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Notes and Comment

Improvements on a new model for choice reaction time

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The past decade witnessed construction of a unique theory of simple and disjunctive response time (RT) based on ideas drawn from both Hull-Spence learning theory and Thurstonian psychophysical scaling (Grice, 1968; Grice, Hunt, Kushner, & Morrow, 1974; Grice, Hunt, Kushner, & Nullmeyer, 1976). Now, in a recent theoretical extension, Grice, Nullmeyer, and Spiker (1977) propose a new theory of choice reaction time (CRT) and provide methods for analyzing RT distributions designed to expose the underlying theoretical mechanisms thought responsible for response time duration and variability. Derivations for this new theory are, regrettably, in error and therefore the conclusions drawn from analyses of RT distributions may require modification. The purpose of the current improvement is to correct the erroneous derivations and suggest a new approach toward testing the theory's RT predictions.

A major assumption in the theory of CRT advanced by Grice et al. (1977) is that a stimulus presentation triggers a deterministic development of response strength which continues until a response criterion is first exceeded. The development is deterministic, but the criterion is a random variable and therefore the time to reach the criterion is a random variable. The model for two-response CRT also assumes that following a stimulus presentation, two independent deterministic processes begin: "It is our conception that there is a separate function of time describing the growth of excitatory strength to each response contingent upon the presentation of a given stimulus" (p. 434).

The first of these functions describes the accrual of excitatory strength with respect to a correct response criterion. Because the criterion is a random variable, the time to accrue excitatory strength equal to the criterion is a random variable with probability density function $f_C(t)$ [in Grice et al., 1977, this density function is labeled $y(c_t)$]. The second process accrues strength until a criterion for an error is reached. Again, the criterion is a random variable and $f_E(t)$ we will define as the density function of the time for the second process to reach the error criterion. The model specifies that the process first to reach its criterion elicits a response—either correct or error. Thus, $f_C(t)$ and $f_E(t)$ are not observed directly, but their proper estimation is of singular theoretical importance.

Up to this point, the formal properties of the theory are similar to race or competition theories proposed by LaBerge (1962), Link (1968), and Gibbon and Rutschmann (1969), to mention a few. The new assumption contained in the theory is that the distributions of criteria have a known form (in fact, normal) and constitute the major source of variability in response time. By also specifying the function that relates the development of, say, correct response strength to time, it may be argued that the percentage of correct response strength accruals reaching criterion prior to a particular time measures the percentage of excitatory strengths less than a value of the criterion distribution. The distribution in time of correct response strength accruals may then be used to determine such properties of the criterion distribution as its mean and variance.

Of course, the percentage of correct response strength accruals reaching criterion prior to a particular time is not observed. Rather, the correct response-time distribution is observed, and this distribution consists of those instances where the accrual of correct response excitatory strength reached a variable criterion prior to the error response excitatory strength reaching its variable criterion. In order to determine properties of the correct and error variable criterion distributions, the observed distributions of correct and error response times must be somehow decomposed into the distributions of time to reach criterion for the correct and error processes when the processes are not racing to their respective criteria. Then determinations of the criteria distributions can proceed. It is the failure to derive this decomposition properly that draws the theory into major difficulties.

Unambiguous derivations of the theory's predictions depend upon a more refined partition of theoretical and observable variables than those contained in Grice et al. (1977). In particular, our discussion of correct and error responses will always intend that derivations are conditioned on a particular stimulus being presented. That is to say, our derivations are not performed for the marginal distributions of correct and error responses, but are conditioned upon the occurrence of a particular stimulus. Moreover, to clearly distinguish between the time required for correct response strength to reach the correct response strength criterion and the correct response time, which depends on the correct response strength cri-

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terion being reached prior to the error strength criterion being reached, the discussion below uses the word "process" to refer to the unobservable mechanism that accrues response strength as a function of time.

THEORETICAL RESULTS

The assumed independence of the correct and error accruals permits a straightforward calculation of the probability of a correct response. We note that if t_c is the time when the accrual of correct response strength reaches criterion and if t_e is the time when the error response strength reaches criterion, then $t_c < t_e$ results in a correct response. Let $f_{C,E}(t_c,t_e)$ be the joint probability density function for the completion times of the two processes. By virtue of independence, this joint density function equals $f_C(t_c)f_E(t_e)$. The probability of a correct response is then calculated as

$$P(C) = \int_0^\infty f_C(t)[1 - F_E(t)]dt \qquad (1)$$

where $F_E(t) = \int_0^t f_E(t_e)dt_e$.

To compute the probability density function for observed correct responses, we first define

$$g_C(t) = f_C(t)[1 - F_E(t)], \qquad (2)$$

and note, by the argument put forth above, that

$$\int_0^\infty g_C(t)dt = P(C),$$

so that the function $g_C(t)$ does not integrate to 1.0 and is therefore a defective probability density function. However, $g_C(t)$ can be transformed into a legitimate density function by division by P(C), producing

$$h_C(t) = \frac{f_C(t)[1 - F_E(t)]}{P(C)}, \qquad (3)$$

which is, given the theory, the density function for correct responses.

We may now compare Equation 3 with the corresponding equation in Grice et al. (1977), also labeled 3, which is

$$y(C_t \cap \bar{e}_{<t}) = y(C_t)[1 - p(e_{<t})], \qquad (3G)$$

where $y(C_t)$ is "the theoretical probability density of the correct response reaching criterion at t, depending purely on the stimulus input and the criterion distribution of that response," and $1 - p(e_{<t})$ "is the probability of the absence of a prior error." The reader will notice that $y(c_t)$ corresponds to $f_C(t)$ in Equation 3 above, but beyond this there is little further correspondence between the two equations.

Obviously, any calculations depending upon Equation 3G are likely to be in error. Thus, it is not a surprise that in determining $y(C_t)$ Grice et al. (1977) obtain (incorrectly),

$$y(C_t) = \frac{y(C_t \cap \bar{e}_{<t})}{1 - p(e_{<t})}, \qquad (4G)$$

whereas Equation 3 gives (correctly)

$$f_C(t) = P(C)\left[\frac{h_C(t)}{1 - F_E(t)}\right]. \qquad (4)$$

The major difference between Equations 4 and 4G, besides the obvious multiplication by P(C) in Equation 4, is the difference in the denominators. For Grice et al. (1977), the denominator is the probability of the absence of an error prior to time t, and is to be estimated by 1 minus the proportion of trials on which errors occurred before t. By contrast, the denominator of Equation 4 is 1 minus the probability that the error response strength reached criterion before time t—a probability that cannot be estimated directly from the observed proportion of trials on which errors occurred before time t.

A second difficulty arises from an attempt to estimate values of the defective density $y(C_t)$. The authors argue that this unknown, but desired, density can be estimated by

$$y(C_t) = \frac{P(C_t)}{1 - P(e_{<t})}, \qquad (5G)$$

where the numerator is "the proportion of trials on which the correct response occurred at t (sic), and $P(e_{<t})$ is the proportion of trials on which errors occurred before t."

There are two objections to these substitutions. First, the density in the numerator of Equation 4G cannot be estimated by the substitution of a proportion, which estimates an integral of the density function, for a value of the density function. Second, the value of $P(e_{<t})$ is the probability that an error occurs before time t, but it does not equal the probability that the error process reaches criterion before time t, which is what, according to Equation 4, should be contained in the denominator. Thus, neither suggestion concerning estimation results in an analysis that can be applied to obtained data.

In the event that Equation 4 is used to estimate $f_C(t)$, we must calculate the unobserved value of $F_E(t)$, the probability that the error process, considered alone, reaches its criterion by time t. To perform this calculation, we note first that the (marginal) probability of an error is

$$P(E) = \int_0^\infty f_E(t)[1 - F_C(t)]dt.$$

The density function for observed errors can be calculated using a method identical to that for obtaining Equation 3, which results in the defective density for observed errors,

$$g_E(t) = f_E(t)[1 - F_C(t)].$$

We find the legitimate density function for observed errors to be

$$h_E(t) = \frac{f_E(t)[1 - F_C(t)]}{P(E)}. \quad (5)$$

Then Equation 5 leads the unknown density for the error process to be represented as

$$f_E(t) = \frac{P(E)h_E(t)}{1 - F_C(t)}.$$

The probability that the error process reaches its criterion by time t, when considered independently from the correct response process is

$$F_E(t) = \int_0^t f_E(t^*)dt^*$$

$$= P(E)\int_0^t \frac{h_E(t^*)}{1 - F_C(t^*)}dt^*.$$

Thus, an appropriate representation for the unknown density for the correct response process, obtained by substitution into Equation 3, is

$$f_C(t) = \frac{P(C)h_C(t)}{1 - P(E)\int_0^t \frac{h_E(t^*)}{1 - F_C(t^*)}dt^*}, \quad (6)$$

where $h_C(t)$ and $h_E(t)$ are the density functions for correct and error responses, respectively.

The difficulty with using Equation 6 to estimate $f_C(t)$ arises from the fact that we must simultaneously estimate both $h_C(t)$ and $h_E(t)$ and also estimate the value of the integral in the denominator of Equation 6. Rather than pursue this method further, it may prove useful to calculate the density function of the time to criterion for the independent correct and error processes by investigating the assumptions of the theory in greater detail.

A NEW APPROACH TO TESTING THE THEORY

An additional assumption contained in Grice et al. (1977) is that the variable criteria are normally distributed. Because the accrual process is deterministic, there corresponds to each value of the criterion a unique value for excitatory strength. Moreover, because excitatory strength is a function of time, there corresponds to each value of excitatory strength a unique value of time. Therefore, for each value of a variable criterion there is a unique, corresponding value of excitatory strength which has a unique corresponding value of time. It follows that the response time density function can be treated as the density function of a transformed normal random variable where the transformation is simply the inverse function for the accrual process.

We may now turn to the computation of $f_C(t)$, the density function for the time at which the accrual of correct process excitatory strength stops. In this regard, it is useful to review a fact concerning functions of random variables. Let X be a continuous random variable with probability density function, say, n(x), and suppose that $t = H(x)$ is a strictly monotone differentiable function of x. Then the random variable T, defined as $T = H(X)$, has probability density function given by,

$$f_C(t) = n(x)\left|\frac{dx}{dt}\right|,$$

where x is expressed in terms of t.

In the simplest form of the CRT theory proposed by Grice et al. (1977), the relationship between excitatory strength and time is

$$X = a - be^{-cT},$$

where X is normally distributed, with mean μ and variance κ. To determine the density of T, we need, first,

$$\left|\frac{dx}{dt}\right| = \left|\frac{d}{dt}(a - be^{-ct})\right|$$

$$= bce^{-ct}.$$

Then,

$$f_C(t) = \frac{1}{\sqrt{2\pi\kappa}}\exp[(-1/2\kappa)(a - be^{-ct} - \mu)^2]bce^{-ct}, \quad (7)$$

where truncation of n(x) necessitated by time being nonnegative is ignored for simplicity.

Now that $f_C(t)$ is known, we have computed the probability density function for the termination times of the correct response process. Were one to test the theory against the data, one would, presumably, seek out through chi-square minimization those values of μ, $\varkappa$, a, b, and c that provide a best fit to the distribution function,

$$F_C(t) = \int_0^t f_C(t^*)dt^*,$$

by using standard procedures. Alternatively, one could apply maximum likelihood or method of moments techniques to the estimation of the parameters. However, in the case of the Grice et al. theory of CRT, estimation of $f_C(t)$ cannot proceed directly for it is not $f_C(t)$ that is observed but, rather, correct responses occurring prior to an error.

The density function for the error process can be determined in a manner similar to that used in obtaining Equation 7. Suppose that the relationship between excitatory strength and time is

$$Y = u - ve^{-wT},$$

where Y is normally distributed with mean m and variance L. The result is:

$$f_E(t) = \frac{1}{\sqrt{2\pi L}} \exp[(-1/2L)(u - ve^{-wt} - m)^2]wve^{-wt}. \tag{8}$$

Since the processes for correct and error excitatory strengths are independent, we can compute probability of a correct response by using Equation 1. Then the density function for the observed correct responses follows by applying Equation 3. Again, a similar method leads to the determination of the observed error distribution by using Equations 7 and 8 in Equation 5.

CONCLUSION

The practical difficulties in pursuing the methods outlined above are disturbingly large. For example, formulating the theory in the simple manner proposed here exposes the fact that there are at least 10 parameters involved in specifying the density functions for the correct and error processes. These are a, b, c, μ, $\varkappa$, u, v, w, m, and L. If the normal distributions for the variable correct and error criteria were to be truncated to ensure that negative response times are impossible, then two additional parameters arise. The frankly alarming number of parameters needed to provide a cogent test of the theory causes at least this investigator to question the theoretical value of the theory, especially when several other current theories of choice reaction time could probably account for the basic empirical facts with far fewer parameters.

The analyses presented in this improvement were conditioned on the presentation of a particular stimulus and do not apply to the analysis of marginal correct and error RT distributions. These marginal distributions are properly considered to be mixtures of distributions. For example, the marginal error RT distribution equals the weighted average of the error RT distributions resulting from presentations of the two stimuli. The weights are estimated by the number of errors in a distribution divided by the total number of errors. The parameters of the resulting distribution of response times are generally not the weighted average of the parameter values obtained by separate analysis of each separate error distribution. In fact, the mixture (of distributions) often results in a distribution which is not a member of the same family as the distributions being mixed. In this case, application of family-dependent estimation methods would lead to questionable results. Furthermore, the idea of simply combining all error responses into a single distribution and proceeding with estimation is identical to analyzing a marginal distribution that is a mixture of distributions and will lead to problematic conclusions.

Beyond these practical problems are theoretical calculations which, in spite of the comments by Grice et al. (1977, p. 432), are best pursued by formulating the theory rigorously and applying straightforward mathematical methods in deriving the predictions of the theory. If the techniques contained in this "improvement" had been applied to the formulation of the proposed theory, then the authors' assertions regarding the value of their theory, as opposed to the value of other current theories, might have merit. But the analyses of data are based on erroneous derivations and hardly support the "scientific optimism" subscribed to by the authors.

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