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PROGRAM ABSTRACTS/ALGORITHMS

BES: An algorithm for estimation in log-normal distributions

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A Bessel function used to obtain uniform minimum-variance unbiased estimates for log-normal distributions is presented in this paper, along with an algorithm (BES) that approximates this function.

In magnitude-estimation experiments, it is common to have a single subject make several judgments at each of the stimulus intensities employed. Let ψ_{ij} denote the j^{th} numerical response of a subject for stimulus intensity I_i. It is well known that the variability of these responses increases with increasing values of I_i. Often, an adequate model to account for such responses in terms of the intensities is provided by a power function of the form

$$\psi_{ij} = \alpha V_{ij} I_i^{\beta}, \tag{1}$$

where α and β denote constants, and V_{ij} denotes a log-normally distributed error component. This model may be linearized by taking logarithms on both sides of Equation 1:

$$y_{ij} = \beta x_i + \gamma + z_{ij}, \tag{2}$$

with $y_{ij} = \ln(\psi_{ij})$, $x_i = \ln(I_i)$, $\gamma = \ln(\alpha)$, and $z_{ij} = \ln(V_{ij})$. Under the assumption that the V_{ij} are log-normally distributed, the z_{ij} are normally distributed. From Equation 2 it is evident that the parameters β and γ may be estimated by least squares regression, and the adequacy of Equation 2 may then be tested through analysis of variance (e.g., Coleman, Graf, & Alf, 1981).

The aim, however, is not to estimate γ but to estimate $\alpha = e^{\gamma}$. The problem is that, if $\hat{\gamma}$ denotes an unbiased estimate of γ, then $e^{\hat{\gamma}}$ is not an unbiased estimate of α. This problem has been discussed amply by several authors. Thomas (1981) presents two methods for estimating α which are highly unstable unless the number of observations is unfeasibly large. Elzinga (1985) describes a method to obtain an estimate of α that is uniform minimum-variance unbiased (UMVU). This method requires an infinite series expansion of a Bessel-function. In this paper I present an algorithm to approximate this series expansion, and I compare this approximation with one proposed by Aitchison and Brown (1957).

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Estimating α. Let $\hat{\beta}$ and $\hat{\gamma}$ denote the least squares regression estimates of β and γ respectively in Equation 2. Furthermore, let s^2 denote the variance of the errors of regression, that is, the residual variance $s^2 = \Sigma(y_j - \hat{\beta}x_j - \hat{\gamma})^2/(n-2)$ with n = the number of intensity levels. The UMVU-estimate $\hat{\alpha}$ of α in Equation 1 is then given by (Elzinga, 1985)

$$\hat{\alpha} = e^{\hat{\gamma}} g_{n-2}(cs^2), \tag{3}$$

with

$$g_n(t) = 1 + \frac{n-1}{n} t + \sum_{k=2}^{\infty} \frac{(n-1)^{2j-1} t^j}{n^j(n+1) \cdots (n+2j-3)j!} \tag{4}$$

and

$$c = 1/2 - [(n-1)\Sigma x_i^2]/[n(n-2)\Sigma(x_i - \overline{x})^2]. \tag{5}$$

Approximating the Bessel function $g_n(t)$. Since $g_n(t)$ is an infinite series, its numerical values can only be approximated. Aitchison and Brown (1957) proposed to use

$$g_n^*(t) = e^t[1 - t(t+1)/n + (3t^4+22t^3+21t^2)/(2n^2)] \tag{6}$$

to approximate Equation 4.

Below I show that Equation 6 considerably overestimates $g_n(t)$. I propose an approximation that arises from a simple recurrence relation in the infinite series in Equation 4. Let A denote the infinite series in Equation 4; then

$$g_n(t) = 1 + \frac{n-1}{n} t + A \tag{7}$$

and

$$A = \sum_{k=2}^{\infty} A_k$$

with

$$A_2 = [(n-1)^3 t^2]/[n^2(n+3)2], \tag{8}$$

$$A_k = A_{k-1}[(n-1)^2 t]/[n(n+2k-3)k] \qquad (k > 2).$$

This approximation may also be used for estimation of the mean μ_v and variance σ_v^2 on the basis of the mean $\overline{v}$ and variance s_v^2 of a sample of a log-normally distributed

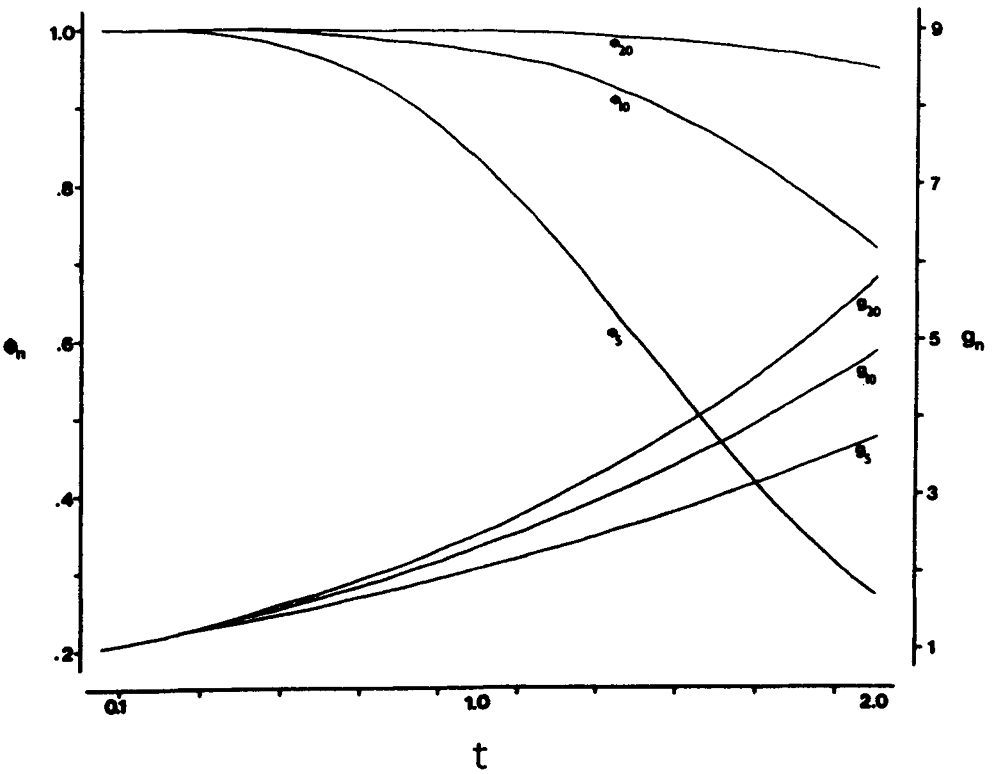

Figure 1. Plots of $\phi_n(t)$ (left ordinate, upper curves) and $g_n(t)$ (right ordinate, lower curves) for different values of t (abcissa) and n.

variable V, since (Finney, 1941) UMVU-estimates for μ_V and σ_V^2 are given by

$$\mu_V = e^{\bar{v}} g_{n-1}(s_V^2/2), \tag{9}$$

$$\sigma_V^2 = e^{2\bar{v}}\left[g_{n-1}\left(2s_V^2\right) - g_{n-1}\left(\frac{n-2}{n-1}\, s_V^2\right)\right]. \tag{10}$$

Description of BES

The listing presented in this paper is that of the FORTRAN-routine BES, which is a double-precision iterative procedure that computes an approximation of $g_n(t)$, based upon Equations 7 and 8. The accuracy of the approximation depends upon two parameters: the maximum number of iterations (MAXIT; i.e., the maximum number of terms in the sum A) and the minimum absolute difference (DMIN) between consecutive values of the approximated value. The algorithm stops if either of these limits is attained.

Obviously, the speed of convergence depends upon the values of n and t: Convergence slows down with increasing values of n and t. My experience indicates that even for large values of n and t, an approximation with an error less than 10^{-8} is easily produced within 100 iterations. In applications, BES will usually do so in less than 15 iterations, which is certainly feasible, even on microcomputers.

I compared the performance of BES with the performance of the approximation suggested by Aitchison and Brown (1957) (Equation 6) by computing the ratio $\phi_n(t) = g_n(t)/g_n^*(t)$ with n = 5, 10, and 20, and t ranging from 0.05-2.0. If Equation 6 were to be considered an acceptable alternative to BES, the ratio $\phi_n(t)$ should be close to 1.0 for relevant values of n and t. In Figure 1, I plotted the values of $\phi_n(t)$. Clearly, $\phi_n(t) < 1$; hence BES is preferable to Equation 6 (most experiments do not involve more than 8-12 different levels of intensity). In this same figure, I plotted the values of $g_n(t)$ as computed by BES. Based on the values shown in Figure 1, I conclude that the use of Equation 6 may lead to a nontrivial overestimation of g_n and, therefore, of α in Equation 1.

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      SUBROUTINE BES(N,T,G,MAXAT,DMIN,IER)
C*****************************************************
C
C     I N P U T  -  P A R A M E T E R S
C
C   N      :  FUNCTION ARGUMENT,
C             POSITIVE INTEGER
C   T      :  FUNCTION ARGUMENT,
C             DOUBLE PRECISION REAL
C   MAXIT  :  MAXIMUM NUMBER OF ITERATIONS,
C             POSITIVE INTEGER
C   DMIN   :  MINIMUM DIFFERENCE IN CONSECUTIVE
C             ITERATIONS, DOUBLE PRECISION REAL
C
C
C     O U T P U T  -  P A R A M E T E R S
C
C   G      :  FUNCTION VALUE AT ARGUMENTS
C             N AND T, DOUBLE PRECISION REAL
C   IER    :  USER-ERROR PARAMETER, INTEGER
C             IER=0  NO USER-ERROR
C             IER=1  N OR MAXIT OR DMIN ARE
C                    NONPOSITIVE
C
C*****************************************************
      IMPLICIT DOUBLE PRECISION (A-H,O-T)
      IER=0
      IF(N.GT.0) GO TO 2
    1 IER=1
      G=1.D0
      RETURN
    2 IF(DMIN.LE.0.D0) GO TO 1
      D=N
      DM1=D-1.D0
      G=1.D0+T*DM1/D
      IF(MAXIT.GT.0) GO TO 3
      IER=1
      RETURN
    3 J=2
      A=(DM1/D)*(DM1/D)*(DM1/(D+1.D0))*(T/2.D0)*T
    4 B=G
      G=G+A
      IF(DABS(G-B).LT.DMIN) RETURN
      J=J+1
      IF(J.GT.MAXIT) RETURN
      RJ=J
      B=(DM1/D)*(DM1/(D+2.D0*RJ-3.D0))*(T/RJ)
      A=A*B
      GO TO 4
      END
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