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Notes and Comment

Probability of being correct with 1 of M orthogonal signals

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A recent research interest led us to compute the expected proportion of correct responses when the signal is one of M orthogonal signals added to Gaussian noise. This is formally the same as the probability of a correct response $P_M(c)$ in an M-alternative forced-choice task, and therefore our efforts duplicate some earlier calculations published by Hacker and Ratcliff (1979). Our calculated values are almost precisely the same as theirs. We suggest a simple approximation that yields the correct value of $P_M(c)$ to within one quarter of a percent for several values of M.

The problem is mathematically equivalent to the following. Suppose there are M Gaussian variables, all with unit variance. One of the variables, the signal, has mean d', while the other noise-alone variables—there are $M-1$ of them—have mean zero. What is the probability, $P_M(c,d')$, that the signal random variable is the greatest of all M samples? In equation form, it is

$$P_M(c,d') = \int_{-\infty}^{+\infty} \phi(x-d')\Phi(x)^{M-1}dx, \qquad (1)$$

where ϕ is the ordinate of the normalized Gaussian density function and Φ is the area from minus infinity to x under ϕ.

This integral was evaluated for d' between 0 and 6 for small M and 0 and 8 for larger M in steps of 0.01, using a software package called *Mathematica* (Wolfram, 1988). For $M = 2$, the signal and noise samples are both Gaussian, and the probability is equivalent to the difference of the signal minus the noise variables being greater than zero. This difference is also Gaussian and tabulated to 15 decimal places in Abramowitz and Stegum (1956). Our calculated values agree with their table out to 12 decimal places for $0 < d' < 2.8$. Thus, we feel that our calculation is reasonably accurate. If we interpolate our calculation to obtain values for d' at 0.01 steps in probability, they agree with Hacker and Ratcliff's (1979) values almost exactly. For $M = 1{,}000$, for example, we disagree at the second decimal place (we obtained $d' = 3.50$ rather than 3.51 at $P_{1,000}[c] = 0.6$). For other values of M, our calculations rarely differ from theirs, and any discrepancy is less than 0.01.

Another way to regard Equation 1 is to consider it the probability that the signal random variable is greater than the extreme value of $M-1$ samples from the noise-alone distribution. The greatest of several independent samples from a Gaussian distribution is also approximately normal (Cramer, 1946). The difference between the signal value and the extreme value must also be approximately normal. We therefore were led to approximate Equation 1 by means of another integral, namely

$$P'_M(c,d') = \int_{-\infty}^{(d'-m)/\sigma} \phi(x)dx = \Phi\left(\frac{d'-m}{\sigma}\right), \qquad (2)$$

where both the mean, m, and the standard deviation, σ, are adjusted to produce the least mean square error between the values of $P_M'(c)$ and $P_M(c)$. Table 1 gives these values of the mean and sigma for several values for M. For example, if $M = 8$ and $d' = 2.06$, then $m = 1.34579$ and $\sigma = 1.18216$; thus, $(d' - m)/\sigma = 0.6042$, so that $P'_8(c) = .7277$. Using Equation 1, for $M = 8$, we calculated $P_8(c) = 0.7280$ when $d' = 2.06$, a difference of about 0.0003. For each value of M, we list in the table the maximum error, either positive or negative, between this approximation and that given by Equation 1. As can be seen, the approximation, $P_M'(c)$, yields a value that differs from $P_M(c)$ by at most 0.25%. We should note that in determining the values of m and σ in Table 1, both values were adjusted to minimize the squared deviations in $P'_M(c)$ over all values of d'. We did not use the constraint, $P_M'(c,0) = \Phi(-m/\sigma) = 1/m$. If that constraint

Table 1

M	Mean (m)	Sigma (σ)	Maximum-Error	
2	0.00012	1.41415	0.0000	−0.0000
3	0.55653	1.30384	0.0014	−0.0006
4	0.83865	1.25471	0.0019	−0.0010
5	1.02194	1.22626	0.0023	−0.0012
6	1.15626	1.20689	0.0023	−0.0015
7	1.26072	1.19261	0.0024	−0.0015
8	1.34579	1.18216	0.0025	−0.0015
16	1.73062	1.14052	0.0021	−0.0016
32	2.05200	1.11466	0.0018	−0.0015
64	2.33356	1.09624	0.0017	−0.0013
100	2.50033	1.08742	0.0016	−0.0013
500	3.03338	1.06527	0.0013	−0.0010
1000	3.23885	1.05917	0.0011	−0.0010

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is used, the absolute error can be sizable for middle values of d'.

Most psychologists will be more interested in the inverse calculation, namely the value of d' corresponding to a given $P_M(c)$ value. The value of d' as a function of $P_M(c)$ is not easily approximated, because d' goes to infinity as $P_M(c)$ approaches one. The best way to determine d' is to interpolate. As we have stated, Hacker and Ratcliff's (1979) tables are apparently quite accurate. If our approximation is used instead of Equation 1, then the interpolated value of d' will be in error. The difference is small ($d' \pm 0.01$) for values of $P_M(c)$ between .95 and $0.05+1/M$. But for extreme values, the error can often be as large as ± 0.05. If such an error can be tolerated, then Equation 2 provides a very simple way to convert $P_M(c)$ to d' for different values of M. If more accurate calculations are required, Hacker and Ratcliff's table can be used. We will be glad to send readers a copy of the tables obtained from our more detailed calculation, Equation 1, if the request will include a high-density diskette that is compatible with IBM microcomputers.

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