right\}. \end{aligned} \tag{3.17}$$

Thanks to (3.5),

$$E_{1\varepsilon} = \int_\Omega f_\varepsilon \cdot \psi(u_\varepsilon - v)\phi \, dx - \int_\Omega \sum_{l=1}^N \sigma_l\Big(x, \frac{\partial u_\varepsilon}{\partial x_l}\Big) \cdot \psi(u_\varepsilon - v) \frac{\partial \phi}{\partial x_l} \, dx. \tag{3.18}$$

To estimate $E_{2\varepsilon}$ note that

$$D\psi(r) = \alpha'(|r|) \frac{r \otimes r}{|r|^2} + \frac{\alpha(|r|)}{|r|} \left( I - \frac{r \otimes r}{|r|^2} \right),$$

so that

$$\sigma_l(x, \xi) \cdot D\psi(r)\xi \geq 0, \quad \forall \xi, r \in \mathbb{R}^m.$$

This follows from (1.5), since

$$\sigma_l(x, \xi) \cdot D\psi(r)\xi = \frac{\alpha(|r|)}{|r|} \sigma_l(x, \xi) \cdot \left( I - \left[ \left( 1 - \frac{\alpha'(|r|)\,|r|}{\alpha(|r|)} \right) \frac{r \otimes r}{|r|^2} \right] \right) \xi,$$

where the term inside the square brackets can be written as $a \otimes a$ for some $a \in \mathbb{R}^m$ with $|a| \leq 1$ (recall that $\alpha'(t)t \leq \alpha(t)$). Hence

$$E_{2\varepsilon} \leq \int_\Omega \sum_{l=1}^N \sigma_l\big(x, \frac{\partial u_\varepsilon}{\partial x_l}\big) \cdot D\psi(u_\varepsilon - v) \frac{\partial v}{\partial x_l} (1 - \eta(u_\varepsilon - v))\phi \, dx. \tag{3.19}$$

Since $u_\varepsilon \to u$ a.e. and $\eta$, $\psi$, $D\psi$ are continuous and bounded functions, we deduce from (3.15), (3.18), (3.19), and (3.6) that

$$\begin{aligned} L_{l1} \leq \sup|\psi| \int_\Omega \phi \, d\mu & - \int_\Omega \sum_{l=1}^N \beta_l \cdot \psi(u - v) \frac{\partial \phi}{\partial x_l} \, dx \\ & + \int_\Omega \sum_{l=1}^N \beta_l \cdot D\psi(u - v) \frac{\partial v}{\partial x_l} \left(1 - \eta(u - v)\right) \phi \, dx \\ & - \int_\Omega \sum_{l=1}^N \sigma_l\big(x, \frac{\partial v}{\partial x_l}\big) \cdot \big(\frac{\partial u}{\partial x_l} - \frac{\partial v}{\partial x_l}\big) \eta(u - v)\phi \, dx \end{aligned} \tag{3.20}$$

where

$$L_{l1} = L_{l1}(\eta,\phi,\psi) := \limsup_{\varepsilon\to 0} \int_{\Omega_1} c_3 \left| \frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial v}{\partial x_l} \right|^{p(x)} \eta(u_\varepsilon - v)\phi \, dx$$

and

$$L_{l2} = L_{l2}(\eta,\phi,\psi) := \limsup_{\varepsilon\to 0} \int_{\Omega_2} \left| \frac{\partial u_\varepsilon}{\partial x_l} - \frac{\partial v}{\partial x_l} \right|^{p(x)} \eta(u_\varepsilon - v)\phi \, dx.$$

Next, we specify the functions $v$, $\eta$, $\psi$, $\phi$. Fix any point $x = a \in \Omega$ that is simultaneously a Lebesgue point of $\frac{\partial u}{\partial x_l}, h_l, \beta_l$, $l = 1, \dots, N$, and the measure $\mu$. Choose $v$ as the first order Taylor polynomial of $u$ around $x = a$:

$$v(x) = u(a) + \nabla u(a)(x-a),$$

and replace $\phi, \eta, \psi$ in the above calculations by the following functions:

$$\eta_\rho(r) = \tilde{\eta}\big(\frac{r}{\rho}\big), \quad \tilde{\eta} \in C_c^\infty(B(0,1)), \quad \tilde{\eta}|_{B(0,\frac{1}{2})} \equiv 1,$$
$$\phi_\rho(x) = \frac{1}{\rho^n}\tilde{\phi}\big(\frac{x-a}{\rho}\big), \quad \tilde{\phi} \in C_c^\infty(B(0,1)), \quad \int \tilde{\phi} = 1,$$

and $\psi_\rho(r) = \rho\widetilde{\psi}\big(\frac{r}{\rho}\big)$. Denote by $L_{l1}(\rho)$ (resp. $L_{l2}(\rho)$) the corresponding $L_{l1}$ (resp. $L_{l2}(\rho)$), that is, $L_{l1}(\rho) := L_{l1}(\eta_\rho, \phi_\rho, \psi_\rho)$ (resp. $L_{l2}(\rho) := L_{l2}(\eta_\rho, \phi_\rho, \psi_\rho)$). We deduce $\limsup_{\rho\to 0} L_{l1}(\rho) = 0$, since as $\rho \to 0$,

$$\frac{1}{|B(a,\rho)|} \int_{B(a,\rho)} \left| \frac{u-v}{\rho} \right| dx \to 0,$$
$$\frac{1}{|B(a,\rho)|} \int_{B(a,\rho)} \sum_{l=1}^{N} \left| \frac{\partial u}{\partial x_l} - \frac{\partial v}{\partial x_l} \right| dx \to 0,$$
$$\frac{1}{|B(a,\rho)|} \int_{B(a,\rho)} \sum_{l=1}^{N} |\beta_l(x) - \beta_l(a)| \, dx \to 0,$$

where the second and third terms in (3.20) tend to zero as we have

$$\psi_\rho(u-v)\frac{\partial \phi}{\partial x_l} = \mathcal{O}\big(\frac{u-v}{\rho}\big), \quad 1 - \eta_\rho(u-v) = \mathcal{O}\big(\frac{u-v}{\rho}\big).$$

The first term tends to zero since

$$\limsup_{\rho\to 0} \mu(B(a,\rho))/\rho^n < \infty,$$

and thus

$$\sup|\psi_\rho| \int_\Omega \phi_\rho \, d\mu \le C\rho\mu(B(a,\rho))/\rho^n.$$

In the case $p(x) < 2$, we also use that the term $\max\left\{w_{2\varepsilon}^{(2-p^-)/2}; w_{2\varepsilon}^{(2-p^+)/2}\right\}$ in (3.17) stays finite in the above localization procedure to obtain $\limsup_{\rho\to 0} L_{l2}(\rho) = 0$. Since

$$\frac{1}{|B(a,\rho)|}\int_{B(a,\rho)}\sum_{l=1}^{N}|h_l(x)-h_l(a)|\,dx \to 0 \quad \text{as } \rho\to 0,$$

it follows, via (3.14), that $h(a) = 0$. This completes the proof of (3.13), and hence the lemma. □

**Remark 3.2.** *Since* $\Omega$ *is bounded then* (3.2) *implies in particular that*

$$u \in \bigcap_{s(\cdot)<\frac{N(p(\cdot)-1)}{N-1}} W_0^{1,s(\cdot)}(\Omega).$$

**Remark 3.3.** *Remark that in the constant case* [8]*, by* (2.7) *and* (3.10)*, we have*

$$\|u_\varepsilon\|_{\mathcal{M}^{\frac{N(p-1)}{N-p}}(\Omega;\mathbb{R}^m)} \le C \quad \textit{and} \quad \left\|\frac{\partial u_\varepsilon}{\partial x_i}\right\|_{\mathcal{M}^{\frac{N(p-1)}{N-1}}(\Omega;\mathbb{R}^m)} \le C, \quad i=1,\dots,N.$$

*Then* (1.1) *has at least one weak solution* $u$*, possesses the regularity*

$$u \in \mathcal{M}^{\frac{N(p-1)}{N-p}}(\Omega;\mathbb{R}^m),\ \frac{\partial u}{\partial x_i} \in \mathcal{M}^{\frac{N(p-1)}{N-1}}(\Omega;\mathbb{R}^m), \quad i=1,\dots,N.$$

*For the nonconstant case, it remains an open problem to show that*

$$u \in \mathcal{M}^{\frac{N(p(\cdot)-1)}{N-p(\cdot)}}(\Omega;\mathbb{R}^m),\ \frac{\partial u}{\partial x_i} \in \mathcal{M}^{\frac{N(p(\cdot)-1)}{N-1}}(\Omega;\mathbb{R}^m), \quad i=1,\dots,N,$$

*where* $p(\cdot)$ *is defined in* (1.4).

## 4. An extension

In this section we show that the results obtained for (1.1) can be extended to more general elliptic systems of the form

$$\begin{aligned} -\sum_{l=1}^{N}\frac{\partial}{\partial x_l}\sigma_l\big(x,\frac{\partial u}{\partial x_l}\big) + g(x,u) &= f, \quad \text{in } \Omega,\\ u &= 0, \quad \text{in } \partial\Omega, \end{aligned} \tag{4.1}$$

where the vector fields $\sigma_1,\dots,\sigma_N$ are as before and $f = (f_1,...,f_m)^T \in L^1(\Omega;\mathbb{R}^m)$. We assume that the nonlinearity $g(x,r) : \Omega\times\mathbb{R}^m \to \mathbb{R}^m$ is measurable in $x\in\Omega$ for all $r\in\mathbb{R}^m$, continuous in $r$ for a.e. $x\in\Omega$, and satisfies the following conditions:

$$g(x,r)\cdot r \ge 0, \quad \forall r\in\mathbb{R}^m, \tag{4.2}$$

$$\sup\{|g(x,r)| : |r|\le\tau\} \in L^1(\Omega;\mathbb{R}^m), \quad \forall\tau\in\mathbb{R}. \tag{4.3}$$

$$g(x,r)\cdot(r-r') \ge 0, \quad \forall\, r\,,r'\in\mathbb{R}^m \text{ with } |r| = |r'|. \tag{4.4}$$

A prototype example of (4.1) is provided by the equation

$$-\sum_{l=1}^{N} \frac{\partial}{\partial x_l}\Big( \Big|\frac{\partial u}{\partial x_l}\Big|^{p(x)-2} \frac{\partial u}{\partial x_l}\Big) + \theta(x)|u|^{\theta(x)}u = f\,,$$

for some positive function $\theta \in L^\infty(\Omega)$.

**Remark 4.1.** *Remark that* (4.2) *and* (4.4) *are equivalent if* $m = 1$.

**Definition 4.1.** A distributional solution of (4.1) is a function $u : \Omega \to \mathbb{R}^m$ such that (3.1) and $g(x,u) \in L^1(\Omega;\mathbb{R}^m)$ hold, and $\forall \varphi \in C_c^\infty(\Omega;\mathbb{R}^m)$

$$\int_\Omega \sum_{l=1}^{N} \sigma_l\big(x, \frac{\partial u}{\partial x_l}\big) \cdot \frac{\partial \varphi}{\partial x_l}\, dx + \int_\Omega g(x,u)\varphi\, dx = \int_\Omega \varphi f(x)\, dx.$$

Our main result is the following.

**Theorem 4.1.** *Let* $f \in L^1(\Omega;\mathbb{R}^m)$. *Then, under the assumptions stated above and in Section 1,* (4.1) *has at least one distributional solution* $u$. *Moreover,* $u$ *has regularity as stated in* (3.2).

*Proof.* Let $(f_\varepsilon)$ be a sequence of bounded functions defined in $\Omega$ that converges to $f$ in $L^1(\Omega;\mathbb{R}^m)$, and which verifies

$$|f_\varepsilon| \le C(\varepsilon) \quad \text{and} \quad |f_\varepsilon| \le |f|$$

Then, by classical arguments, there exists a sequence of approximate solutions $(u_\varepsilon)_{0<\varepsilon\le 1}$ satisfying for all $\varphi \in W_0^{1,p(\cdot)}(\Omega;\mathbb{R}^m)$:

$$\int_\Omega \sum_{l=1}^{N} \sigma_l\big(x, \frac{\partial u_\varepsilon}{\partial x_l}\big) \cdot \frac{\partial \varphi}{\partial x_l}\, dx + \int_\Omega g(x,u_\varepsilon)\cdot\varphi\, dx = \int_\Omega f_\varepsilon \cdot \varphi\, dx. \tag{4.5}$$

Setting

$$\alpha_i = \begin{cases} \frac{|g_i(x,u_\varepsilon)|}{g_i(x,u_\varepsilon)}, & \text{if } g_i(x,u_\varepsilon) \neq 0 \\ 1, & \text{if } g_i(x,u_\varepsilon) = 0 \end{cases}, \quad \alpha = (\alpha_1,\dots,\alpha_m)^T,\ g = (g_1,\dots,g_m)^T,$$

by (4.4), we obtain

$$g(x,u_\varepsilon)\frac{Nu_\varepsilon}{|u_\varepsilon|} - |g(x,u_\varepsilon)| = \frac{N}{|u_\varepsilon|}\left( g(x,u_\varepsilon)\cdot\left(u_\varepsilon - \frac{|u_\varepsilon|}{N}\alpha\right)\right) \ge 0. \tag{4.6}$$

Choosing $\varphi = T_\gamma(u_\varepsilon)$, $\gamma > 0$ as test function in (4.5), it follows from (4.2) and (4.6), that

$$c_1 \sum_{l=1}^{N} \int_{\{|u_\varepsilon|\le\gamma\}} \left|\frac{\partial u_\varepsilon}{\partial x_l}\right|^{p(x)} dx + \frac{\gamma}{N}\int_{\{|u_\varepsilon|>\gamma\}} |g(x,u_\varepsilon)|\, dx \le \int_\Omega |f||T_\gamma(u_\varepsilon)|. \tag{4.7}$$

Combining (4.7) and Lemma 3.2, we have for all $t > 0$

$$\int_{|u_\varepsilon|>t} t^{q(x)} dx \le C, \quad \int_{|\frac{\partial u_\varepsilon}{\partial x_l}|>t} t^{\widetilde{q}(x)} dx \le C, \quad l = 1,\dots,N,$$

where $C$ is a constant independent of $\varepsilon$ and $q(\cdot)$ and $\widetilde{q}(\cdot)$ are defined in (3.3). Consequently, we can assume without loss of generality that the convergence in (3.11) hold for our sequence $(u_\varepsilon)_{0<\varepsilon\le 1}$. Using (4.7) and (4.3), we deduce

$$\int_\Omega |g(x,u_\varepsilon)|\,dx \le C,$$

where $C$ is a constant independent of $\varepsilon$. By (4.7) and (4.3) we obtain that $g(x,u_\varepsilon)$ is equi-integrable in $\Omega$. Then in view of Vitali's theorem, $g(x,u_\varepsilon)$ converges strongly in $L^1(\Omega;\mathbb{R}^m)$ to $g(x,u)$. The proof of Lemma 3.3 remains more or less unchanged, except that the term $E_{1\varepsilon}$ rewrites in our problem (4.1) as

$$\begin{aligned} E_{1\varepsilon} &= \int_\Omega f_\varepsilon \psi(u_\varepsilon - v)\phi\,dx - \int_\Omega g(x,u_\varepsilon)\psi(u_\varepsilon - v)\phi\,dx \\ &\quad - \int_\Omega \sum_{l=1}^N \sigma_l\big(x,\frac{\partial u_\varepsilon}{\partial x_l}\big)\psi(u_\varepsilon - v)\frac{\partial \phi}{\partial x_l}\,dx, \end{aligned} \tag{4.8}$$

and estimate (3.20) rewrites as

$$\begin{aligned} L_{l1} &\le \sup|\psi| \Big(\int_\Omega \phi\, f dx + \int_\Omega |g(x,u)|\,\phi\,dx\Big) \\ &\quad - \int_\Omega \sum_{l=1}^N \beta_l\cdot\psi(u-v)\frac{\partial\phi}{\partial x_l}\,dx \\ &\quad + \int_\Omega \sum_{l=1}^N \beta_l\cdot D\psi(u-v)\frac{\partial v}{\partial x_l}\,(1-\eta(u-v))\,\phi\,dx \\ &\quad - \int_\Omega \sum_{l=1}^N \sigma_l\big(x,\frac{\partial v}{\partial x_l}\big)\cdot\big(\frac{\partial u}{\partial x_l} - \frac{\partial v}{\partial x_l}\big)\eta(u-v)\phi\,dx. \end{aligned} \tag{4.9}$$

Letting $x = a$ be a Lebesgue point simultaneously of $f$, $g(x,u)$, $h$, $u$, $Du$, and $\beta = (\beta_1,\dots\beta_N)$, we can proceed as in the proof of Lemma 3.3. □

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