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Proximity model of perceived numerosity

Jüri Allik [1,2] · Aire Raidvee [1]

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Abstract
The occupancy model (OM) was proposed to explain how the spatial arrangement of dots in sparse random patterns affects their perceived numerosity. The model's central thesis maintained that each dot seemingly fills or occupies its surrounding area within a fixed radius r_o and the total area collectively occupied by all the dots determines their apparent number. Because the perceptual system is not adapted for the precise estimation of area, it looks likely that the OM is just a convenient computational algorithm that does not necessarily correspond to the processes that actually take place in the perceptual system. As an alternative, the proximity model (PM) was proposed, which instead relies on a binomial function with the probability β characterizing the perceptual salience with which each element can be registered by the perceptual system. It was also assumed that the magnitude of β is proportional to the distance between a dot and its nearest neighbor. A simulation experiment demonstrated that the occupancy area computed according to the OM can almost perfectly be replicated by the mean nearest neighbor distance. It was concluded that proximity between elements is a critical factor in determining their perceived numerosity, but the exact algorithm that is used for the measure of proximities is yet to be established.

Keywords Perceived number · Numerosity illusions · The occupancy model · Spatial statistics · Nearest neighbor distance · Visual crowding

Introduction

Stanley Jevons, one of the founders of neoclassical economics, ran one of the most elegant experiments in the history of psychology, using only a round paper box and a handful of black beans. He observed that he had the ability to estimate, with sufficient accuracy, the number of randomly scattered black beans even if there was no time or opportunity for them to be counted exactly (Jevons, 1871). He made no mistakes when the number of beans was less than five, and even with higher quantities of beans, errors were mainly towards underestimation. However, this method of identifying the exact number of displayed elements presumes that the subjects are familiar with the concept of numerals. There are languages with a very small lexicon of numbers, such as Munduruku or Pirahã, whose speakers can nevertheless compare large approximate numbers far beyond their naming range (Gordon, 2004; Pica, Lemer, Izard, & Dehaene, 2004). Additionally, small children have a number sense, which allows them to work with numbers before they have acquired the semantics of numerals (Halberda & Feigenson, 2008).

One way that enables us to avoid using numerals is to instead ask to determine which of two sets of elements is more numerous (Allik & Tuulmets, 1991; Burgess & Barlow, 1983; Raphael, Dillenburger, & Morgan, 2013; Van Oeffelen & Vos, 1982b). This simple task bypasses verbal or symbolic coding, making this protocol more functional for not only young children but also for various animal species, including pigeons (Honig & Matheson, 1995), treefrogs (Lucon-Xiccato, Gatto, & Bisazza, 2018), angelfishes (Gomez-Laplaza & Gerlai, 2020), and of course chimpanzees (Woodruff & Premack, 1981). Although the number of elements does not change if their order or spatial arrangement is changed, like young children, our perceptual system can be fooled by a different spatial configuration of elements. Many so-called numerosity illusions, which arise from differing arrangements of elements, have been observed and described. For instance, Christopher and Uta Frith described the solitaire illusion, which creates the impression that one large cluster of dots appears to contain more elements than several small clusters (Frith & Frith, 1972). It was also observed that a regular or even arrangement of elements appeared to have a different

✉ Jüri Allik
juri.allik@ut.ee

1 Institute of Psychology, University of Tartu, Näituse 2, 50409 Tartu, Estonia

2 Estonian Academy of Sciences, Tallinn, Estonia

number of elements than the same number of randomly positioned elements (Burgess & Barlow, 1983; Ginsburg, 1976; Taves, 1941). One factor that is known to affect this perceived number is the grouping of elements into clusters (Bertamini, Zito, Scott-Samuel, & Hulleman, 2016; Chakravarthi & Bertamini, 2020; Ginsburg & Goldstein, 1987; Van Oeffelen & Vos, 1982a; Vos, Van Oeffelen, Tibosch, & Allik, 1988). However, an ability to estimate numerosity depends substantially on whether dots were configured in the same pattern throughout the experiment or randomly generated before each trial to avoid memorable configurations (Krajcsi, Szabó, & Mórocz, 2013; Wolters, van Kempen, & Wijlhuizen, 1987). In order to avoid confusing numerosity estimation with recognition of already familiar patterns, we need stochastic processes that place elements into random positions in the image. This is a good opportunity to give a reminder of the remarkable progress made in the study of stereoscopic perception thanks to the invention of random dot patterns (Julesz, 1971).

Since the spatial configuration of elements has an effect on their apparent number, it clearly indicates that it is not the number per se but another attribute that is actually estimated when there is an instruction to judge the number of elements. This speaks to the constraints that limit visual processing in situations where observers are instructed to carry out a task the visual system is not adapted to (Morgan, Hole, & Glennerster, 1990). The French psychologist Alfred Binet was probably the first to notice that young children, before they acquire an understanding of the concept of numbers, confuse numbers with the extent to which elements are stretched in space (Binet, 1890).

To explain how spatial arrangement of elements affects their perceived numerosity, the Occupancy Model (OM) was proposed (Allik & Tuulmets, 1991). The development of this model was based on the knowledge provided by spatial statistics (Ripley, 1981), which is a special field of statistics that describes how objects are distributed in space. For example, if a plane area A^2 is randomly bombarded by N dots – usually called the Poisson process – then the probability that the nearest neighbor is in the radius r can be found with the following formula: $1-e^{-N\pi r^2}$ (Schachter & Ahuja, 1979). However, by deviating from this totally random Poisson process, one can force the nearest neighboring dots to be statistically closer or more distant to each other than in the case of a random bombardment. In order to shorten the nearest neighboring distance, it is possible to use a satellite process in which half of all the elements are parents, each having a child or satellite positioned randomly around the parent within the specified radius r_s (Schachter & Ahuja, 1979). The opposite process uses the inhibitory radius r_i surrounding all already positioned elements for protection so that no new elements can be placed closer than a specified critical distance (Schachter & Ahuja, 1979).

Typically, the Poisson distribution (Fig. 1B) appears less numerous than the pattern generated by the inhibitory process (Fig. 1A), which controls that no two dots are more closely positioned than the inhibitory radius r_i, but does seem more numerous than the satellite process (Fig. 1C), which guarantees that no dot has its nearest neighbor within the satellite radius r_s. It is important to notice, however, that it makes sense to talk about inhibitory or satellite radii only if the dots' distribution is sparse enough to allow identifying dots as individuated single items (Anobile, Cicchini, & Burr, 2016). There is convincing evidence that with densely packed stimuli, which look like textures, the proximity between neighbor dots is not a relevant characteristic (Anobile et al., 2016). This is very likely because the numerosity of differently colored elements in densely packed textures is perceived differently from the number of dots in sparsely distributed patterns (Anobile, Cicchini, & Burr, 2014; Anobile, Turi, Cicchini, & Burr, 2015; Pomè, Anobile, Cicchini, Scabia, & Burr, 2019).

In order to explain why inhibitory patterns look more numerous than patterns generated by the Poisson process, which, in turn, are typically judged to be more numerous than satellite patterns, the OM assumed that each dot (or element) had an impact upon its neighborhood, which can be depicted as a spread of influence in the occupancy radius r_o. In other words, each dot appears to fill or occupy an area in its neighborhood within a fixed radius r_o. This idea that numerosity may be judged on the basis of an apparently filled area was originally proposed by Piet Vos and colleagues (Vos et al., 1988). They further proposed that the occupancy areas of each dot may change size depending on the dot's distance to its nearest neighbor. This assumption, however, was incorrect since the best agreement between the model's predictions and empirical data was obtained when the occupancy radius r_o was assumed to have a fixed value (Allik & Tuulmets, 1991). If two occupancy areas overlap, then the total overall area occupied by these two individual occupancy areas decreases by the size of their overlap. Thus, if two elements are less than $2r_o$ apart, then their occupancy areas overlap, which leads to a reduction of the total occupancy in proportion to the size of overlap. Although the occupancy radius seems to change as a function of image size and the density of visual elements (Allik, Tuulmets, & Vos, 1991), the typical value for the occupancy radius was around 0.5° of visual angle (Allik & Tuulmets, 1991; Im, Zhong, & Halberda, 2016).

Surprisingly, the OM with only one free parameter r_o has survived all known attempts at falsification that have been undertaken in the last 30 years. For example, there was a very meticulous attempt to explain how perceptual grouping affects human number estimation in random dot arrays by the proposition of a k-means clustering algorithm (Im et al., 2016). In addition to their own model, the authors also wrote a code to implement the OM and compared the predictions of these two models. They reached the following conclusion:

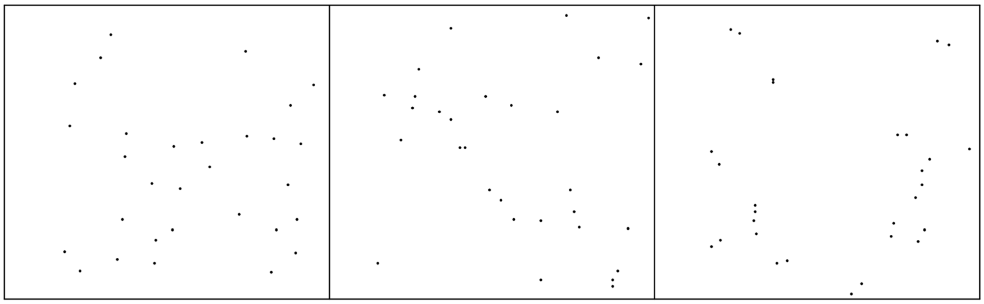

Fig. 1 Three examples of random pattern generation processes distributing N=30 elements over an observation area: (**A**) Inhibitory process, (**B**) Poisson process and (**C**) Satellite process

> "To test whether the occupancy model also generalizes to our current dataset, and to test whether the occupancy radius r is a relatively universal parameter across studies, we exploited the occupancy model to find the best occupancy radius r for our data. /…/ We also computed a group average r by fitting the entire dataset as a single group. These agreed well, and the best-fit occupancy radius r for our dataset was about 0.34° of visual angle, which is very close to the occupancy radius r (18–22' arc corresponding to 0.30–0.37° of visual angle) that was found in the original paper on the occupancy model by Allik and Tuulmets (1991), thereby replicating their study. Although experimental settings and parameters were different between Experiment 2 and the methods of Allik and Tuulmets (e.g., the size of the stimuli and the display areas involved), the human responses during the dot enumeration task in Experiment 2 can be explained by an occupancy radius r quite similar to the original occupancy model. The occupancy model was designed to generate predictions for human number estimation. Using the best-fit value for the occupancy radius r (0.34°) we next found that the occupancy model predictions correlated with human number estimations (R = .485, p < .01). This is somewhat less than the correlation noted earlier for our cluster model predictions of human number estimation (R = .75, p < .01). /…/ Rather than claiming that the cluster model is superior to the occupancy model as a model of number estimation behavior, we would like to highlight the similarity in the approach of these two models – both focus on the overlap and relationships across elements within a display." (Im et al., 2016, pp. 302-303)

The observation that two different algorithms are capable of providing very similar predictions may suggest that the OM could be just an elegant technical tool, but not an adequate description for how the numerosity of random dot arrays is judged in reality. It should also be noted that the concept of occupancy area was never tested by asking whether or not the perceptual system has the necessary mechanisms for the accurate estimation of areas of shapes on the plane.

In one of our previous studies, we spotted that when an observer was instructed to compare the sizes of a set of circles, the comparison was never made in terms of the areas of the circles but instead on the basis of the size of cross-sections or diameters (Allik, Toom, Raidvee, Averin, & Kreegipuu, 2014). We also conducted a study in which the observer was instructed to compare a set of circles on the basis of mean diameter or summary area (Raidvee, Toom, Averin, & Allik, 2020). The results clearly demonstrated that the precision to which a summary area of geometric figures can be judged was much worse than discriminations based on radius or diameter. Other studies have also demonstrated that observers have no direct access to a two-dimensional area and decisions are instead based on a variety of one-dimensional cues (Morgan, 2005; Nachmias, 2008). Ironically, Sverker Runeson proposed the existence of smart visual mechanisms, which can directly register complex perceptual variables, illustrating this proposal with an example of the polar planimeter (Runeson, 1977). Thus, the OM presumes a perceptual mechanism, for judging the area of plane regions, for which the visual system may not have developed the proper tools (Fig. 2).

Another unusual property of the OM is its purely deterministic character: the model's only parameter, occupancy radius r_o, was assumed to be a constant, not a random, variable (Allik & Tuulmets, 1991). It was not the authors' intention to propose a model in which there was no place for representational sensory noise. However, the pressure to use random continuously changing patterns also produced variation in the stimulus attribute that was supposedly used for judgements. Because there was no direct method of separating representational sensory variation from purely physical stimulus variation, the sensory noise remained undisclosed within observable summary noise. However, it would be realistic to assume the occupancy radius r_o is a random variable, which has a mean value and standard deviation. If we assume the occupancy radius r_o is a random variable, then we have to conclude the total area A_o occupied by all dots is also a variable even if the

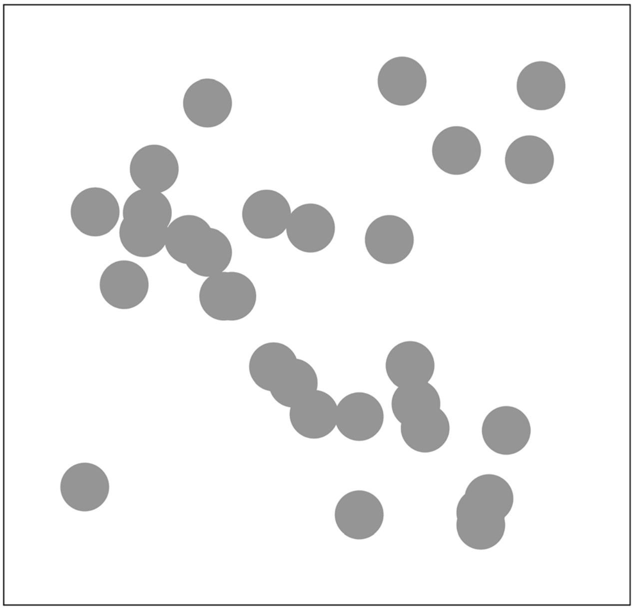

Fig. 2 An illustration of the occupancy area surrounding each dot. Apparent numerosity decreases with the overlapping occupancy areas. The dot pattern is the same as in the middle panel in Fig. 1(B)

same fixed pattern is presented repeatedly. In general, a noisy transmission transforms the number of presented visual elements stochastically into a continuum of psychological states representing their apparent number. This also implies that Thurstone's law of comparative judgement (Thurstone, 1927) is the proper theoretical language for describing how two numerosities can be discriminated. As expected, Thurstonian analysis becomes standard apparatus for the analysis of numerosity discrimination data (Burgess & Barlow, 1983; Halberda, Mazzocco, & Feigenson, 2008; Tokita & Ishiguchi, 2009; Van Oeffelen & Vos, 1982b).

However, it must be noted that application of the Thurstonian approach to discrimination of the number of elements may be problematic (Raidvee, Lember, & Allik, 2017). Normal, or Gaussian, distribution was chosen to represent the continuum of internal states primarily because of computational convenience, not because there was irrefutable empirical evidence in favor of Gaussian distribution. Also, the suitability of a continuum of internal states for representing a discrete variable, such as integers, was not discussed seriously. Although the accurate identification and discrimination of small numbers – also called subitizing – can be explained by assuming that representational noise is fractional (Van Oeffelen & Vos, 1982b), it is still unclear what segments between two integers mean psychologically and how these fractions are experienced by the observer (Raidvee et al., 2017).

Because representing whole numbers on a continuous scale looked somewhat unnatural, an alternative binomial model was proposed (Raidvee et al., 2017). Let us suppose that a perceptual process transforms each element presented on the screen into a corresponding perceptual state. Unlike normal distribution, these states have only two values: zero when the presence of the element was not recorded, or one when the element was recorded. This process can be described as a binomial distribution, where parameter β is the probability that each stimulus element can be noticed and recorded for further processing. This allows us to determine how many elements out of N were registered by the perceptual system provided that the probability of noticing each of them was β. If two patterns are compared, then the one in which more elements were registered is the one that is expected to be chosen.

Remarkably, within approximation, the proposed binomial model is formally identical to the more conventional Thurstonian models, which use normal distribution as a representation of internal perceptual states. For every binomial model with a certain parameter value β there is an approximately equivalent Thurstonian-Gaussian model with the corresponding standard deviation value for the normal distribution (Raidvee et al., 2017, Appendix 2).

Indeed, the binomial model with the parameter β provided an excellent explanation for how two random arrays of elements were discriminated by the number of their elements (Raidvee et al., 2017). However, since the binomial probability β was assumed to be the same for all elements, this model is incapable of explaining the effect of any arrangement or configuration on perceived numerosity. In order to make perceiving probability β sensitive to spatial arrangement of elements, it is necessary to transform this probability β into a function, where β depends on some spatial statistics characterizing proximity between elements.

The main purpose of this study was to find an attribute in the spatial distribution of elements, which could control the binomial parameter β in such a way as to explain how apparent numerosity depends on the configuration of elements. We call this amendment the Proximity Model (PM), which has the central thesis that proximity between elements has an impact on their perceived numerosity. The thesis that needs to be tested is that the spatial proximity between the nearest neighbors is a coefficient that modulates the probability with which each visual element can be registered by the perceptual system. It is proposed that the probability of noticing a dot is higher when there are no neighbors in close proximity. If the distance between an element and its nearest neighbor diminishes the probability of it being counted, it also reduces the apparent number as a consequence.

Methods

A computer simulation was adopted as the approach for this study. There is sufficient evidence to indicate that the OM provides a fairly accurate description of how numerosity in variously distributed random dot patterns is perceived (Allik & Tuulmets, 1991; Bertamini et al., 2016; Durgin, 1995; Im et al., 2016). The strategy is to demonstrate that if the proposed PM will be able to produce predictions that are sufficiently close to the predictions of the OM then this will be proof of the PM also being a good fit for the empirical discrimination data that was collected in previous studies.

Stimulus materials

Random dot patterns were generated and distributed within a virtual observation area of 560 × 560 pixels, which was surrounded by a 120-pixel wide safety belt that was needed for computational purposes. In all cases, 30 dots (N = 30) were stochastically distributed over 313,600 available positions. All patterns were generated using one of the three processes described in the original paper (Allik & Tuulmets, 1991) based on the previously described principles for the generation of random patterns (Schachter & Ahuja, 1979).

The Poisson process (Fig. 1B) Two random numbers x and y were generated ranging from 1 to 560 in order to specify the coordinates of a dot. To avoid overlap it was checked before placing a dot that there were no dots already placed in the inhibitory radius of $r_i = 4$ pixels. If a dot was already in this area, then new pairs of x and y coordinates were generated until an empty space was found.

The inhibitory process (Fig. 1A) This process prohibited any two dots from being closer to each other than allowed by an inhibitory radius r_i. Three different values for the inhibitory process were used: r_i = 20, 40, and 60 pixels.

The satellite process (Fig. 1C) Half of all the dots (N = 15) were randomly distributed in the plane using the Poisson process. Each of the remaining 15 satellite dots were assigned to their respective parent. The satellite dot was randomly positioned within the satellite radius r_s from the parent dot but could not be closer than the already mentioned distance of 4 pixels. Three different satellite values were used: r_s = 30, 50, and 70 pixels.

In total, there were seven types of random dot patterns: a completely random Poisson pattern, and three inhibitory and three satellite patterns.

Occupancy model

The OM, as already mentioned, has only one parameter, which is that the radius of occupancy r_o within which each dot influences or fills its immediate surrounding (Allik & Tuulmets, 1991). For our computer experiments, the occupancy radius had a fixed value, r_o = 40 pixels. A filled circle was drawn over each of the 30 dots with the center coinciding with the dot's x and y coordinates. After that the number of filled pixels was counted to determine the total area A_o occupied (or filled) collectively by all the dots. The total occupancy area was expressed as the percentage of filled pixels in relation to the total number of pixels in the observation area.

Proximity model

With the proposed binomial model each dot is counted (or noticed) with the probability β (Raidvee et al., 2017). One way how to make this model sensitive to the spatial arrangement of dots is to postulate that the element's counting probability β decreases as the distance to its neighbor dots diminishes. Thus, we assume that for each j^{th} dot there is an individual coefficient c_j, which diminishes the probability of this dot being counted. Whereas in the original binomial model the expected number of noticed elements is $E(N) = N\beta$, in comparison in the PM, the coefficient depending linearly or in some other form on the distance to the nearest neighbor d_{nn} is used:

$$E(N) = c_1\,\beta + c_2\,\beta + \ldots + c_N\,\beta = \beta\textstyle\sum_{j=1}^{N} c_j.$$

The way in which the coefficients c_j are distributed among individual elements is irrelevant, since what matters is the sum of these coefficients. Thus, after summing up binomial functions it becomes irrelevant how individual coefficients c_j, are distributed among individual elements. Assuming that the nearest neighbor d_{nn} is linearly related to these coefficients c_j, it would allow us to know the mean value of the nearest neighbor distances.

Thus, our goal was to demonstrate that the occupancy area A_o can be predicted with sufficient accuracy from the mean nearest neighbor distance d_{nn}, which was computed on the same random pattern.

Results

One thousand realizations for each of the seven different random processes were generated. The mean occupancy area A_o for arrays generated by the Poisson process was 27.8% of the viewing area. As expected, the inhibitory processes with the inhibitory radiuses r_i = 20, 40, and 60 pixels filled a larger percentage of the viewing area, covering 28.8%, 31.0%, and 33.3%, respectively. Because the satellite processes created more overlap between individual occupancy areas, the occupancy areas A_o for the satellite radiuses r_s = 30, 50, and 70 pixels were smaller – 19.0%, 21.8%, and 24.2% respectively.

For each random array of dots, we also computed the distance d_{nn} from each dot to their nearest neighbor. We sorted these 30 distances so that the first ranking position was reserved for the shortest distance, the second ranking position for the next shortest distance, and so forth until the final 30th ranking position was allocated to the dot that had the longest distance to its the nearest neighbor. Figure 3 demonstrates the distribution of the d_{nn} values for the pattern-generation processes used. As expected, for inhibitory processes the mean nearest neighbor distances are larger (with the exception of the last two positions) than for the patterns generated purely by the Poisson process. In arrays generated by the satellite process, the nearest neighbors were found to be in close proximity to another dot.

Next we aggregated the nearest neighbor distance across all the 30 positions of ranking. Figure 4 demonstrates the relationship between the average nearest distance d_{nn} and the total occupancy area A_o that was computed for the same type of pattern. The regression function of the mean nearest neighbor distance d_{nn} and the total occupancy area A_o is very close to a straight line (r = .9996). Although the mean nearest neighbor distance and the occupancy area for the same pattern are not necessarily mathematically identical, they for all intents and purposes seem to be two practically interchangeable measures. From knowing the mean nearest neighbor distance, we can almost exactly predict the value of the total occupancy area for this particular distribution of dots.

Since it may not be initially obvious that the occupancy area and mean distance to the nearest neighbor are identical or nearly identical statistics, Fig. 5 demonstrates all the nearest neighbor distances for the same Poisson pattern, which is

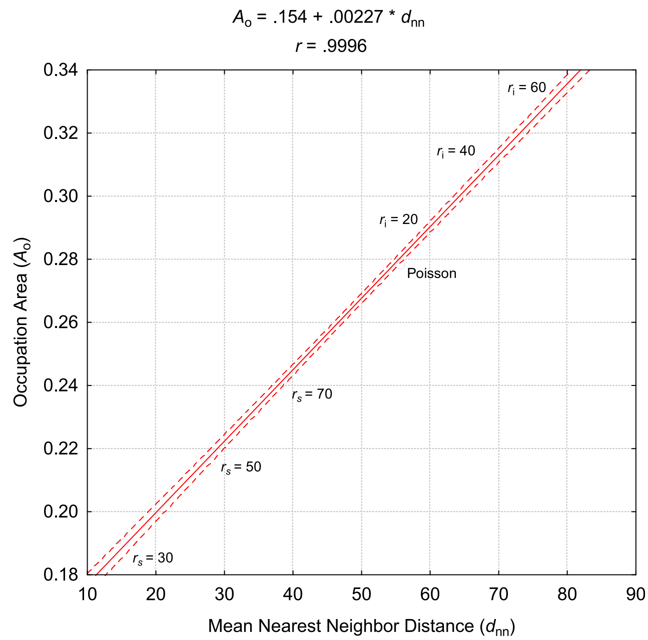

Fig. 4 The relationship between the total occupation area A_o and the mean distance to the nearest neighbor d_{nn}

shown on the middle panel of Fig. 1 and for which Fig. 2 illustrates how the total occupancy area is found. In Fig. 5, all 30 nearest neighbor distances are shown as dipoles connecting two closest neighbors. The average length of these 30 dipoles can be accurately predicted from the total occupied area demonstrated in Fig. 2. The equivalence of these two measures indicates that all predictions that are made on the basis of the OM can also be made, without any loss of precision, on the basis of the PM.

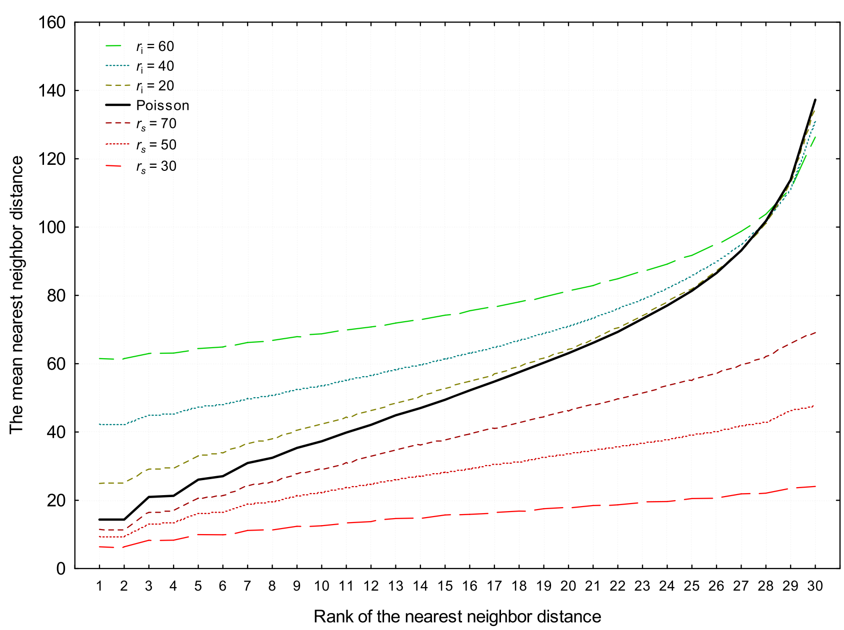

Fig. 3 The mean nearest neighbor distances for three different pattern generation processes – inhibitory (i), Poisson, and satellite (s) – as a function of each element's distance ranking position

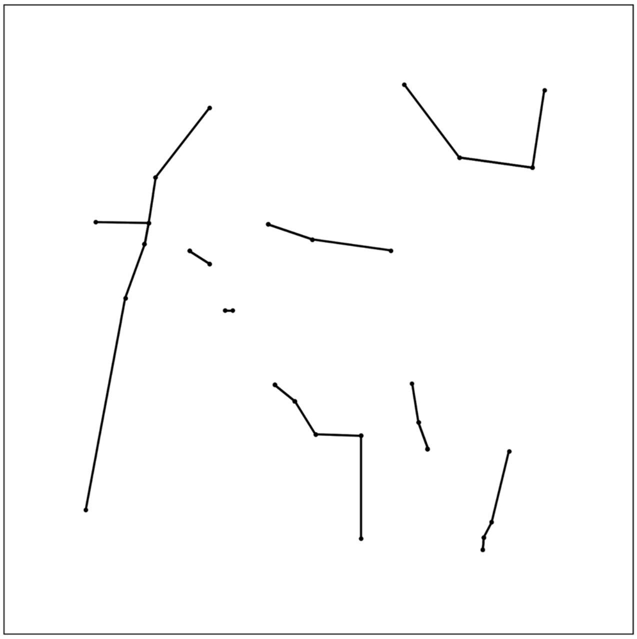

Fig. 5 The nearest neighbor distances shown by dipoles connecting two dots for the same pattern portrayed in Figs. 1B and 2

It is possible that not all the nearest neighbor distances equally influence the relationship between occupation area and nearest neighbor distance. For example, one possibility is that the proximity of neighbors starts to diminish perceived numerosity if they are closer than a critical distance. If this is true, we can expect the total occupancy area to be predicted more on the basis of the nearest neighbors that are closest together and less on the basis of pairs that have the largest values. This predicts that the first few positions on the distance rankings contribute more significantly to the correlation than the latter ones. However, data portrayed a different story. The strongest correlation around .97 between A_o and the mean d_{nn} was observed for middle-rank positions, not for either the shortest or the longest distances between the nearest neighbors. These correlations can be approximated with a negative quadratic function, with the maximum being between the 15th and 16th ranking positions. This appears to be further proof of the mean nearest neighbor distance across all dots being the best predictor of total occupancy area.

Discussion

Although the OM has provided a good fit to a majority of the numerosity discrimination data that has been collected over the last 30 years (Allik & Tuulmets, 1991; Durgin, 1995; Im et al., 2016), it may not offer an adequate description of how the human observer extracts numerosity information from sparsely distributed random dot patterns. After considering the evidence, it seems very unlikely that the central postulate of the model – the occupied or filled area is what is judged when observers are instructed to judge numerosity – is valid. There is a lot of indication that the area of forms (or regions) cannot be directly coded in the perceptual system, causing the area to be judged with considerable inaccuracy (Morgan, 2005; Nachmias, 2008; Raidvee et al., 2020). Furthermore, the idea of judging an occupied area was assumed, yet never tested with sufficient thoroughness. It seems that while the OM remains a simple and elegant computational algorithm, it nevertheless cannot be considered to be an actual description of what really happens in the perceptual system when impressions about numerosity are formed.

A remarkable coincidence between the occupied area A_o and the mean nearest neighbor distance d_{nn} demonstrates that the two different algorithms are able to compute what is essentially the same statistical property characterizing the spatial distribution of patterns. This is also the main reason for why the k-mean model and the OM provide very similar predictions (cf. Im et al., 2016). Although the occupied area and the mean distance to the nearest neighbor may not be mathematically identical, from a practical point of view, it is difficult to find any meaningful differences between them. This was the main reason why no new data were collected: if the occupancy area A_o accurately predicts numerosity discriminations then the mean nearest neighbor distance d_{nn} predicts them equally well.

Nevertheless, there is no question about which of these two statistical parameters – the occupied area or the nearest neighbor distance – is more fundamental, as the nearest neighbor distance is one of the key concepts of spatial statistics (Gelfand, Diggle, Fuentes, & Guttorp, 2010; Ripley, 1981). For example, nearest neighbor analysis is also a practical method for discovering patterns in plant ecology (Dixon, 2002; Perry, Miller, & Enright, 2006) and archeology (Pinder, Shimada, & Gregory, 1979). To confirm, the above-presented analysis demonstrated how the distribution of nearest neighbor distances, as well its mean value, is sensitive to the inhibitory and satellite processes, which both bias distance distribution in opposite directions from the non-constrained Poisson distribution (see Fig. 3). Although computing the total occupancy area is easy to imagine, it does require making some auxiliary assumptions, including about the size of the occupancy radius.

If the mean nearest distance is judged then we are in fact talking about extraction of summary statistics or what is often called ensemble perception (Allik, Toom, Raidvee, Averin, & Kreegipuu, 2013; Ariely, 2001; Chong & Treisman, 2003; Whitney & Leib, 2018). Because the perceptual system cannot count the exact number of elements without support from a symbolic system, it needs to rely on some sort of summary statistics characterizing how close or distant on average elements are from one another. If they are less distant as could happen in the case of random bombarding then their perceived number diminishes. If the nearest neighbors are sufficiently far away then their number seems to increase. Although we are aware of studies in which the link between perception of the mean size and numerosity was examined (e.g., Utochkin & Vostrikov, 2017), there are no studies where perceived proximity between elements was distinguished from their average density (cf. Anobile et al., 2014; Dakin, Tibber, Greenwood, Kingdom, & Morgan, 2011; Durgin, 1995; Raphael & Morgan, 2016).

It is relevant to note that the nearest neighbor distance is a rudimentary statistic. In many situations, the knowledge of distances to at least k-nearest neighbors is more informative (Dudani, 1976). Based on nearest neighbor distances alone it is impossible, for instance, to reconstruct even the spatial frequency content of that image, if not picture itself. As early as 1962, Bela Julesz observed that texture pairs with identical second-order (or dipole) statistics but different third- and higher-order statistics were usually not discriminable without scrutiny (Julesz, 1962). Thus, for two textures or patterns to be discriminated they need to have different second-order or dipole statistics (Julesz, 1980). Dipoles connecting the nearest neighbors form only a tiny fraction of the whole family of dipoles involving all lengths and orientations. This is the reason why it seems unlikely that numerosity perception can be explained in the framework of texture perception (cf. Anobile et al., 2016; Durgin, 1995; Morgan, Raphael, Tibber, & Dakin, 2014; Raphael & Morgan, 2016; Victor, Conte, & Chubb, 2017).

The ease and simplicity with which the OM could be rewritten in terms of mean nearest neighbor distances inclines one to think that neither can be considered to be an empirically falsifiable model (cf. Im et al., 2016). For example, the observed accuracy with which the mean size of an array of elements can be perceived may be explained by an assumption that a small number of elements was taken into account for averaging (Myczek & Simons, 2008). The distinction between universal theoretical languages and empirically falsifiable models was thoroughly explained by Jones and Dzhafarov (2014). The proposed binomial version of psychophysical analysis (Raidvee et al., 2017) is a universal language for describing results when two random arrays of dots are required to be discriminated by their numbers. The model's only parameter β is open to many plausible psychological or neuronal interpretations. However, as was conjectured, this parameter may change its value depending on the proximity of the closest elements. It was surprising that the degree of overlap between different individual occupancy areas corresponds almost perfectly to the mean nearest neighbor distance across all dipoles. This implies the existence of a perceptual mechanism that is capable of effortlessly and accurately estimating the mean proximity between all or a sample of the nearest neighbor pairs. However, upon reviewing Fig. 5 nothing seems to tell us that the perceptual system has a machinery that is capable of extracting dipoles connecting nearest neighbors and estimating their mean length.

One plausible mechanism capable of taking proximity between elements into account is crowding. Crowding is usually defined as the deleterious influence of nearby elements on visual discrimination (Levi, 2008; Pelli, 2008; Whitney & Levi, 2011). This definition seems to embrace the fact that shortening the mean distance to the nearest neighbor decreases perceived numerosity. This decrease signifies how the proximity between neighbor dots exerts a deleterious impact which results in reduced visibility or salience of the dots. Several studies have scrutinized a possible link between crowding and perceived numerosity (Anobile et al., 2015; Balas, 2016; Chakravarthi & Bertamini, 2020; Valsecchi, Toscani, & Gegenfurtner, 2013). For example, it was noticed that the perceived numerosity of a peripheral cloud of dots was judged to be less numerous in comparison to a central cloud of dots, particularly when the dots were highly clustered (Balas, 2016; Valsecchi et al., 2013). On the other hand, it was found that these results were incompatible with a crowding account of numerosity underestimation and instead point to separate mechanisms for object identification and number estimation (Chakravarthi & Bertamini, 2020). In another study, the authors reached the conclusion that perception of numerosity was not mediated by crowding (Anobile et al., 2015). However, the largest amount of skepticism over associating a proximity effect with crowding arose from the fact that crowding only operates over a limited distance. When separation between visual elements exceeds a certain critical distance then the effect of crowding vanishes (Whitney & Levi, 2011). It was proposed that this critical spacing between two visual elements corresponds to a fixed cortical distance (Pelli, 2008). Since our modelling results demonstrated that very short nearest neighbor distances contributed to the predictions of the OM even less than intermediate distances, they are not compatible with the observation that the effect of crowding disappears beyond a critical distance. We can conclude this discussion with the notion that there are not enough data to decide whether the proximity effects that have been observed are based on an entirely different mechanism or due to a new category of crowding.

Without new critical experiments, the PM remains a universal language of description that is compatible with many specific mechanisms for implementing the idea of proximity. In this respect the PM is not very different from its predecessor the OM, which has survived critical testing for nearly 30 years. Following the admission that there are multiple ways to operationalize the concept of proximity between elements in random arrays of dots (Im et al., 2016), the next task is to invent an experimental protocol, which is capable of identifying the perceptual attribute that is actually used in forming impressions about numerosity.

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