

Computing the exact probability of an r by c contingency table with fixed marginal totals

DAVID C. HOWELL and LAWRENCE R. GORDON
University of Vermont, Burlington, Vermont 05401

Several computer programs have recently been published (Hays, 1975; Trichter & Pedrini, 1975) for the computation of Fisher's exact test for the probability of a 2 by 2 table with marginal totals fixed. This procedure is simply a special case of a general method for the r by c table with fixed marginals which was described by Freeman and Halton (1951). Several programs for this more general procedure have also been published (Hancock, 1975; March, 1972). They have been used in Monte Carlo investigations of the chi-square approximation to the probability of such tables (March, 1970) and the effects on statistical decisions of violating the standard "assumptions" of the chi-square approximate test (Howell & Gordon, Note 1).

The intent of this note is to call the attention of psychologists to this more general procedure and to make our subroutine available to that group. This subroutine is on the order of five times faster than that of March, and is about equivalent to that of Hancock.

The method of Freeman and Halton (1951) employs the following formula for the exact probability of a single contingency table, given the marginal distributions:

$$p_K = \frac{\prod_{i=1}^r (R_i!) \prod_{j=1}^c (C_j!)}{N! \prod_{ij} (X_{ij}!)},$$

where R_i and C_j are the (fixed) row and column marginal totals, N is the total number of observations, X_{ij} are the observed cell frequencies, and p_K is the probability of Contingency Table K .

The procedure generates all tables more extreme (i.e., with lower probability) than the data table and sums their probabilities and that of the observed table to produce the probability of obtaining a table as extreme as the one observed, given the fixed marginal totals. This procedure is directly analogous to saying that the probability of at least 8 heads out of 10 flips equals $p(8) + p(9) + p(10)$.¹

Until recently, this method was impractical for contingency tables larger than 2 by 2 because of the number of tables which must be generated, and the number of computations required for each. With the increased availability of computer facilities, it is now reasonable to apply this method to the general r by c table, provided r, c, and N are not too large.

The CONTIN subroutine which implements the above procedure has two basic subprocedures. One of these generates all possible tables having the observed marginal frequencies, and sums the required probabilities. The second carries out the actual calculation of the p_K using logarithms.

As an illustrative example, the observed 2 by 3 table

4	4	0	8
0	4	3	7
4	8	3	15

is one of 20 with the same marginal totals. The probability of obtaining this particular table under the null hypothesis of independence is .01087, and five tables have smaller probabilities (.00870, .00497, .00373, .00124, .00016). The exact probability

The authors would like to acknowledge the assistance of Mrs. Cathy Thompson of the University of Durham, England, who made several helpful contributions concerning the calculation of factorials.

of a table as extreme is $p = .01087 + \dots + .0016 = .02967$. The chi-square statistic for this table has probability = .03074.

We recommend use of this procedure in preference to the chi-square approximation for any contingency analysis where the expected frequencies are small, and the assumption of fixed marginal frequencies is not unreasonable (see Cochran, 1952, 1954).

The CONTIN subroutine can handle any r by c contingency table up to 10 by 10. On a 32-bit Xerox Sigma-6 computer running a time-sharing system, the computation of the 4 by 4 table with $X_{ii} = 3$, and $X_{ij} = 0$ ($i \neq j$) required approximately 10 sec of CPU time, generating and evaluating 2,008 tables in the process. CONTIN is coded in FORTRAN. It receives the number of rows, the number of columns, and the observed cell frequencies from the calling program, and prints the input table and marginal totals, the probability of the specific input table, and the probability of a table as extreme as the input table.

Availability. Listings of the subroutine and a calling program which cycles the analysis until an "end-of-file" is encountered may be obtained by writing to either author at The Department of Psychology, University of Vermont, Burlington, Vermont 05401. No cost is involved.

REFERENCE NOTE

1. Howell, D. C. & Gordon, L. R. *The chi-square test with small expected frequencies*. Paper presented at the meeting of the Eastern Psychological Association, New York, N. Y., April 1975.

REFERENCES

- COCHRAN, W. G. The χ^2 test of goodness of fit. *Annals of Mathematical Statistics*, 1952, **23**, 315-345.
- COHRAN, W. G. Some methods for strengthening the common χ^2 test. *Biometrics*, 1954, **10**, 417-451.
- FREEMAN, G. H., & HALTON, J. H. Note on an exact treatment of contingency, goodness of fit and other problems of significance. *Biometrika*, 1951, **38**, 141-149.
- HANCOCK, T. W. Remark on Algorithm 434: Exact probabilities for R by C contingency tables. *Communications of the ACM*, 1975, **18**, 117-119.
- HAYS, J. E. A FORTRAN procedure for Fisher's exact probability test. *Behavior Research Methods & Instrumentation*, 1975, **7**, 481.
- LANCASTER, H. O. Significance tests in discrete distributions. *Journal of the American Statistical Association*, 1961, **56**, 223-234.
- MARCH, D. L. *Accuracy of the chi-square approximation for 2 by 3 contingency tables with small expectations*. (Doctoral dissertation, Lehigh University, 1970). *Dissertation Abstracts International*, 1971, **32**, 6329B. (University Microfilms No. 71-10, 519)
- MARCH, D. L. Algorithm 434: Exact probabilities for R by C contingency tables. *Communications of the ACM*, 1972, **15**, 991-992.
- RADLOW, R. & ALF, E. F., JR. An alternate multinomial assessment of the accuracy of the χ^2 test of goodness of fit. *Journal of the American Statistical Association*, 1975, **70**, 811-813.
- TRICHLER, D. L., & PEDRINI, D. T. A computer program for Fisher's exact probability test. *Educational and Psychological Measurement*, 1975, **35**, 717-719.

NOTE

1. An alternative procedure proposed by Lancaster (1961) and Radlow and Alf (1975) is also available in CONTIN. This procedure orders tables by their chi-square values and sums the exact probabilities of tables whose chi-square values are equal to or greater than that for the data table.