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Efficient estimation of probabilities in the t distribution

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The probability (p) of a value in the t distribution can be calculated exactly by means of Student's (1908) cosine series, which Fisher (1935) elucidated and which Zelen and Severo (1964, sections 26.7.3-4) reprinted.[1] Cooper (1968), Morris (1968), Levine (1969), Hill (1970a), and Dudewicz and Dalal (1972) prepared computer functions based on these series or on a closely related method. Precise computation also can be achieved by numerical integration (Wood & Wood, 1984, 1986). Alternatively, exact calculations can be made by evaluation of the F distribution (e.g., Dorrer, 1968; Dunlap & Duffy, 1975; Lackritz, 1984; Morris, 1969) or the incomplete beta function (e.g., Kennedy & Gentle, 1980, chap. 5; Selvin & Wong, 1975), after the given t value is converted to its equivalent value in the other distribution.

When precise calculation is not required, numerous methods are available to approximate the p of a t value (Johnson & Kotz, 1970, chap. 27). O'Grady (1981), Ogasawara (1982), von Collani (1983), and Evans and Gilfillan (1986) have provided recent examples. The most popular approximation may be Jaspen's (1965), which transforms an F ratio to an approximate normal deviate (z) by Palson's (1942) formula, which itself is based on Wilson and Hilferty's (1931) normalizing approximation of the chi-square distribution, and then estimates the p corresponding to z by an approximation from Zelen and Severo (1964) that was adapted from Hastings (1955). When Jaspen's method is used with the t distribution, the t value is first converted to its equivalent F value ($F_{1,n} = t_n{}^2$; the p of F is equivalent to the two-tailed p of t).

Computer routines that use exact methods to determine the p of a t value are relatively long and slow, and they do not ordinarily handle nonintegral degrees of freedom (n), whereas many of the approximation routines are fairly inaccurate. Futhermore, generic routines such as Jaspen's (1965) are unnecessarily complex if they are to be used only with the t distribution. There is need for a short, fast routine that is sufficiently accurate for applications such as determining the significance of an obtained t value. A fast routine would be particularly advantageous for real-time processing in the laboratory. Reasonable target accuracy is four decimal places.

Ling (1978) found that approximations of Wallace (1959, Approximation u_4) and Peizer and Pratt (1968, Equation 4.7) were the most accurate, simplest methods of several tested, and Prescott (1974) confirmed the superiority of Wallace's approximation to four other approximations. Moreover, Wallace provided an even more accurate approximation (Approximation u_5) that was not investigated by either Ling or Prescott. The present study tests both of Wallace's approximations, Peizer and Pratt's approximation, and four newer approximations. In addition, comparisons are made with a Jaspen-type approximation.

Listing 1 shows the approximations, in BASIC, with a brief driver program to input values of t and n, call a selected method, and print the calculated one-tailed p. Each approximation is a normalizing transformation that estimates the z corresponding to t; the p of z is assessed by an approximation from Zelen and Severo (1964, section 26.2.17) that is accurate to six decimal places (Brophy, 1983a).

The first two approximations are by Wallace (1959). He adapted a normalizing approximation from Chu (1956):

$$z = [n \ln(1+t^2/n)]^{1/2}, \tag{1}$$

which is an upper bound on the z corresponding to t. Wallace constructed other bounds and empirically developed two related approximations. The simpler of the approximations (Approximation u_4) is

$$z = (8n+1)/(8n+3)[n \ln(1+t^2/n)]^{1/2}. \tag{2}$$

His other approximation (Approximation u_5) is

$$z = u - 2u/(8n+3)[1-\exp(-s^2)]^{1/2}, \tag{3}$$

where $s = 0.368(8n+3)/(2n^{1/2}u)$, and u is the z of Equation 1. The third approximation is one of a family of methods developed by Peizer and Pratt (1968, Equation 4.7; Pratt, 1968) for major sampling distributions,

$$z = [n-2/3+1/(10n)][1/(n-5/6) \ln(1+t^2/n)]^{1/2}. \tag{4}$$

The fourth approximation, by Hill (1970a), is a three-term Cornish-Fisher form of expansion with the third-term divisor adjusted to increase accuracy:

$$z = w + (w^3+3w)/b - (4w^7+33w^5+240w^3 +855w)/[10b)b+0.8w^4+100)], \tag{5}$$

where $w = [a \ln(1+t^2/n)]^{1/2}$, $a = n-1/2$, and $b = 48a^2$. Hill, who sought very high accuracy, combined this asymptotic expansion in an exceptionally elegant program with two other methods: Student's (1908) cosine series for small values of n, and a precise tail series expansion for large values of t. These enhancements are not necessary to attain the more modest accuracy desired here. Nor is it necessary to adopt Hill's (1970a, 1981) or el Lozy's

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Listing 1
Approximations of the *t* Distribution

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10 CLS
20 INPUT"t, df "; T,N
30 INPUT"Which approximation (1-8) "; AP
40 ON AP GOSUB 110,210,310,410,510,610,710,810
50 PRINT"p (one-tailed) =" P
60 PRINT: GOTO 20
70 END
90 '
100 ' Wallace. Approximation u4
110 Z=(1-.25/(N+.375))*SQR(N*LOG(1+T*T/N))
120 GOTO 1010
190 '
200 ' Wallace. Approximation u5
210 Z=SQR(N*LOG(1+T*T/N)): IF Z=0 THEN 1010
220 W=(4*N+1.5)/Z: Z=Z-SQR(1-EXP(-.135*W*W/N))/W
230 GOTO 1010
290 '
300 ' Peizer & Pratt, Equation 4.7
310 Z=(N-2/3+.1/N)*SQR(LOG(1+T*T/N)/(N-5/6))
320 GOTO 1010
390 '
400 ' Hill, Asymptotic approximation
410 A=N-.5: B=48*A*A: W=A*LOG(1+T*T/N): Z=(((((-.4*W-3.3)
    *W-24)*W-85.5)/(.8*W*W+100+B)+W+3)/B+1)*SQR(W)
420 GOTO 1010
490 '
500 ' Mickey, Approximation Pc (modified)
510 Z=SQR((N-.475)*LOG(1+T*T/N))
520 GOTO 1010
590 '
600 ' Bailey, Equation 5
610 W=N+1/12: Z=(1-1/(N+1.125))*SQR((W+1.5)*LOG(1+T*T/W))
620 GOTO 1010
690 '
700 ' Gaver & Kafadar, Equation 2.7 (inverted)
710 Z=(N-1)*SQR(LOG(1+T*T/N)/(N-1.5))
720 GOTO 1010
790 '
800 ' Exact calculation for df = 1,2
810 IF N>2 OR N>INT(N) THEN 840
820 IF N=1 THEN P=.3183099*ATN(T) ELSE P=T/SQR(T*T+2)/2
830 P=.5-ABS(P): RETURN
840 STOP: ' (GOTO routine for df>2 or noninteger df)
990 '
1000 ' Approximation of p of z (Zelen & Severo, #26.2.17)
1010 Y=.3989423*EXP(-Z*Z/2): S=1/(1+.2316419*Z)
1020 P=((((1.330274*S-1.821256)*S+1.781478)*S-.3565638)*S
     +.3193815)*S*Y: RETURN
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(1979) methods for evaluating $\ln x$ in calculating w when x is close to unity.

The fifth approximation is based on a slight modification of Equation 1 proposed by Mickey (1975, Approximation Pc):

$$z = [(n-\tfrac{1}{2})\ln(1+t^2/n)]^{1/2}. \qquad (6)$$

Empirical tests showed that substitution of 0.475 for ½ in the equation provided greater overall accuracy (albeit somewhat larger errors with moderate and large n), so the approximation was used with 0.475. The sixth approximation is by Bailey (1980), who pointed out that Equations 2 and 6 suggest a general class of approximations:

$$z = (n+b)/(n+c)\{(n-a)\ln[1+t^2/(n+h)]\}^{1/2}, \qquad (7)$$

for which he used the constants $a = -{}^{12}\!/_{9}$, $b = {}^{1}\!/_{8}$, $c = {}^{9}\!/_{8}$, and $h = {}^{1}\!/_{12}$. Bailey asserted that this approximation is more accurate uniformly than any previous normalizing transformation. The seventh approximation, obtained by inverting an approximation of percentage points of t from Gaver and Kafadar (1984, Equation 2.7), is

$$z = (n-1)[1/(n-\tfrac{3}{2})\ln(1+t^2/n)]^{1/2}. \qquad (8)$$

All the approximations except Hill's (1970a) are similar in structure, differing primarily in the multiplier of the logarithmic expression. Pratt (1968) observed that there is little difference in the multiplier of Equations 2 and 4 unless n is very small; that is true to a lesser extent of Equations 6 and 8.

Listing 1 includes a routine for exact calculation when $n = 1$ or 2 that was derived from Student (1908, p. 13). The Jaspen-type approximation used in the study was similar to Ogasawara's (1982) program, but it contained two improvements: (1) substitution of the Zelen and Severo (1964, section 23.2.17) approximation shown in Listing 1 for the less accurate approximation (Zelen & Severo, 1964, section 26.2.18) employed by Jaspen (1965) and Ogasawara, and (2) addition of Brophy's (1983b) correction for values near the middle of the F distribution (i.e., in the region of one-tailed $p = .25$ in the t distribution).

Tests of the Approximations. The approximations were tested with critical t values at 11 one-tailed levels of p from .00005 to .4 for all integral n from 1 through 20 and for $n = 30, 40, 60$, and 120. Several sources were used in an effort to obtain t values of high precision over a wide range of ps. Three-decimal t values for $p = .00005$ and .0001 were taken from Federighi (1959); four-decimal t values for $p = .0005, .0025, .1$, and .4 from Smirnov (1961, p. 125), with two corrections supplied by Hill (1970b) and two additional corrections (for $n = 1$ and 2 at $p = .0025$) made from exact calculations; five-decimal t values for $p = .005, .01, .025$, and .05 from Owen (1965); and five-figure t values for $p = .25$ from Merrington (1942). (Federighi's table does not give values for $n = 120$, so values for $n = 100$ were used from that source.)

Table 1
Maximum Absolute Error of Approximations at Selected Probability Levels for Two Ranges of Degrees of Freedom

Approximation	Probability Level .25–.40	.005–.100	.0001–.0025	.00005
Degrees of Freedom = 1,2				
Wallace				
Approximation u_4	.002260	.007515	.002927	.000258
Approximation u_5	.002260	.000499	.000205	.000012
Peizer & Pratt	.061574	.048102	.002025	.000046
Hill	.011657	.013988	.001777	.000049
Mickey (modified)	.023173	.033405	.009538	.001162
Bailey	.004304	.003406	.001666	.000140
Gaver & Kafadar*	.025933	.023561	.001300	.000032
Jaspen (improved)	.004372	.004372	.002264	.002220
Degrees of Freedom = 3–120				
Wallace				
Approximation u_4	.000776	.000678	.000351	.000027
Approximation u_5	.000776	.000155	.000054	.000002
Peizer & Pratt	.001322	.000602	.000290	.000024
Hill	.000008	.000011	.000005	.000001
Mickey (modified)	.001200	.002068	.000581	.000038
Bailey	.000491	.000274	.000189	.000017
Gaver & Kafadar	.004675	.004070	.000060	.000009
Jaspen (improved)	.003607	.002042	.000608	.000183

*For 2 degrees of freedom; approximation is not applicable with 1 degree of freedom.

Table 2
Maximum Percentage Relative Error of Approximations for Two Ranges of Degrees of Freedom

Approximation	Degrees of Freedom 1,2	3–120
Wallace		
Approximation u_4	516	54
Approximation u_5	24	4
Peizer & Pratt	91	48
Hill	97	1
Mickey (modified)	2,324	77
Bailey	280	34
Gaver & Kafadar	63*	18
Jaspen (improved)	4,438	365

*For 2 degrees of freedom; approximation is not applicable with 1 degree of freedom.

Because of discrepancies among the tables, as well as limited precision of some t values, the nominal tabled p values were not used as the test values with which to compare the results of the approximations. Instead, test values of p corresponding to the t values were computed by Wood and Wood's (1986) numerical integration program. (Wood and Wood's program was modified to extend its range through $n = 120$, to accept values of t rather than F, and to yield one-tailed rather than two-tailed ps.) For $p \leq .05$, all computed p values agreed with the tabled p to at least six decimal places, presumably testifying to the accuracy of both the tables and Wood and Wood's program. For larger ps, the discrepancies were as great as .00002, probably reflecting the limited precision of the tabled t values.

The maximum absolute value of the absolute error $[(p_a - p_t)$, where p_a is the approximated p, and p_t is the true $p]$ and the maximum absolute value of the percentage relative error $[100(p_a - p_t)/p_t]$ were determined for each approximation. Absolute errors were analyzed separately for four ranges of p. Because errors for $n \leq 2$ are substantially greater than those for larger n, the errors for the two ranges of n also were analyzed separately.

Tables 1 and 2 summarize the results for absolute error and relative error, respectively. For $n \geq 3$, Hill's (1970a) asymptotic series is the most accurate of the approximations, providing at least four-decimal-place accuracy in p for all t values tested and five-decimal-place accuracy when $p \leq .0005$. Wallace's (1959) Approximation u_5 is second in accuracy; it is the most accurate of all for $n \leq 2$. Bailey's (1980) approximation also is relatively accurate, although the results do not support the claim of overall superiority. The accuracy of Gaver and Kafadar's (1984) approximation varies substantially with the level of p, but its relative error is low. On the other hand, Jaspen's (1965) approximation is relatively poor, particularly at low p. It should not be used to make differential decisions regarding $p < .01$. For $n \geq 3$, even Mickey's (1975) simple approximation, as modified, is as accurate as Jaspen's rather complex method. The latter does, however, provide two-decimal-place accuracy for $n \leq 2$, as well as for larger n; of the other approximations tested, that is true only of Bailey's approximation and Wallace's Approximation u_5.

To illustrate the performance of the approximations and to clarify the nature of the measures used in the error analysis, Table 3 shows the p values estimated by each approximation for t values with $n = 5$ and 10 at $p = .05$ and .00005. The absolute error $(p_a - p_t)$, a simple measure of the discrepancy between the approximated value and the true value, helps to answer a common question: To how many decimal places is an approximation correct? It is sometimes all that the applied statistician wants to know in evaluating an approximation. The relative error $[(p_a - p_t)/p_t]$ is valuable, however, as an index of accuracy in relation to the magnitude of the true value. It is particularly informative when the true value is very small (or very large, although that does not occur with p). An approximation that has a large relative error when p_t is small may be sufficiently accurate for use when p_t is larger. Nevertheless, the maximum percentage relative error, as shown in Table 2, offers a useful assessment of the adequacy of an approximation over a range of values of p_t.

All of the approximations can be used with nonintegral values of n, such as occur in tests of means from populations with unequal variances and in some multiple comparison procedures. However, accuracy with nonintegral n was not tested.

In conclusion, Hill's (1970a) asymptotic approximation is recommended as a reasonably accurate, short routine to estimate the p of a t value. If necessary, it can easily be supplemented by the exact routine for $n \leq 2$ given in Listing 1. If two-decimal-place accuracy is acceptable, when $n \geq 3$ any of the approximations can be used, and a shorter method of estimating the p of z, such as Zelen and Severo's (1964, section 26.2.18) method or Brophy's (1983a) modification of Cadwell (1951), will be adequate.

Table 3
Performance of Approximations with Selected Values of t

Approximation	Estimated p	Absolute Error	Percentage Relative Error
$t = 2.01505,\ n = 5,\ p = .05$			
Wallace			
Approximation u_4	.050101	.000101	.202
Approximation u_5	.050039	.000039	.078
Peizer & Pratt	.050051	.000051	.102
Hill	.050000	.000000	.000
Mickey (modified)	.050489	.000489	.978
Bailey	.049984	−.000016	−.032
Gaver & Kafadar	.049623	−.000377	−.754
Jaspen (improved)	.049315	−.000685	−1.370
$t = 11.17771,\ n = 5,\ p = .00005$			
Wallace			
Approximation u_4	.000060	.000010	20.
Approximation u_5	.000048	−.000002	−4.
Peizer & Pratt	.000059	.000009	18.
Hill	.000050	.000000	0.
Mickey (modified)	.000062	.000012	24.
Bailey	.000055	.000005	10.
Gaver & Kafadar	.000057	.000007	14.
Jaspen (improved)	.000133	.000083	166.
$t = 1.81246,\ n = 10,\ p = .05$			
Wallace			
Approximation u_4	.050009	.000009	.018
Approximation u_5	.050008	.000008	.016
Peizer & Pratt	.050013	.000013	.026
Hill	.050000	.000000	.000
Mickey (modified)	.049999	−.000001	−.002
Bailey	.049995	−.000005	−.010
Gaver & Kafadar	.049959	−.000041	−.082
Jaspen (improved)	.048635	−.001365	−2.730
$t = 6.21105,\ n = 10,\ p = .00005$			
Wallace			
Approximation u_4	.000052	.000002	4.
Approximation u_5	.000050	.000000	0.
Peizer & Pratt	.000052	.000002	4.
Hill	.000050	.000000	0.
Mickey (modified)	.000052	.000002	4.
Bailey	.000051	.000001	2.
Gaver & Kafadar	.000052	.000002	4.
Jaspen (improved)	.000063	.000013	26.

Language and Execution Time. The approximation routines are written in GW-BASIC, but they use only statements common to most BASIC dialects. The approximations were tested in single-precision arithmetic on a Tandy 1000 microcomputer. They run without modification in IBM Personal Computer BASIC on an IBM PC microcomputer. Representative execution time for each approximation is 0.1 sec.

Availability. A listing of the approximations, as shown in Listing 1, can be obtained without charge from the author.

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NOTE

1. Student's (1908) cosine series involve the terms $\sin x$ and $\cos x$, where $x = \arctan(t/n^{1/2})$ and n is the number of degrees of freedom. Computation can be simplified by substituting $t/(n+t_2)^{1/2}$ for $\sin x$ and $[n/(n+t^2)]^{1/2}$ for $\cos x$. Thus the sine and cosine functions are not required. (The arctangent function is required when n is odd.) Computer routines utilizing the series often make these substitutions.

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