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A new method for determining the personal constants in the Luneburg theory of binocular visual space

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Based on the Luneburg theory of binocular visual space, the equation for a visual circle (an apparent circle) was derived to estimate the personal constants, σ and K, in this theory. Using small points of light in an otherwise dark room, five observers were asked to construct a visual circle on the horizontal plane with a standard radius of 50 or 100 cm in the median plane. The observation distance for the small circle was 250, 450, or 700 cm, and that for the large circle was 700, 1,030, 1,300, or 1,600 cm. The personal constants calculated from the radii of six directions in each circle were found to be inconsistent with those expected from this theory. The σs obtained were remarkably larger than those observed in earlier studies and increased systematically as the observation distance increased. Almost all Ks were negative, with most being less than minus one. Possible factors responsible for these inconsistencies are discussed with reference to the results of previous experiments.

The Luneburg theory of binocular visual space (Luneburg, 1947, 1950) may be looked upon as an epochal theory designed to describe and predict the characteristics of an individual's visual space in situations in which experiential and other so-called psychological factors are excluded. This theory has encountered some difficulties, however, especially with its assumptions of constant negative curvature of visual space and of the form of the mapping function which relates physical space to visual space. With respect to the mapping function, Blank (1953, 1958a, b, 1959) proposed a modified theory without altering the main structure of the Luneburg theory. Many workers have since concerned themselves with the original or modified theory from theoretical and experimental viewpoints. Some of the results of various experiments carried out to test the validity of the theory confirmed it (e.g., Blank, 1961; Foley, 1964; Hagino, Yoshioka, & Hirabayashi, 1963; Hardy, Rand, Rittler, Blank, & Boeder, 1953; Indow, Inoue, & Matsushima, 1962a, b; Zajaczkowska, 1956a, b), while others failed to do so (e.g., Foley, 1972; Ishii, 1962, 1964; Squires, 1956).

Luneburg proposed two personal constants, σ and K. Here σ indicated the precision of depth perception of the individual and K the curvature of his visual space, with the curvature being hyperbolic, Euclidean, or elliptical, when K was less than, equal to, or larger than zero, respectively. To determine these personal constants, the parallel and distance alley experiments and the 3- and 4-point experiments have been used. However, as has been pointed out, in both of these experiments some problems have been encountered in experimental or computational procedure to determine the personal constants. The construction of the parallel and distance alleys were so difficult that Hardy, Rand, and Rittler (1951) concluded that the alley experiment was inadequate as a test of personal constants because of the effect on the results of preconception, experience, and judgment habits of the observer. To estimate the personal constants from the results of the alley experiments, it is necessary to determine the y-axis intercepts of the tangents to a parallel and a distance curve at the farthest point of each curve (Luneburg, 1950). Despite the extreme sensitivity of the personal constants to the slightest change in these intercepts, as noted by Indow et al. (1962a), the methods proposed to determine the intercepts were neither precise nor sufficient. Hardy et al. (1951) obtained the intercepts by drawing the best tangent to the more remote parts of each curve. Zajaczkowska (1956a) recommended that the intercepts be calculated by passing a straight line through only the two farthest points of each curve. The 4-point experiment

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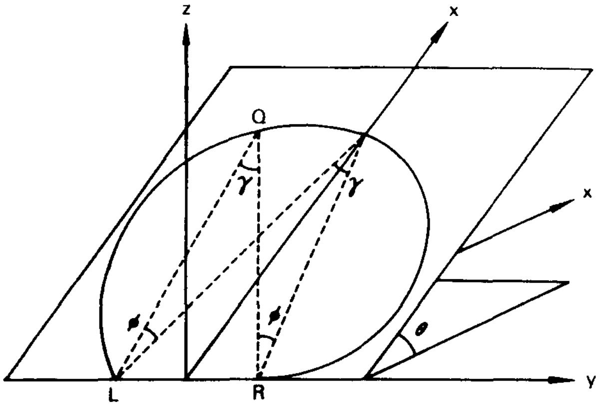

Figure 1. Schematic illustration of Cartesian coordinates (x, y, z) and bipolar coordinates (γ, ϕ, θ) in physical space.

was originally devised by Luneburg (1950) to overcome the insensitivity of the 3-point experiment in estimating both constants, with the intention that the latter was for obtaining σ and the former for K. However, Zajaczkowska (1956a) presented a different procedure in which the results of the 4-point experiment were used to modify the tentatively adopted values of both constants obtained by the 3-point experiment. The underlying logic of this computational procedure seemed to be so vague that Indow et al. (1962a) criticized the logic inherent in the procedure as being beyond the authors' comprehension.

As remarked above, both the alley experiments and the 3- and 4-point experiments involved experimental and computational difficulties and possibly, as a consequence, the personal constants varied considerably even within an individual (Yoshioka, 1965). Although the equilateral triangle and the isosceles right-triangle experiments were proposed by Blank (1958b) and were used by Ishii (1964) and Foley (1968, 1972), what could be determined by these experiments was only the sign of K of two personal constants. Therefore, it seemed useful to develop a simple and more feasible method for determining these constants. For this purpose, a point of light was presented slightly below the eye level in the median plane, and the observer was asked to position other points of light on the same horizontal plane so as to form an apparent circle (the visual circle) using the first point of light at its center (Figure 5). This task was understood and completed without much difficulty by most observers. From each of the adjusted positions of the light points, σ and K could be calculated readily, using equations derived from the equation of the visual circle which was based on the Luneburg theory. Because as many pairs of both constants as the number of adjusted points of light can be determined in a single visual circle constructed, this method is efficient and effective not only in determining the personal constants but also in examining whether or not any change developed in the constants with change of experimental conditions.

DERIVATION OF THE THEORETICAL FORMULA

The Luneburg Theory

Before deriving the equation for the visual circle, it is necessary to describe the Luneburg theory briefly. Luneburg introduced Cartesian coordinates (x, y, z) and bipolar coordinates (γ, ϕ, θ) into the physical space which contained the visual object (Figure 1). In Figure 1, L and R represent the center of rotation of each eye and the distance between them is the unit of length being defined as $\overline{LR} = 2$. Q is the binocularly seen stimulus point. The line to x′ is the projection of the coordinate x to the elevated plane forming the angle of θ with the horizontal plane. The circle in the elevated plane is the Vieth-Müller circle which passes through point Q. If the convergence angle, γ, is sufficiently small, the following equations relate the two coordinate systems:

$$\left.\begin{aligned} x &= (2\cos^2\phi\cos\theta)/\gamma \\ y &= (2\sin\phi\cos\phi)/\gamma \\ z &= (2\cos^2\phi\sin\theta)/\gamma. \end{aligned}\right\} \quad (1)$$

In addition to these coordinates, Luneburg also introduced the Cartesian coordinates (ξ, η, ζ) and the polar coordinates (ϱ, φ, ϑ) of points into the visual space (Figure 2) which correspond to the points in the physical space. Point P in the visual space corresponds to the point Q in the physical space (Figure 1). O represents the subjective egocenter of the observer and ϱ is the distance from O to P.

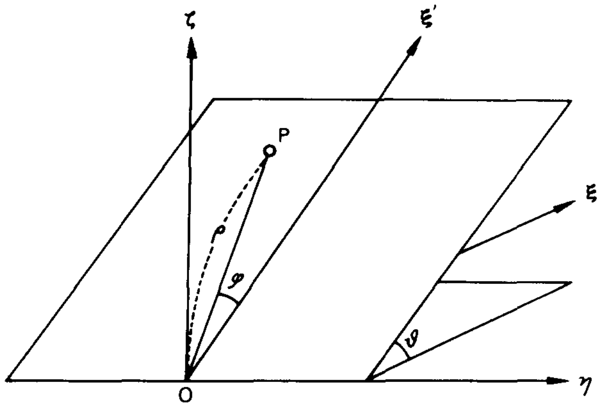

Figure 2. Schematic illustration of Cartesian coordinates (ξ, η, ζ) and polar coordinates (ϱ, φ, ϑ) in visual space.

The relation between these two coordinate systems is given by the following equations:

$$\left.\begin{aligned} \xi &= \varrho \cos\varphi \cos\vartheta \\ \eta &= \varrho \sin\varphi \\ \zeta &= \varrho \cos\varphi \sin\vartheta. \end{aligned}\right\} \quad (2)$$

The visual distance, D, between point $P_1(\xi_1, \eta_1, \zeta_1)$ and $P_2(\xi_2, \eta_2, \zeta_2)$ is given by the following equation:

$$\frac{1}{(-K)^{1/2}} \sinh\left[\frac{1}{2}(-K)^{1/2}\frac{D}{C}\right] = \frac{[(\xi_1-\xi_2)^2+(\eta_1-\eta_2)^2+(\zeta_1-\zeta_2)^2]^{1/2}}{\left[\left(1+\frac{K}{4}\varrho_1^2\right)\left(1+\frac{K}{4}\varrho_2^2\right)\right]^{1/2}}, \quad (3)$$

where $\varrho_i = \xi_i^2 + \eta_i^2 + \zeta_i^2$, $(i = 1, 2)$, K is one of the personal constants, and C is an arbitrary constant.

Finally, Luneburg proposed the following relations between the physical and the visual spaces:

$$\varrho = 2e^{-\sigma\gamma}, \ \varphi = \phi, \text{ and } \vartheta = \theta. \quad (4)$$

Derivation of the Equation of the Visual Circle

The foregoing equations can be applied to the spherical surface in the visual space described above and in Figure 3. Figure 3 shows a physical space, in which there are three points, i.e., $Q_1(s, 0, 0)$, $Q_0(s+t, 0, 0)$, and $Q_3(x, y, z)$. The points in the visual space, which correspond to the three points in the physical space, are designated as $P_1(\xi_1, 0, 0)$, $P_0(\xi_0, 0, 0)$, and $P_3(\xi, \eta, \zeta)$, respectively. If the distances $\overline{Q_1Q_0}$ and $\overline{Q_1Q_3}$ in the physical space are perceptually equal, then the distances $\overline{P_1P_0}$ and $\overline{P_1P_3}$ in the visual space are equal. Since the two terms, , which correspond to $\overline{P_1P_0}$ and $\overline{P_1P_3}$, in the right side of Equation 3 are equal, it follows that:

$$\frac{(\xi-\xi_1)^2+\eta^2+\zeta^2}{\left(1+\frac{K}{4}\varrho^2\right)\left(1+\frac{K}{4}\varrho_1^2\right)} = \frac{(\xi_1-\xi_0)^2}{\left(1+\frac{K}{4}\varrho_0^2\right)\left(1+\frac{K}{4}\varrho_1^2\right)}, \quad (5)$$

where $\varrho^2 = \xi^2 + \eta^2 + \zeta^2$, $\varrho_0^2 = \xi_0^2$, and $\varrho_1^2 = \xi_1^2$. Simplifying Equation 5,

$$\frac{(\xi-\xi_1)^2+\eta^2+\zeta^2}{1+\frac{K}{4}\varrho^2} = \frac{(\xi_1-\xi_0)^2}{1+\frac{K}{4}\varrho_0^2}. \quad (6)$$

Using Equation 2, Equation 6 in Cartesian coordinates can be transformed into Equation 7 in polar coordinates:

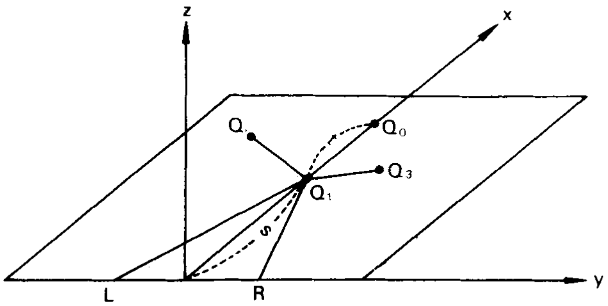

Figure 3. Schematic representation of three points, $Q_1(s, 0, 0)$, $Q_0(s+t, 0, 0)$, and $Q_3(x, y, z)$, in physical space.

$$\varrho^2 + 2\cos\varphi\cos\vartheta\, A\varrho + B = 0, \quad (7)$$

where A and B are defined by the following equations:

$$A = \frac{4\varrho_1}{KT-4}, \qquad B = \frac{4(T-\varrho_1^2)}{KT-4}. \quad (7)'$$

In Equation 7′, T is defined as follows:

$$T = \frac{(\xi_1-\xi_0)^2}{1+\frac{K}{4}\varrho_1^2} = \frac{(\varrho_1-\varrho_0)^2}{1+\frac{K}{4}\varrho_1^2}.$$

Equation 7 is the equation of a visually spherical surface. Consider the visual circle which results from the intersection of the horizontal plane at eye level with this visually spherical surface. In this case, $\vartheta = 0$, and the equation of the visual circle is as follows:

$$\varrho^2 + 2\cos\varphi\, A\varrho + B = 0. \quad (8)$$

Estimation of Personal Constants

The personal constants σ and K can be obtained by using Equation 8. In Figure 4, the flat circle represents the visual circle in the physical space, and Q_0, Q_2, and Q_3 are three points on this circle (γ is the convergence angle of each point). When Q_0, Q_2, and Q_3 correspond to the visual points, P_0, P_2, and P_3, the following equations define each of three points:

for P_0, $\varrho_0^2 + 2A\varrho_0 + B = 0$ (because $\varphi = 0$) (9)

for P_2, $\varrho_2^2 + 2A\varrho_2 + B = 0$ (because $\varphi = 0$) (10)

for P_3, $\varrho_3^2 + 2\cos\varphi\, A\varrho_3 + B = 0.$ (11)

Eliminating A and B from Equations 9 through 11 results in:

$$\varrho_0\varrho_2 - \cos\varphi\, \varrho_3(\varrho_0 + \varrho_2) + \varrho_3^2 = 0. \quad (12)$$

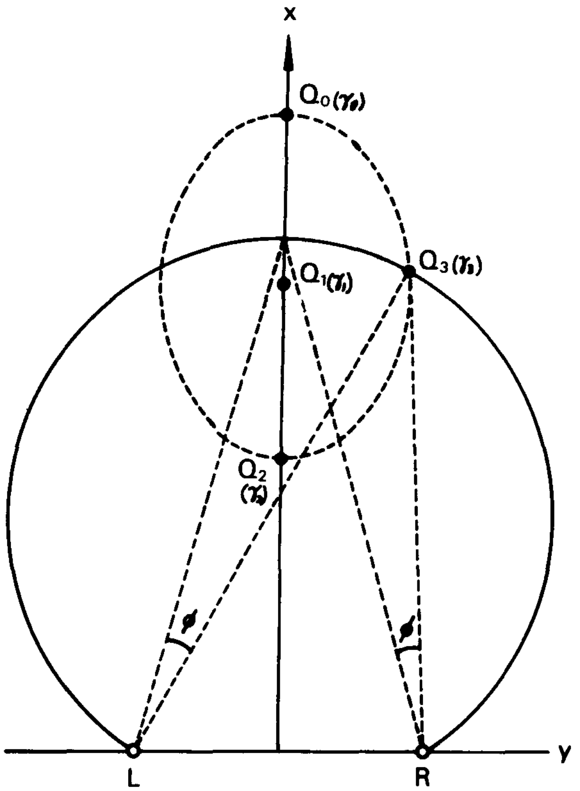

Figure 4. Schematic illustration of the visual circle in physical space as seen from above. The flat dotted line circle represents the visual circle in which Q_0, Q_2, and Q_3 lie, and γ is the convergence angle for each point. The solid line circle which passes through L (left eye) and R (right eye) is the VMC passing through Q_3, a point in the visual circle.

Substituting $\varrho_i = 2e^{-\sigma\gamma_i}$, (i = 0, 2, 3), into Equation 12 gives:

$$e^{-\sigma(\gamma_0+\gamma_2)} + e^{-2\sigma\gamma_3}$$

$$= \cos\phi[e^{-\sigma(\gamma_0+\gamma_3)} + e^{-\sigma(\gamma_2+\gamma_3)}]. \quad (13)$$

In order to calculate σ from Equation 13, it is necessary to determine γ_0, the convergence angle of the farthest point in the configuration, by the experimental restrictions, and γ_2, γ_3, and ϕ by the experimental results.

The other constant, K, can be determined as follows. From Equations 9 and 10, it is evident that ϱ_0 and ϱ_2 are the two roots of the equation $\varrho_2 + 2A\varrho + B = 0$. Therefore,

$$A = (\varrho_0 + \varrho_2)/2, \quad B = \varrho_0\varrho_2. \quad (14)$$

Eliminating T from the two equations in 7′, we get:

$$K = \frac{4(\varrho_1 + A)}{(A\varrho_1 + B)\varrho_1}. \quad (15)$$

Substituting A and B in Equation 14 into Equation 15, we obtain the value of K.

EXPERIMENTAL INVESTIGATION

Method

Procedure. Seventeen small points of light (1 mm in diameter) were presented simultaneously, as shown in Figure 5, in a horizontal plane from 2.9 to 18.6 cm below the observer's eye level, varying with observation distance, in an otherwise dark room. Throughout this experiment, the observer's head was fixed by a chin- and headrest, and his line of regard to the point Q_0 intersected the horizontal plane at about 40 min of arc.[1] The luminance of these light points was adjusted by rheostats so that each of these light points would be apparently equal and be dimly visible. Points Q_0 and Q_1 in the median plane were fixed and the distance between Q_0 and Q_1, $\overline{Q_0Q_1}$, was designated as the standard radius of the visual circle to be constructed. Other points of light were movable on the radial tracks having Q_1 as their center. The distance from the observer to the farthest point of light, Q_0 (the observation distance), was 250, 450, 700, 1,030, 1,300, or 1,600 cm and $\overline{Q_0Q_1}$ was either 50 or 100 cm.

The observer was asked to construct a visual circle using the points of light with $\overline{Q_0Q_1}$ as its standard radius. At first, the observer adjusted Q_2 so that the distance $\overline{Q_1Q_2}$ was apparently equal to the distance $\overline{Q_0Q_1}$. Then, after adjusting Q_6 so as to make $\overline{Q_1Q_6}$ apparently equal to $\overline{Q_0Q_1}$, he adjusted the other points of light on the right of the median plane. The points of light to the left of Q_0 were adjusted in a similar fashion following the completion of the adjustments on the right. Before the

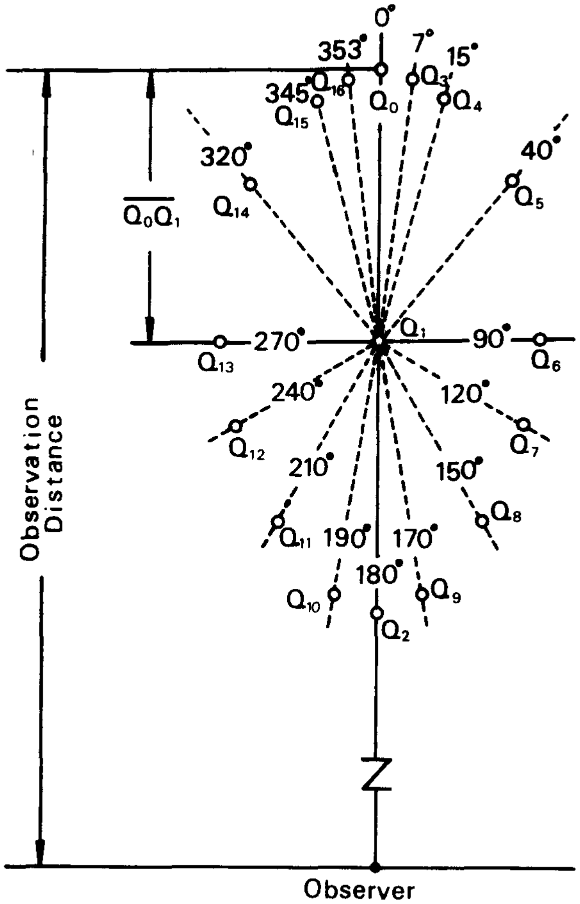

Figure 5. Illustration of stimulus configuration viewed from above. Tiny circles represent small points of light (1 mm in diameter). Q_0 and Q_1 are the fixed points, and other points are movable back and forth on the radial tracks.

observer made any adjustment, the experimenter placed all the adjustable points of light either far inside or far outside the expected positions. For each observer, the experimenter completed four series of settings under the direction of the observer, and after finishing the settings of each series, the observer was allowed to readjust the points as a group to satisfy the visual task.

Points of subjective equality (PSEs) of the standard radius of visual circle for eight directions were calculated by averaging the two mean values of a pair of points of light symmetrically located with respect to the median plane, after finding the means of the four values for each point.

For the alley experiments, nine pairs of light points were presented to the observer simultaneously to construct the parallel alley, and a pair of light points were presented in random order with the farthest pair of light points at a time to form the distance alley. The main dimensions of the alley used were: $x = 500$ cm, $y = \pm 35$ cm, and $\phi_0 = 4.0°$. For each of the parallel and distance alleys, four series of settings were completed by the experimenter under the direction of the observer. The computation of the personal constants from the parallel and distance curves was done according to Zajaczkowska's procedure (1956a).

Observers. Five male undergraduate students, 19 to 21 years in age and majoring in psychology, were used as observers. They all had normal or corrected normal visual acuity, and, using a space eikonometer, they showed less than 1% aniseikonia between the left and right retinal images. All the observers except T.Y. participated in both the visual circle and the alley experiments.

Results

Form of visual circle. For all observers and in all conditions, the resultant physical form of the visual circle was elliptical, with the radius in the median (depth) plane much larger than in the frontoparallel plane. This is illustrated by the results of S.T. in Table 1 and Figure 6. With some observers, there was a tendency to overestimate the far distance up to 10 m, that is, physically $\overline{Q_0Q_1} < \overline{Q_1Q_2}$ though apparently $\overline{Q_0Q_1} = \overline{Q_1Q_2}$.

Personal constants. The personal constants, σ and K, were calculated from the visual circles constructed with an electronic computer using Equations 13, 14, and 15 for the six radii for each observer. ϱ_i, γ_i, and ϕ_i were determined from the experimental restrictions regarding γ_0 and the experimental results. Since Equation 13 contains the transcendental functions of σ, it must be solved by numerical calculation. Table 2, for example, shows all the personal constants in six directions for S.T. Since the standard deviations of PSEs in the direction of 90° were smaller than those in other directions and the observers reported that they were able to make judgments with greatest ease and confidence in this direction, for the sake of simplicity, only the PSEs and standard deviations in the direction of 90° are shown in Table 3 for all observers. The personal constants calculated from the PSEs shown in Table 3 are presented in Table 4, including the same observers' constants from the alley experiment. The ratios of ϕ'_{16} (visual angle subtended by $\overline{Q_1Q_6}$) to Γ_{01} (the angle of disparity between Q_0 and Q_1, $\gamma_1 - \gamma_0$) are also presented.

Variations in the personal constants. It can be seen in Table 2 that the values of σ and the absolute values of K tend to increase as the observation distance increases, except for K for the largest observation distance. However, values of σ and K in each column tend to be, with a few exceptions, relatively of the same magnitude, irrespective of the directions of radii of the visual circle. Therefore, it can be concluded that the personal constants, especially σ estimated from the visual circles have a tendency to vary systematically with the distance

Table 1
PSE and Standard Deviation (SD, in Centimeters) of Radius of the Visual Circle in Eight Directions (7 through 180 Deg)

		Radius of Visual Circle ($\overline{Q_0Q_1}$)						
		50 cm			100 cm			
		Observation Distance (cm)			Observation Distance (cm)			
Directions		250	450	700	700	1030	1300	1600
7 Deg	PSE	48.57	48.18	43.05	76.56	66.21	75.68	77.81
	SD	2.77	2.75	6.11	5.91	3.83	9.40	12.10
15 Deg	PSE	45.00	46.42	41.12	69.82	65.67	70.51	71.25
	SD	2.74	3.95	4.03	5.92	3.69	10.10	14.68
40 Deg	PSE	31.01	33.72	23.86	46.20	37.21	45.25	43.40
	SD	3.61	4.50	3.13	5.00	3.10	5.65	7.79
90 Deg	PSE	24.71	24.75	16.11	29.71	25.78	29.41	27.52
	SD	.96	3.24	1.11	2.89	.88	3.33	6.27
120 Deg	PSE	26.74	26.57	17.72	32.91	27.63	*	32.00
	SD	1.95	3.55	1.37	3.71	1.95		5.08
150 Deg	PSE	44.51	44.30	29.85	55.30	41.43	55.25	51.35
	SD	3.16	5.88	3.25	3.07	3.32	6.58	10.04
170 Deg	PSE	52.31	52.15	35.22	75.72	53.52	73.62	63.23
	SD	3.21	4.68	2.88	5.48	8.10	6.76	11.95
180 Deg	PSE	55.17	55.97	41.87	81.30	65.32	83.15	68.57
	SD	3.39	5.22	3.60	6.39	4.49	2.14	9.08

Note—Observer: S.T. **Determination was not made.*

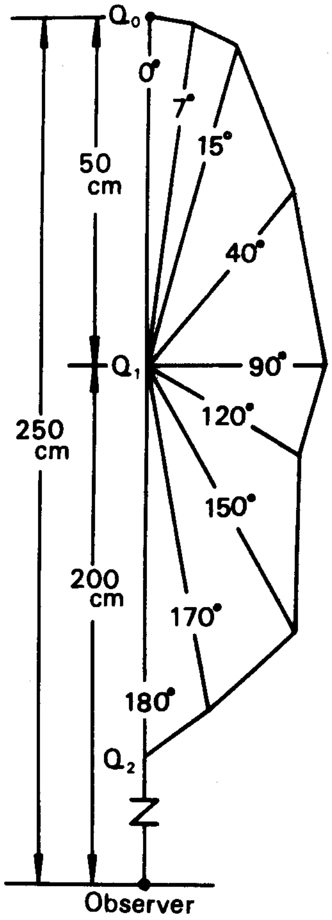

Figure 6. Physical form of the visual circle obtained by observer S.T. Observation distance is 250 cm, and the radius of the visual circle is 50 cm.

from the observer to the visual circle and to remain constant in every direction of radii in the same visual circle.

The σs in Table 4, estimated from the radius in the direction of 90° from each visual circle of five observers, show the same tendency as shown in Table 2. Their values increase as the observation distance of circles for each $\overline{Q_0Q_1}$ increases and remain relatively the same for circles of the same observation distance. The variation of σs within an observer is larger than that between observers. It can be observed from the *K*s in Table 4 that there are no remarkable systematic changes with observation distance and that each observer retains relatively the same rank order in terms of values of *K* within each observation distance. The largest value of *K* in each row most frequently appears in the column of J.S., the third largest in the columns of S.T. and T.Y., the fourth largest in the column of H.S., and the fifth largest in the column of H.T. In comparison with the personal constants in the last two rows that were obtained from the alley experiment, almost all the values of σ and the absolute values of *K* in the visual circle experiment are larger than those in the alley experiments.

DISCUSSION

Although most of the obtained visual circles had some irregularities like those of the visual circle shown in Figure 6, they were less irregular and more stable within each observer than those obtained in the previous experiments (Hagino, Yoshioka, & Hirabayashi, 1962, 1963), in which only three points

Table 2
Personal Constants, σ and K, in Six Directions (15 through 170 Deg)

		Radius of Visual Circle ($\overline{Q_0Q_1}$)						
		50 cm			100 cm			
		Observation Distance (cm)			Observation Distance (cm)			
Directions	Personal Constants	250	450	700	700	1030	1300	1600
15 Deg	σ	13.964	20.895	37.620	24.529	30.754	43.884	67.240
	K	−1.669	−1.542	−12.740	−.685	−1.765	−5.492	−1.994
40 Deg	σ	13.224	19.403	35.846	32.309	43.086	62.822	79.526
	K	−1.625	−1.495	−10.703	−.568	−2.217	−16.364	−2.262
90 Deg	σ	14.089	28.827	35.888	31.024	47.005	60.716	78.549
	K	−1.677	−1.815	−10.398	−.589	−2.340	−13.689	−2.243
120 Deg	σ	13.373	17.916	35.436	30.024	46.019	*	83.762
	K	−1.634	−1.450	−10.380	−.591	−2.340		−2.363
150 Deg	σ	17.071	19.119	43.417	36.754	48.101	72.008	77.550
	K	−1.865	−1.487	−24.962	−.492	−2.432	−41.680	−2.217
170 Deg	σ	13.586	8.843	25.168	35.708	31.109	56.981	78.798
	K	−1.647	−1.202	−4.670	−.511	−1.779	−10.982	−2.246

Note—Observer: S.T. **Personal constants were not calculated (cf. the footnote in Table 1).*

Table 3
PSE and Standard Deviation (SD, in Centimeters) of Radius of Visual Circle in the Direction of 90 Deg

Radius of Visual Circle ($\overline{Q_0Q_1}$)	Observation Distance		Observers				
			S.T.	H.S.	H.T.	J.S.	T.Y.
50 cm	250 cm	PSE	24.71	32.71	35.85	27.93	29.45
		SD	.91	4.20	3.84	2.49	2.65
	450 cm	PSE	24.75	32.08	38.40	27.44	21.84
		SD	3.24	4.99	4.46	1.69	1.26
	700 cm	PSE	16.11	28.56	38.65	23.44	18.95
		SD	1.11	1.06	2.31	2.46	1.56
100 cm	700 cm	PSE	29.71	44.35	49.98	33.11	30.51
		SD	2.89	1.85	4.53	3.45	1.66
	1030 cm	PSE	25.78	39.50	51.81	37.70	31.93
		SD	.88	3.47	3.94	3.22	1.55
	1300 cm	PSE	29.41	32.87	56.83	37.56	32.14
		SD	3.33	6.41	4.02	4.56	2.94
	1600 cm	PSE	27.52	36.60	50.63	42.78	21.98
		SD	6.27	1.97	3.45	3.56	2.04

of light (Q_0, Q_1, and Q_i) were presented at one time. However, the personal constants calculated from the visual circles in the present study were not consistent with the values expected from the Luneburg theory.

First, the values of σ increased systematically as the observation distance increased and were considerably larger than those obtained in other studies, the latter being usually not greater than 25. Exceptions are the studies by Ishii (1962) and Nishikawa (1969), who reported values from 10.6 to 44.3 and from 25.0 to 46.9, respectively. The large values of σ in the present study (ranging from 13 to 120) may be ascribable to a kind of constancy phenomenon in which the physical magnitude of the radii of the configuration perceived as a circle remained essentially constant despite the increase in the observation distance (Tables 1 and 3). Since the standard deviations of PSEs were smallest in the direction of 90°

Table 4
Personal Constants, σ and K, and ϕ'_{16}/Γ_{01} in the Direction of 90 Deg

Radius of Visual Circle ($\overline{Q_0Q_1}$)	Observation Distance	Personal Constants	Observers				
			S.T.	H.S.	H.T.	J.S.	T.Y.
50 cm	250 cm	σ	14.089	18.800	21.371	17.658	19.004
		K	−1.677	−2.013	−2.042	−1.606	−1.766
		(ϕ'_{16}/Γ_{01})	1.93	2.69	2.80	2.22	2.46
	450 cm	σ	28.827	39.968	46.261	33.901	32.854
		K	−1.815	−2.111	−2.358	−1.812	3.497
		(ϕ'_{16}/Γ_{01})	4.05	4.76	5.59	4.71	3.28
	700 cm	σ	35.888	59.012	75.587	46.076	42.788
		K	−10.398	−1.981	−2.660	−1.821	−.832
		(ϕ'_{16}/Γ_{01})	3.55	6.25	8.41	5.21	4.43
100 cm	700 cm	σ	31.024	48.896	49.502	36.791	38.422
		K	−.589	−.096	−1.189	2.251	−4.253
		(ϕ'_{16}/Γ_{01})	3.27	4.34	5.47	3.68	3.57
	1030 cm	σ	47.005	69.364	86.474	51.344	58.203
		K	−2.340	−8.093	−22.633	−1.610	−3.792
		(ϕ'_{16}/Γ_{01})	4.14	6.64	8.33	6.17	5.46
	1300 cm	σ	60.716	78.123	119.902	78.249	73.629
		K	−13.689	−2.842	−.487	9.545	−3.397
		(ϕ'_{16}/Γ_{01})	6.01	6.98	11.55	8.15	6.92
	1600 cm	σ	78.549	108.192	118.422	111.731	62.977
		K	−2.243	−2.957	−1.641	−8.794	−2.029
		(ϕ'_{16}/Γ_{01})	7.01	9.43	12.61	10.89	5.86
Alley Experiment	500 cm	σ	3.131	34.226	8.204	7.626	
		K	−.388	−.194	−.102	.000	

and judgments were made with greatest ease and confidence in this direction for all observers, it seems reasonable to choose the two radii of 0° (the standard radius) and 90° to characterize the physical forms of the visual circles. If their magnitudes were expressed in angular size as Γ_{01}, the angle of disparity between Q_0 and Q_1 and ϕ'_{16}, the visual angle subtended by $\overline{Q_1Q_6}$, then Γ_{01} should decrease, theoretically, in inverse proportion to the square of the observation distance, while ϕ'_{16} should decrease inversely proportional to the observation distance. This differential decrease between angular sizes of the two radii should result in an increase of the ratio between them, ϕ'_{16}/Γ_{01}, as the observation distance increases. As shown in Table 4, for almost all observers in this experiment, ϕ'_{16}/Γ_{01} increased as the observation distance increased in each of the two kinds of the visual circle with different standard radius. This implies that the radii in the frontoparallel plane became larger, in angular size, relative to the radii in the median plane as the observation distance increased. Also, for each observer, the values of σ in Table 4 varied as a linear function of ϕ'_{16}/Γ_{01}. This variation corresponds to the results of a theoretical study by Hagino et al. (1962). In that study, the authors varied σ in their theoretical equations for the visual circle, holding K constant, and calculated the radii of various circles. The results showed that as σ became larger, frontoparallel radii of the visual circles increased considerably and the form of the visual circles approached the geometric physical circle.[2]

Secondly, in the present study, almost all the values of K were negative, and this supports Luneburg's proposition that visual space is hyperbolic. However, most of the absolute values of K were larger than one. According to Luneburg (1950), K should be within minus one to zero, that is, $-1 < K < 0$. Therefore, the obtained Ks that are less than -1 are not consistent with his theory. Values of K more negative than -1 imply a logical contradiction that we can perceive sizes and distances larger than infinity (Blank, 1958b). Ks smaller than -1 in previous studies (Blank, 1958b; Indow et al., 1962a; Zajaczkowska, 1956b) may be attributable to experimental errors. However, the magnitude by which some of the Ks in the present study are less than -1 is too large to be ascribable to experimental errors. As mentioned before, some observers in the present study showed a tendency to overestimate the far distance up to 10 m. This phenomenon, called overconstancy of distance, was also observed in other studies carried out in reduced stimulus conditions using small light points as stimuli in a dark room (e.g., Tada, 1956). However, it must be noted that there was no relation between the overestimation of the far distance and the values of K smaller than -1 or the remarkably large σs in the present study. It is interesting to compare the PSEs in Table 1, especially the entries of the two bottom rows and the third and fourth columns, which indicate overestimation of the far distance, with the corresponding Ks and σs in Table 2 (personal constants corresponding to the first and last rows in Table 1, 0° and 180°, were not calculated). No conspicuous relationship was observed between the overestimation and the personal constants.

The constant K in the Luneburg theory relates to the geometry of the primary visual space, that is, as Foley (1972) termed it, the space defined by perceptual relations when only the primary cues of accommodation, convergence, and disparity are available. Shipley (1957) claimed that this constant was so important that the determination of the sign of K was prerequisite in future research. In the present study, most of the Ks obtained in the visual circle experiment were negative, but not consistent with the Luneburg theory. However, Ks that are anomalous according to the Luneburg (1950) theory have been observed elsewhere. In the alley experiments, K was larger than zero for 6 of 15 observers (Hardy et al., 1951) and for 7 of 9 observers (Ishii, 1962). This implies that the visual spaces of those observers were not hyperbolic but elliptical. Further, Ishii (1964) reported $k > 0$ for 5 of 12 observers and for 5 of 17 observers in the first and the second equilateral triangle experiments, respectively, and also for 9 of 17 observers in the isosceles right-triangle experiment. Examining the Desarguesian property of binocular visual space, Foley (1964) concluded that the visual spaces of a significant proportion of his five observers were Desarguesian. However, Foley (1968) observed that in the equilateral triangle experiment K was negative for two observers but for the third observer K varied from positive to negative as the size of configuration increased. Recently, Foley (1972) reported on three experiments, the results of which seemed to be significant for the assumption of homogeneity, or constant negative curvature, of binocular visual space of the Luneburg-Blank theory. From the results of these experiments, which involved construction of one or two apparent isosceles right triangles, the following conclusions were drawn. The primary visual space was neither Euclidean nor homogeneous; two kinds of judgment, size-distance and angle, were the product of independent processes; geometry might approach Euclidean geometry with the introduction of additional cues to distance; and no one geometrical model can be appropriate for all stimulus situations.

Regarding the curvature of visual space, Indow (1968, 1974), Matsushima and Noguchi (1967), and Nishikawa (1967), applying the method of multidimensional scaling to their data, reported the possibility that the intrinsic geometry of visual space might be Euclidean. On the other hand, Ishii (1972) proposed that the geometry of visual space might not be so exclusively hyperbolic as Luneburg claimed but might vary from hyperbolic to elliptical and from individual to individual, that is, $-1 \leqslant K \leqslant +1$, based on the statistics of signs of Ks obtained by the earlier researchers, including himself (Blank, 1961; Hagino et al., 1963; Hardy et al., 1951, 1953; Indow et al., 1962a, b; Ishii, 1962, 1964; Shipley, 1957; Zajaczkowska, 1956a, b). According to his statistics, Ks were positive in 13 of 52 cases in the alley experiments and in 8 of 68 cases in the 3- and 4-point and in other experiments.

Thirdly, systematic changes in the personal constants were observed in the present study, that is, for the same observers, not only did σs increase as the observation distance increased, as mentioned before, but also almost all of the values of σ and absolute values of K were larger in the visual circle experiment than in the alley experiment. According to the Luneburg theory, the personal constants should remain the same and within experimental errors irrespective of variations in kinds of or varied conditions in the same kind of experiment to yield the constants. However, it has been reported occasionally that systematic changes in both constants occurred concurrently not only with the difference in the kind of experiment but also with the varied conditions of the experiment. Comparing the values of σ and K obtained from the 3- and 4-point experiments with those from the alley experiment, Zajaczkowska (1956b) reported that the values of σ were larger in the latter experiment, though the absolute values of K were larger in the former. The same tendencies were observed in the studies of Indow et al. (1962a, b) and Hagino et al. (1963). Zajaczkowska (1956b) used three kinds of alleys: classic, intermediate, and broad. Referring to the results described in her Table IV (p. 523), Foley (1964) pointed out that for the majority of observers the absolute values of K increased with the width of the alleys and that systematic changes in σ could be observed, that is, the wider the alleys, the smaller the values of σ. However, Hagino et al. (1963) observed opposite tendencies in that the values of σ and the absolute values of K obtained in the narrow and long alley were larger than those obtained in the broad and short alley. Yoshioka (1965) reported that in the alley experiment, when the observation distance was varied with ϕ_0 being held constant, no systematic change of either constant was found, and when ϕ_0 was decreased with the observation distance being held constant σs tended to increase. According to the studies of Indow et al. (1962a, b), in the alley experiment σs obtained in the vertical plane were smaller than those in the horizontal plane and Ks in the vertical plane were closer to -1 than those in the horizontal plane, while in the 3- and 4-point experiments exactly reversed tendencies were observed.

It has been reported that in the Blank system the values conceived to be constant also varied systematically with change in experimental conditions. In his system (Blank, 1953, 1958a, b; Hardy et al., 1953), $r(\Gamma)$, where $\Gamma = \gamma - \gamma_0$, r corresponds to ϱ, the visual radial distance in the Luneburg system, and ω, defined as $\omega = r(0)$, is used as the measure of visual radial distance of the farthest point, $Q_0(\gamma_0)$, in a given stimulus configuration. By definition, ω is conceived to be constant for an observer irrespective of the value of Γ. The value of ω is determined by the following equation:

$$\omega = \operatorname{arc\,cosh} (S/T)^{1/2}, \tag{16}$$

where S/T is an index reflecting the degree of the hyperbolic discrepancy between the parallel and the distance alley and is considered to be the same irrespective of the value of Γ. However, Zajaczkowska (1956a) noted that there were systematic changes in the values of ω with the variation of γ_0. The values of ω became larger as the values of γ_0 decreased, while Hagino et al. (1963) observed an opposite tendency of ω, that is, the smaller the values of γ_0, the smaller the values of ω. But Indow, Inoue, and Matsushima (1963) reported that they found neither tendency consistently. Reanalyzing the data of Blumenfeld (1913) and Zajaczkowska (1956a), Shipley (1957) pointed out that S/T became smaller as Γ increased with 7 of 33 analyzed cases varying randomly. On the other hand, Indow et al. (1963) reported that with two of three observers the ratio seemed to become larger as the values of Γ increased.

Thus, in the Blank system, too, values conceived to be constant for an observer varied systematically with the difference in experimental condition. However, so far as mapping functions are concerned, it was reported that the Blank system might be better than the Luneburg system (Hagino et al., 1963). Using the equations of the visual circle and the necessary values of parameters obtained for each observer in the 3- and 4-point and the alley experiments, the authors constructed various theoretical visual circles for each observer based on the mapping functions in the Blank and the Luneburg systems. The resulting theoretical visual circles based on the Blank system fitted the experimentally obtained visual circles better than those based on the Luneburg system, although the agreement between the theoretical circles and experimental ones was far from being satisfactory.

The Luneburg theory has been of heuristic value in specifying problems in binocular space perception, and this theory could be experimentally confirmed if an ideal experimental situation were set up where factors of experience, preconception, and judgment habits of the observer, as mentioned by Hardy et al. (1951), are completely excluded. However, the experimental results, including those of the present study, were not consistent with the Luneburg or the Blank theory. In view of the foregoing discussion, inconsistent results are of two kinds. The first kind is the finding that the so-called personal constants are not invariant but change systematically with changes in experimental conditions. The second, equally important, result is the finding that visual space is consistently neither negative nor homogeneous. It is not certain at present what factors are responsible for these inconsistencies and whether these two kinds of inconsistency have the same responsible factors in common or not. However, two points, one theoretical and the other experimental, should be considered in future investigations of binocular visual space. The theoretical point is that the mathematical model of binocular visual space should be a flexible one which can vary its curvature and/or its mapping functions with changes in the perceived egocentric distance and orientation of the stimuli. In this respect, Gogel (1958), in developing and testing a theoretical approach to the perception of depth from purely binocular cues, concluded that the amount of frontal constancy (*n* in his designation), as well as the observer constant, C, is important in determining the perceived depth resulting from a binocular disparity. The experimental consideration is that it is important to exclude the effects of cognitive factors from the results if the concern is to analyze purely perceived visual space. It is likely that all the experimental situations which have been used to estimate personal constants are more or less susceptible to the observer's cognitive or inferential processes. As Gogel (1973) has pointed out, cognitive as well as perceptual factors can influence spatial responses, and the relative contribution of each of these can vary as a function of instructions and stimulus conditions.

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NOTES

1. Since cos 40′ = 0.99993, the difference between intersecting angle of the observer's line of regard at the horizontal plane and the angle of ϑ, which was specified as zero to derive Equation 8 for the visual circle from Equation 7 for the visual sphere, was not considered to have any effect on the value of σ and K in the practical sense.

2. Concerning depth and lateral (vernier) acuities, Berry (1948) reported that when the angular separation between the two vertical rods was 133.7 sec of arc or more, depth threshold was smaller than vernier threshold for all of his three subjects.

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