Multifractal analyses of response time series: A comparative study

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Abstract Response time series with a non-Gaussian distribution and long-range dependent dynamics have been reported for several cognitive tasks. Conventional monofractal analyses numerically define a long-range dependency as a single scaling exponent, but they assume that the response times are Gaussian distributed. Ihlen and Vereijken (Journal of Experimental Psychology: General, 139, 436–463, 2010) suggested multifractal extensions of the conventional monofractal analyses that are more suitable when the response time has a non-Gaussian distribution. Multifractal analyses estimate a multifractal spectrum of scaling exponents that contain the single exponent estimated by the conventional monofractal analyses. However, a comparison of the performance of multifractal analyses with behavioral variables has not yet been addressed. The present study compares the performance of seven multifractal analyses. The multifractal analyses were tested on multiplicative cascading noise that generates time series with a predefined multifractal spectrum and with a structure of variation that mimics intermittent response time variation. Time series with 1,024 and 4,096 samples were generated with additive noise and multiharmonic trends of two different magnitudes (signal-to-noise/trend ratio; 0.33 and 1). The results indicate that all multifractal analysis has individual pros and cons related to sample size, multifractality, and the presence of additive noise and trends in the response time series. The summary of pros and cons of the seven multifractal analyses provides a guideline for the choice of multifractal analyses of response time series and other behavioral variables.

Keywords Multifractal · Multiplicative interaction · Wavelet · Scaling laws · Response series · Behavior · Variability

Introduction

Analyses of response time series have given insight into cognitive processes and mental organization in a wide variety of tasks such as simple reaction, word naming, choice decision, visual search, memory search, and lexical decision. Contemporary response time analyses have been developed within two paradigms: analyses of the response time distribution and analyses of the trial dependency of response time dynamics. Response time distributions are typically unimodal and positively skewed toward long response latencies. Consequently, distributions like the ex-Gaussian (Burbeck & Luce, 1982), shifted Wald (Schwarz, 2001), lognormal (Ulrich & Miller, 1993), Gamma (Van Zandt & Ratcliff, 1995), Weibull (Logan, 1992), and power law (Holden, Van Orden, & Turvey, 2009; Moscoso del Prado Martin 2009) distributions have been suggested as models to parameterize the response time distribution. However, all models assume that the response times are trial-independent random variables. In contrast, several studies conducted through the last decades have found that response times possess long-range trial dependency (Gilden, 2001; Gilden, Thornton, & Mallon, 1995; Kello, Beltz, Holden, & Van Orden, 2007; Kello et al., 2010; Van Orden, Holden, & Turvey, 2003). The long-range trial dependency of response times is numerically defined by a scaling exponent obtained by monofractal analyses such as detrended fluctuation analysis, scaled window variance analysis, or rescaled range analysis, to mention but a few (cf. Delignières et al., 2006). All these analyses numerically define long-range trial

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dependency of response time series by a single exponent that is estimated by the scaling of the root-mean square, variance, or range. This scaling exponent is found to decrease with increased difficulty and an increased level of external perturbation of the response time task (cf. Diniz et al., 2011) and has been suggested to reflect the self-organization of cognitive processes (e.g., Van Orden et al., 2003).

The conventional numerical definition of a single scaling exponent assumes that response times are Gaussian distributed such that the response time variation is described by the second-order statistical moment (i.e., variance) alone. However, response time distributions are found to be unimodal and right-skewed toward long response latencies across a range of cognitive tasks (Luce, 1986). The presence of a non-Gaussian response time distribution has two fundamental implications for the analyses of long-range trial dependency of the response time series. First, the presence of a non-Gaussian response time distribution indicates that the response time variation cannot be exclusively described by the scaling of variance alone but that the scaling of higher order statistical moments such as skewness and kurtosis has to be considered. Thus, the single scaling exponent defined by conventional monofractal analyses will define only the fraction of the trial dependency related to the variance of the response time series. Second, the presence of a non-Gaussian response time distribution indicates intermittent changes in the magnitude of response time variation. These intermittent changes might be due to modulation of the task difficulty, sensory feedback, or changes in participants' intention to act according to the task instruction. Furthermore, intermittent changes in variability have been shown to be a central feature for phase transition in a bimanual coordination task (Kelso, 1995) and other complex systems (Schaffer et al., 2009). Intermittent changes in the response time variation will provide temporal modulation of both the width and shape of the response time distribution and, consequently, temporal modulation of the scaling exponent.

Ihlen and Vereijken (2010) introduced multifractal analyses that are able to numerically define the scaling exponents of higher order statistical moments and the temporal modulation of a local scaling exponent. They suggested that the multifractal structure of response time series reflected phase transitions in the mental organization driven by multiplicative interactions across the temporal scales of the response time dynamics. Later, multifractal analyses of response times revealed temporal changes in the trial-dependent structure of the response series by modulation in task difficulty and performance feedback (Kuznetsov & Wallot, 2011) and the presence of strong anticipation (Stephen & Dixon, 2011). The presence of multifractal variation is also a central feature of healthy regulation of physiological processes like heart rate dynamics (Ivanov et al., 1999), and the absence of multifractal variation is related to several pathological conditions (Wang, Huang, Xie, Wang, & Hu, 2007). However, none of the

studies within behavioral research have compared the performance of the multifractal analyses a priori with their employment for behavioral variables.

The aim of the present study is to compare the performance of seven multifractal analyses and, consequently, to provide a guideline for the choice of multifractal analyses of response time series and other behavioral variables. The article describes and compares multifractal analyses and assumes that the reader is familiar with basic statistical concepts of fractal analyses. Readers unfamiliar with the basic concepts of fractal analyses are referred to Ihlen and Vereijken (2010) and Ihlen (2012) for an introduction to the topic.

Method

Multifractal analyses of a response time series are all conducted by a step-wise procedure illustrated in Fig. 1. First, the response time series is decomposed into a scale-dependent measure $\mu_{s,t}$ (see the top panel in Fig. 1). Second, the multifractal spectrum is computed either directly from $\mu_{s,t}$ (right panels along B arrows in Fig. 1) or indirectly by the q -order statistics of $\mu_{s,t}$ (left panels along A arrows in Fig. 1). The scale-dependent measure $\mu_{s,t}$ and q -order statistics differentiate between the different types of multifractal analyses and, consequently, between their potential pros and cons when applied to response time series. The seven multifractal analyses compared in the present study are introduced in the next three subsections according to their scale-dependent measure $\mu_{s,t}$ and their q -order statistics.

Scale-dependent measure

Response time series are decomposed into both time and scale domains by (1) computing a scale-dependent measure μ_{s,t_0} in a floating trial interval $[t_0 - s/2, t_0 + s/2]$ centered at trial t_0 with sample size s (i.e., scale) or by (2) computing a scale-dependent measure $\mu_{s,v}$ in the nonoverlapping trial interval $[(v-1)s+1, vs]$ with interval number v and sample size s . The analyses that estimate the multifractal spectrum directly from $\mu_{s,t}$ use method 1, whereas the analyses based on q -order statistics of $\mu_{s,t}$ use method 2. The scale-dependent measure of the seven multifractal analyses compared in the present study is defined below.

Multifractal detrended fluctuation analysis

Multifractal detrended fluctuation analysis (MFDFA; Kantelhardt et al., 2002) defines the scale-dependent measure $\mu_{s,v}$ as a root-mean square fluctuation of the integrated profile $y_t = \sum_{i=1}^t (x_i - \bar{x})$ of the response time series x_i around a scale-dependent trend $\hat{y}_{s,t}$ within a nonoverlapping time interval $[(v-1)s+1, vs]$:

Fig. 1 A flow chart of the estimation of the multifractal spectrum D_h directly from the scale-dependent measure μ_{s,t_0} (B arrows) and indirectly by the q -order statistics of $\mu_{s,v}$ (A arrows). The basis for all multifractal analyses within both formalisms is the scale-dependent measure (upper contour plot) that decomposes the response time series or behavioral variable into both the time and scale domains. The red contours indicate large scale-dependent measures of the response time series that coincide with the time periods of intermittent large variations. In contrast, the blue contours indicate small scale-dependent measures that coincide with the time periods of intermittent small variations. The panel below the top arrow A indicates that the scale-dependent measure is summarized by its q -order statistical moment (see Eq. 5). The statistical moments with

positive qs amplifies the large $\mu_{s,v}$ (i.e., red contours), whereas the statistical moments negative qs amplifies the small $\mu_{s,v}$ (i.e., blue contours). The scaling exponent ζ_q numerical defines the power law relation of the intermittent periods with large (i.e., positive qs) and small variations (i.e., negative qs). The panel below the bottom arrow A illustrates a multifractal spectrum D_h estimated by the Legendre transformation of ζ_q (i.e., Eq. 6). The multifractal spectrum D_h is the plot of D_q versus h_q where the mode $\bar{h} = h_{q=0}$ is defined at the maxima $D_{q=0} = 1$. The panel below the top arrow B illustrates the direct estimation of the local singularity exponent h_{t_0} as the local slope of $\log(\mu_{s,t_0})$ versus $\log(s)$ for each time instant t_0 . The panel below the bottom arrow B illustrates the multifractal spectrum D_h defined by Eq. 9 in the main text

$$\mu_{s,v} = \sqrt{\frac{1}{s} \sum_{t=(v-1)s+1}^{vs} [y_t - \bar{y}_t]^2} \quad (1)$$

The present study compares the performance of MFDFA with linear (MFDFA_{lin}), moving average (MFDFA_{ma}), and empirical mode decomposition (MFDFA_{emd}) detrending procedures (see technical details in Appendix 1).

Entropy analysis

Entropy analysis (Chhabra & Jensen, 1989) defines the scale-dependent measure as a probability measure of the total magnitude of the response time fluctuation (i.e., sum of x_t) within the nonoverlapping time interval $[(v-1)s+1, vs]$:

$$\mu_{s,v} = \frac{\sum_{t=(v-1)s+1}^{vs} x_t}{\sum_{t=1}^N x_t} \quad (2)$$

The definition of $\mu_{s,v}$ in entropy analysis contains no detrending procedure and, therefore, assumes that the response time series is a stationary process.

Leader wavelet transformation

Leader wavelet transformation (LWT; Wendt, 2008) defines the scale-dependent measure $\mu_{s,v}$ by a recursive high-pass and low-pass filtering procedure of the response time series x_t illustrated in Fig. 2a. The scale-dependent measure $\mu_{s=2,v}$ on the smallest scale $s=2$ is the convolution product $\otimes$ of response time series x_t and the high-pass filter ψ_t^{hi} subsampled by 2. The subsampled convolution product $\otimes$ of the response time series x_t and low-pass filter ψ_t^{lo} are used in another convolution and subsample procedure to define the statistical measure $\mu_{s,v}$ on the coarser scales $s=4$. The detrending of the response time series in LWT is dependent on the choice of ψ_t^{hi} and ψ_t^{lo} where the present study uses a

third-order Daubechies filter (Daubechies, 1992). The subsampling by 2 leads to dyadic intervals $\lambda_{s,v}$ that double the sample sizes $s=2^j$ for the nonoverlapping intervals (width of the boxes in Fig. 2b) for each new level j , but halves the frequency band $f=1/2^j$ (height of the boxes in Fig. 2b). The local maximum of $\mu_{s,v}$ (i.e., wavelet leader illustrated in Fig. 2b) is used as a scale-dependent measure to prevent a degeneration of the multifractal spectrum by $\mu_{s,v} \rightarrow 0$.

Gradient modulus wavelet projection

Gradient modulus wavelet projection (GMWP; Turiel, Perez-Vicente, & Grazzini, 2006) defines a scale-dependent measure by a continuous wavelet transformation:

$$\mu_{s,t_0} = \sum_{t=t_0-s/2}^{t_0+s/2} |x_t| \psi_{s,t_0-t} \quad (3)$$

The continuous wavelet transformation is the convolution of the response time series and a waveform ψ_{s,t_0-t} scaled to the floating time interval $[t_0-s/2, t_0+s/2]$ (Mallat, 1999). The detrending of the response series is dependent on the shape of the waveform. The present study used a Laplacian wavelet $\psi_{s,t_0-t} = (1 + (t^2/s))^{-2}$ that performs well on the smallest scales (Turiel & Perez-Vicente, 2003).

Local detrended fluctuation analyses

Local detrended fluctuation analyses (DFA_{loc}; Ihlen, 2012) defines the scale-dependent measure μ_{s,t_0} as a

Fig. 2 Schematic representation of the leader wavelet transformation. **a** The discrete wavelet transformation computes the statistical measure $\mu_{s,v}$ by recursive high-pass and low-pass filtering procedure of the response time series x_t . **b** Twenty-eight samples of a time series (upper panel) and the dyadic intervals $\mu_{s,v}$ of the statistical measure $\mu_{s,v}$ (lower panel) defined by the recursive filtering procedure. The wavelet leaders is defined as the local maxima of three subsequent intervals $\lambda_{s',v}$ on scale s' and for all $s < s'$ (gray colored boxes)

root-mean square fluctuation of the integrated response time series $y_t = \sum_{i=1}^t (x_i - \bar{x})$ around a polynomial trend $\hat{y}_{t,m}$ of order m within a floating trial interval $[t_0 - s/2, t_0 + s/2]$:

$$\mu_{s,t_0} = \sqrt{\frac{1}{s} \sum_{t=t_0-s/2}^{t_0+s/2} [y_t - \hat{y}_{t,m}]^2} \quad (4)$$

The present study used a second-order polynomial detrending (i.e., $m = 2$) of the response time series.

Indirect estimation from q -order statistics of $\mu_{s,v}$

The q -order statistical moments $E(\mu_{s,v}^q)$ are defined for the scale-dependent measure of MFDFA and LWT by the following equation (Parisi & Frisch, 1985):

$$E(\mu_{s,v}^q) = \frac{1}{N_s} \sum_{v=1}^{N_s} \mu_{s,v}^q \propto s^{\zeta_q} \quad (5)$$

where N_s is the number of nonoverlapping intervals at each scale s . $E(\mu_{s,v}^q)$ satisfies the power law on the right-hand side when the response time series has a multifractal structure. The q -order scaling exponent ζ_q is estimated as the linear regression slope of $E(\mu_{s,v}^q)$ versus s in log-log coordinates (see middle left panel in Fig. 1). Next, the q -order singularity exponent h_q and singularity dimension D_q are defined by the following Legendre transformation of ζ_q (Parisi & Frisch, 1985):

$$h_q = \frac{d\zeta_q}{dq} \quad (6)$$

$$D_q = 1 + qh_q - \zeta_q$$

Finally, the multifractal spectrum D_h is defined by plotting h_q versus D_q (see lower left panel in Fig. 1).

In entropy analysis, the scale-dependent measure of Eq. 2 is used to estimate h_q and D_q directly from the following q -order entropies $F_h(\mu_{s,v}^q)$ and $F_D(\mu_{s,v}^q)$, respectively, without the Legendre transformation of Eq. 6 (Chhabra & Jensen, 1989):

$$F_h(\mu_{s,v}^q) = \sum_{v=1}^{N_s} P_{s,v}^q \log(\mu_{s,v}) \propto s^{h_q} \quad (7a)$$

$$F_D(\mu_{s,v}^q) = \sum_{v=1}^{N_s} P_{s,v}^q \log(P_{s,v}^q) \propto s^{D_q}$$

where

$$P_{s,v}^q = \frac{\mu_{s,v}^q}{\sum_{v=1}^{N_s} \mu_{s,v}^q} \quad (7b)$$

Similar to the estimation of ζ_q in MFDFA and LWT, h_q and D_q are estimated as the linear regression slopes of the q -

order entropies $F_h(\mu_{s,v}^q)$ and $F_D(\mu_{s,v}^q)$, respectively, and scale s in log-log coordinates.

Direct estimation from μ_{s,t_0}

GMWP and DFA_{loc} estimate the multifractal spectrum directly from the scale-dependent measure of Eqs. 3 and 4, respectively. The temporal variation in the trial dependency of a response time series can be defined by the local singularity exponents h_{t_0} when scale-dependent measure μ_{s,t_0} satisfies the following power law (Canus, Lévy Véhel, & Tricot, 1998; Turiel, Yahia, & Pérez-Vicente, 2008):

$$\mu_{s,t_0} \propto \lim_{s \rightarrow 0} s^{h_{t_0}} \quad (8)$$

The local scaling exponent h_t is estimated as the linear regression slope of log-log plot of the scale-dependent measure μ_{s,t_0} versus scale s (see the middle right panel in Fig. 1), but the present study uses a method suggested by Struzik (2000) that provides more numerically stable results (see technical details in Appendix 2). The multifractal spectrum D_h is defined by the normalized distribution P_h of h_{t_0} in log-coordinates by the following equation (Turiel et al., 2008):

$$D_h = \lim_{\varepsilon \rightarrow 0} \left(1 - \frac{\log(P_h/P_h^{\max})}{\log(\varepsilon)} \right) \quad (9)$$

where ε is the bin size of the histogram used to define P_h and $P_h^{\max}$ is the maximum probability at the mode $\bar{h}$ of h_{t_0} (see the lower right panel in Fig. 1). The most common trial dependency in the response series is the mode $\bar{h}$ and is assigned $D_h = 1$ by Eq. 9.

In summary, the present article compares the performance of the seven multifractal analyses represented in Table 1 and defined by the choice of scale-dependent measure and q -order statistics. Table 1 also summarizes the parameter settings used to compare the performance of the multifractal analyses. All multifractal analyses in Table 1 were conducted in MATLAB and are available at www.ntnu.edu/inm/geri/software.

Multiplicative cascading noise

The seven multifractal analyses represented in Table 1 were tested on the stationary increments $\Delta B_H(A(t)) = B_H(A(t+1)) - B_H(A(t))$ of a multifractal random walk $B_H(A(t))$ (Bacry, Muzy, & Delour, 2001; Ihlen & Vereijken, 2010). The main feature of $\Delta B_H(A(t))$ is intermittent periods with variation of large and small magnitudes (see red and blue intervals, respectively, in Fig. 3). The increasing width of the multifractal spectrum D_h of $\Delta B_H(A(t))$ reflects increasing differences in the magnitudes of intermittent small and large variations (compare the variations within the red and blue intervals for light to dark blue traces in Fig. 3). The central tendency $\bar{h}$ reflects the long-range

Table 1 Summary of definitions and parameter settings for the seven multifractal analyses compared in the present study

Analysis (abbreviation)	Formalism	Measure $\mu_{s,t}$	Scaling range	q -range
MF DFA	Eqs. 5 and 6	Eq. 1	$16 \leq s \leq N/4$	$-3 \leq q \leq 3$
MF DFA _{ma}	Eqs. 5 and 6	Eqs. 14 and 15	$16 \leq s \leq N/4$	$-3 \leq q \leq 3$
MF DFA _{cmd}	Eqs. 5 and 6	Eqs. 14 and 16	$16 \leq s \leq N/4$	$-3 \leq q \leq 3$
Entropy	Eq. 7a	Eq. 2	$16 \leq s \leq N/4$	$-3 \leq q \leq 3$
LWT	Eqs. 5 and 6	Fig. 2	$16 \leq s \leq N/4$	$-3 \leq q \leq 3$
GMWP	Eqs. 8 and 9	Eq. 3	$7 \leq s \leq 17$	–
DFA _{loc}	Eqs. 8 and 9	Eq. 4	$7 \leq s \leq 17$	–

dependency H of the fractional Brownian motion $B_H(t)$. The present performance comparison utilized an algorithm developed by Chainais, Riedi, and Abry (2005) to generate 100 realizations of $\Delta B_H(A(t))$ of 1,024 and 4,096 samples with $H = 0.5, 0.6, 0.7, 0.8$, and 0.9 and with a multifractal spectrum width $h_{\max} - h_{\min} = 0, 0.1, 0.2, 0.3, 0.4, 0.5$, and 0.6 . A symmetric shape of the multifractal spectrum was provided by a lognormal cascade $A(t)$ constructed by random multipliers with a lognormal distribution.

Response time series are influenced by noise and nonstationary trends caused by differences in task difficulty, external perturbations, or internal mental strategies. Thus, Gaussian-distributed random noise with a signal-to-noise ratio of either 0.33 or 1 was added to $\Delta B_H(A(t))$ to test for the robustness of the

seven multifractal analyses shown in Table 1 to additive noise (Delignières et al., 2006). Furthermore, a complex multiharmonic trend $T(t)$ given by the following equation was added to $\Delta B_H(A(t))$ to test the robustness of the multifractal analyses to complex additive trends:

$$T(t) = C \sin(0.1250\pi t) + C \sin(0.0313\pi t) + C \sin(0.0078\pi t) + C \sin(0.0020\pi t) \quad (10)$$

where t is the sample number of $\Delta B_H(A(t))$. The amplitudes C of oscillations were 0.33 and 1 times the root-mean square variation of $\Delta B_H(A(t))$ (i.e., signal-to-trend ratios of 0.33 and 1). Figure 4 illustrates $\Delta B_H(A(t))$ with $H = 0.9$ (see top panel) with additive white noise (see red trace in left panels) and multiharmonic trend

Fig. 3 Intermittent variations in the multifractal process $\Delta B_H(A(t))$ with $H = 0.9$ for the seven multifractal spectrum widths $h_{\max} - h_{\min}$ used in the present comparative study of multifractal analyses (traces from light to dark blue in the left panel with corresponding colored spectrum in the right panel). $h_{\max} - h_{\min}$ are directly related to the width of the distribution of interaction multipliers used to generate the cascade $A(t)$ within $\Delta B_H(A(t))$. The blue interval and arrow indicate the intermittent period

of small variation with local scaling exponents in the right tail of D_h close to $h_{\max}$. The red interval and arrow indicate intermittent period of large variation with local exponent in the left tail of D_h close to $h_{\min}$. The difference between the magnitude of small and large variations within $\Delta B_H(A(t))$ increases with its multifractal spectrum width $h_{\max} - h_{\min}$ indicated by an increase in the large variation within the red intervals and a decrease in variation within the blue intervals

Fig. 4 Multiplicative cascading noise $\Delta B_H(A(t))$ with $H = 0.9$ (top panel) with additive white noise and multiharmonic trend (red traces). Right panels show $\Delta B_H(A(t))$ with white noise with a moderate signal-to-noise ratio of 0.33 (middle panel) and a high signal-to-noise ratio of

1 (bottom panel). Left panels show $\Delta B_H(A(t))$ with multiharmonic trends with a moderate signal-to-trend ratio of 0.33 (middle panel) and a high signal-to-trend ratio of 1 (bottom panel)

(see red trace in right panels). The white noise and multiharmonic trend generate minor distortions of the intermittent variation in $\Delta B_H(A(t))$ for the signal-to-noise/trend ratio of 0.33 (see middle panels in Fig. 4), whereas major distortion of the intermittent variation in $\Delta B_H(A(t))$ is generated for the signal-to-noise/trend ratio of 1 (see bottom panels in Fig. 4).

Performance comparison of the multifractal analyses

The performance of the multifractal analyses in Table 1 was evaluated by two error measures. The following error measure defines the deviations in the width and shape of the estimated multifractal spectrum D_h^e and the predefined D_h of $\Delta B_H(A(t))$ (Turiel et al., 2006):

$$E_D = \frac{\langle |D_h - D_h^e| \rangle_h}{h_{\max} - h_{\min}} \quad (11)$$

where $\langle | \dots | \rangle_h$ defines the mean absolute difference between D_h and D_h^e and is taken over all values h of D_h and

normalized to the width $h_{\max} - h_{\min}$ of D_h . D_h^e was set to zero for the h of the model where the estimated D_h^e has no defined value (Turiel et al., 2006). Furthermore, the estimated central tendency $\bar{h}^e$ and the central tendency of the model $\bar{h}$ were aligned (i.e., $\bar{h}^e = \bar{h}$) to exclude the influence of errors in estimated $\bar{h}^e$ on the error measure in Eq. 11. The root-mean square error of the estimated central tendency $\bar{h}^e$ was defined by the following equation:

$$E_{\bar{h}} = \sqrt{\langle (\bar{h} - \bar{h}^e)^2 \rangle} \quad (12)$$

where $\bar{h}$ is the central tendency of model D_h and $\bar{h}^e$ is the estimated central tendency by multifractal analyses in Table 1. The mean $\langle \dots \rangle$ was computed for the entire ensemble of 100 model series of $\Delta B_H(A(t))$ for each parameter setting. The following color convention was used to improve the readability of the tabulated results in the Results section below (Turiel et al., 2006):

$E \leq 0.10$	$0.10 < E \leq 0.20$	$0.20 < E \leq 0.30$	$E > 0.30$
White box	Yellow box	Orange box	Red box

A paired sample t -test was employed to identify a significantly different E_D . If one or two analyses had a significantly

smaller or larger E_D ($p < .05$), as compared with the rest of the analyses, then an asterisk * and a crucifix †, respectively, was

marked behind value. Significant differences for two analyses were also considered because the performance of $\text{MF DFA}_{\text{lin}}$ and $\text{MF DFA}_{\text{ma}}$ together with DFA_{loc} and GMWP were expected to be more related than the other analyses. The error measure E_D was also visually evaluated with the mean and standard deviation of D_h from the ensemble of 100 series of $\Delta B_H(A(t))$ where D_h was aligned at their central tendency.

Results

Comparison of the estimated D_h

Figure 5 shows the mean D_h and ± 1 standard deviation (see bars) estimated by the seven multifractal analyses in Table 1 for the ensemble of $\Delta B_H(A(t))$ with $H = 0.5$ and $h_{\text{max}} - h_{\text{min}} = 0$ (see red bars) and $h_{\text{max}} - h_{\text{min}} = 0.6$ (see blue bars). The best performance of analyses based on q -order statistics (i.e., MF DFA , LWT, and entropy analysis) was provided by $\text{MF DFA}_{\text{lin}}$ and $\text{MF DFA}_{\text{ma}}$ when the multifractal analyses were employed to $\Delta B_H(A(t))$ with $H = 0.5$ (see Table 2). However, the LWT and entropy analyses performed more consistently across H and were superior to the MF DFA s for $H = 0.9$, as indicated by error measures in Table 3. The LWT and entropy analyses were also superior to the GMWP and DFA_{loc} that showed similar inconsistency across H as the MF DFA s. Furthermore, LWT had more consistent errors

(i.e., systematic underestimation of D_h), as compared with the other analyses, based on the q -order statistics indicated in Fig. 5 by the smaller standard deviation for the left tail of the multifractal spectrum D_h .

Comparison of the estimated central tendency $\bar{h}$ of D_h

All multifractal analyses estimated the central tendency $\bar{h}$ of the multifractal spectrum with high precision. LWT had the largest estimation error $E_{\bar{h}}$, whereas the GMWP had the smallest $E_{\bar{h}}$ for both sample sizes, as indicated in Table 4. However, GMWP had larger variation in $\bar{h}$, as compared with DFA_{loc} . The estimation of central tendency was consistent across H of $\Delta B_H(A(t))$ but was influenced by the presence of white noise and multiharmonic trends.

Influence of sample size

The error measure E_D decreased in magnitude and was more consistent with an increased sample size (i.e., $N = 4,096$) for all multifractal analyses. In addition, all analyses performed more consistently across H of $\Delta B_H(A(t))$, and the differences between the performance of multifractal analyses decreased. The largest decrease in magnitude of E_D was observed for $\text{MF DFA}_{\text{emd}}$, where the performance became comparable to those for $\text{MF DFA}_{\text{lin}}$ and $\text{MF DFA}_{\text{ma}}$ (compare upper and lower parts of Table 2). Nevertheless, $\text{MF DFA}_{\text{emd}}$ had less

Fig. 5 Mean D_h and ± 1 SD (bars) estimated by the seven multifractal analyses in Table 1 for the ensemble of $\Delta B_H(A(t))$ with 4,096 samples and $h_{\text{max}} - h_{\text{min}} = 0$ (red bars) and $h_{\text{max}} - h_{\text{min}} = 0.6$ (blue bars)

Table 2 Mean error E_D for six different multifractal spectrum widths $h_{\max}-h_{\min}$ for $\Delta B_H(A(t))$ with $H = 0.5$ and with no noise or trends present

	0	0.1	0.2	$h_{\max}-h_{\min}$ 0.3	0.4	0.5	0.6
$N = 1024$							
MF DFA	0.08*	0.10*	0.14*	0.14*	0.14	0.10	0.09
MF DFA _{ma}	0.07*	0.09*	0.14*	0.14*	0.14	0.10	0.09
MF DFA _{emd}	0.10	0.12	0.17	0.16	0.14	0.12†	0.13†
Entropy	0.11	0.13	0.17	0.15	0.12	0.09	0.07
LWT	0.12	0.14	0.17	0.16	0.12	0.09	0.08
GMWP	0.20†	0.21†	0.20	0.17	0.13	0.09	0.06*
DFA _{loc}	0.20†	0.21†	0.20	0.17	0.13	0.09	0.06*
$N = 4096$							
MF DFA	0.05	0.06	0.07	0.06	0.05	0.06	0.08†
MF DFA _{ma}	0.05	0.05	0.06	0.05	0.05	0.07	0.09†
MF DFA _{emd}	0.06	0.06	0.06	0.07	0.07	0.06	0.05
Entropy	0.08	0.09	0.09	0.08	0.05	0.03	0.05
LWT	0.10	0.11	0.12	0.11	0.07	0.05	0.05
GMWP	0.20†	0.17†	0.14†	0.11	0.06	0.03	0.05
DFA _{loc}	0.20†	0.17†	0.14†	0.11	0.06	0.03	0.05

†, * represents a significantly larger and smaller mean error E_D (paired t -test; $p < 0.05$)

consistent error E_D , as compared with the other analyses, indicated by large standard deviations in the tails of D_h in Figs. 5 and 6. The largest increase in consistency of E_D with sample size was observed in DFA_{loc} that had a twofold lower standard deviation of E_D , as compared with the analyses based on the q -order statistics (i.e., MF DFA, LWT, and entropy analysis).

Influence of additive noise

Figure 6 shows the mean D_h and ± 1 SD (see bars) estimated by the seven multifractal analyses in Table 1 for the ensemble of $\Delta B_H(A(t))$ with additive white noise. The blue, green, and red markers in Fig. 6 are the estimated D_h for $\Delta B_H(A(t))$ with signal-to-noise ratios of 0, 0.33, and 1, respectively. Comparison of blue, green, and red markers in Fig. 6 shows that the increasing magnitude of additive white noise truncates the

estimated multifractal spectrum D_h and translates its central tendency toward $\bar{h} = 0.5$. This result was especially apparent for the MF DFA, LWT, and entropy analyses.

All multifractal analyses were quite robust to additive white noise in the case of the moderate signal-to-noise ratio of 0.33, as is shown in Table 5. The robustness to noise appeared visually as the preservation of the left side (i.e., positive q s) of the multifractal spectra for the MF DFA, LWT, and entropy analyses (compare blue and green D_h in Fig. 6). However, the same analyses yield the above-mentioned changes for both the left and right sides of D_h for large noise levels (i.e., signal-to-noise ratio of 1), as is shown by the increased error measure E_D in the upper part of Table 6 and the truncated red multifractal spectra in Fig. 6. GMWP and DFA_{loc} had a more noise-resistant performance, as compared with the analyses based on the q -

Table 3 Mean error E_D for five different H of $\Delta B_H(A(t))$ with multifractal spectrum width $h_{\max}-h_{\min} = 0.6$ and sample sizes 1,024 (upper panel) and 4,096 (lower panel)

	$H = 0.5$	$H = 0.6$	$H = 0.7$	$H = 0.8$	$H = 0.9$
$N = 1024, h_{\max}-h_{\min} = 0.6$					
MF DFA	0.09	0.10	0.12	0.13	0.17
MF DFA _{ma}	0.09	0.11	0.11	0.13	0.13
MF DFA _{emd}	0.07	0.09	0.10	0.12	0.14
Entropy	0.07	0.09	0.11	0.11	0.12
LWT	0.07	0.10	0.11	0.11	0.11
GMWP	0.06	0.11	0.11	0.13	0.16
DFA _{loc}	0.06	0.11	0.11	0.13	0.16
$N = 4096, h_{\max}-h_{\min} = 0.6$					
MF DFA	0.08	0.05	0.06	0.08	0.09
MF DFA _{ma}	0.09	0.06	0.06	0.07	0.07
MF DFA _{emd}	0.05	0.04	0.06	0.07	0.08
Entropy	0.05	0.04	0.05	0.06	0.06
LWT	0.04	0.04	0.06	0.07	0.06
GMWP	0.02	0.05	0.06	0.08	0.09
DFA _{loc}	0.02	0.05	0.06	0.08	0.09

Table 4 Error $E_{\bar{h}}$ of the position of the central tendency for $\Delta B_H(A(t))$ with five different H and multifractal spectrum width $h_{\max} - h_{\min} = 0.6$ and sample sizes 1,024 (upper part) and 4,096 (lower part)

	$H = 0.5$	$H = 0.6$	$H = 0.7$	$H = 0.8$	$H = 0.9$
$N = 1024, h_{\max} - h_{\min} = 0.6$					
MFDFA	0.08	0.09	0.08	0.08	0.10
MFDFA _{ma}	0.09	0.09	0.09	0.09	0.09
MFDFA _{emd}	0.08	0.08	0.08	0.08	0.10
Entropy	0.09	0.09	0.05	0.01	0.06
LWT	0.09	0.11	0.12	0.14	0.15
GMWP	0.01	0.03	0.01	0.01	0.01
DFA _{loc}	0.08	0.08	0.06	0.07	0.08
$N = 4096, h_{\max} - h_{\min} = 0.6$					
MFDFA	0.04	0.04	0.04	0.04	0.04
MFDFA _{ma}	0.04	0.04	0.05	0.05	0.05
MFDFA _{emd}	0.04	0.04	0.04	0.04	0.04
Entropy	0.05	0.04	0.02	0.01	0.07
LWT	0.06	0.07	0.08	0.10	0.11
GMWP	0.01	0.01	0.01	0.02	0.02
DFA _{loc}	0.05	0.05	0.05	0.05	0.05

order statistics, for both the small $N = 1,024$ and large $N = 4,096$ sample sizes, especially for large multifractal spectrum widths $h_{\max} - h_{\min} = 0.5$ and 0.6 . However, the red traces in Fig. 6 for GMWP and DFA_{loc} are comparable to the red monofractal traces in Fig. 5 and indicate that the superior performance of GMWP and DFA_{loc} is related to their overestimation of the monofractal spectrum width.

The estimation of the central tendency $\bar{h}$ of D_h was quite robust to the moderate signal-to-noise ratio of 0.33, but the MFDFA, LWT, and entropy analyses showed less consistency across the different H of $\Delta B_H(A(t))$, as compared with GMWP and DFA_{loc}, for the large signal-to-noise ratio of 1 (see red traces in Fig. 6 and comparison of MFDFA_{emd} and GMWP in the upper part of Table 7). GMWP was the most

Fig. 6 Mean D_h and ± 1 SD (bars) estimated by the seven multifractal analyses in Table 1 for the ensemble of $\Delta B_H(A(t))$ with white noise. The estimated D_h represented for no additive noise (blue bars), a signal-to-

noise ratio of 0.33 (green bars), and a signal-to-noise ratio of 1 (red bars). The parameter settings for $\Delta B_H(A(t))$ are $h_{\max} - h_{\min} = 0.6$ and $H = 0.9$, and the sample size is 4,096

Table 5 Mean error E_D for six different multifractal spectrum widths $h_{\max}-h_{\min}$ for $\Delta B_H(A(t))$ with $H = 0.5$ and with signal-to-noise ratio = 0.33, but with no trends present

	0	0.1	0.2	$h_{\max}-h_{\min}$ 0.3	0.4	0.5	0.6
N = 1024, Signal-to-noise = 0.33							
MFDFA	0.09*	0.09	0.09	0.09	0.09	0.11	0.14
MFDFA _{ma}	0.09*	0.08*	0.09	0.08	0.10	0.12	0.14
MFDFA _{emd}	0.11	0.11	0.12	0.11	0.10	0.12	0.13
Entropy	0.11	0.11	0.10	0.07	0.06	0.07	0.11
LWT	0.12	0.12	0.10	0.09	0.07	0.07	0.11
GMWP	0.20†	0.19†	0.16†	0.12	0.07	0.03*	0.05*
DFA _{loc}	0.20†	0.19†	0.16†	0.12	0.07	0.03*	0.05*
N = 4096, Signal-to-noise = 0.33							
MFDFA	0.05	0.04	0.04	0.05	0.07	0.12	0.15
MFDFA _{ma}	0.05	0.03*	0.04	0.06	0.08	0.12	0.14
MFDFA _{emd}	0.05	0.04	0.04	0.05	0.07	0.10	0.12
Entropy	0.08	0.06	0.04	0.03	0.04	0.08	0.13
LWT	0.10	0.08	0.06	0.04	0.05	0.09	0.14
GMWP	0.20†	0.17†	0.14†	0.11†	0.06	0.03*	0.05*
DFA _{loc}	0.20†	0.17†	0.14†	0.11†	0.06	0.03*	0.05*

†, * represents a significantly larger and smaller mean error E_D (paired t -test; $p < 0.05$)

robust analysis to the presence of additive white noise when estimating the central tendency $\bar{h}$ of D_h .

Influence of additive multiharmonic trends

Figure 7 shows the mean D_h and ± 1 SD (see bars) estimated by the seven multifractal analyses in Table 1 for the ensemble of $\Delta B_H(A(t))$ with additive multiharmonic trends. The blue, green, and red markers in Fig. 6 are the estimated D_h for $\Delta B_H(A(t))$ with signal-to-trend ratios of 0, 0.33, and 1, respectively. Comparison of blue, green, and red markers in Fig. 7 indicates that the increasing magnitude of multiharmonic trends truncates the estimated multifractal spectrum D_h and translates its central tendency towards $\bar{h} = 1$.

Most analyses were quite robust to the moderate signal-to-trend ratio of 0.33, as is indicated by less truncation in the green spectra, as compared with the red spectra, in Fig. 7. MFDFA_{lin} and MFDFA_{ma} had the best performance when estimating the multifractal spectra with narrow spectrum widths (see stars in Table 8), but the performance declined for the large spectrum width by a truncation of the right side of D_h (compare blue and green spectra in Fig. 7). GMWP and DFA_{loc} had the best performance when estimating multifractal spectra with wider spectrum widths (see stars in Table 8 and the lack of truncation in the green spectra in Fig. 7). However, entropy analysis had the most consistent performance across the range of multifractal spectrum widths, especially for the larger sample size of 4,096. The performance declined for the MFDFA, LWT, and entropy

Table 6 Mean error E_D for seven different multifractal spectrum widths $h_{\max}-h_{\min}$ for $\Delta B_H(A(t))$ with $H = 0.5$ and with signal-to-noise ratio = 1 (upper part) and signal-to-trend ratio = 1 (lower part)

	0	0.1	0.2	$h_{\max}-h_{\min}$ 0.3	0.4	0.5	0.6
N = 1024, Signal-to-Noise = 1							
MFDFA	0.09	0.05	0.07	0.13	0.19	0.26	0.32
MFDFA _{ma}	0.08*	0.05	0.08	0.13	0.19	0.26	0.33
MFDFA _{emd}	0.10	0.06	0.06	0.11	0.17	0.23	0.30
Entropy	0.11	0.05	0.05	0.11	0.18	0.25	0.32
LWT	0.12	0.05	0.04	0.10	0.18	0.24	0.29
GMWP	0.20†	0.13†	0.07	0.03*	0.07*	0.14*	0.20*
DFA _{loc}	0.20†	0.13†	0.07	0.03*	0.07*	0.14*	0.20*
N = 1024, Signal-to-Trend = 1							
MFDFA	0.14	0.05	0.05	0.13	0.21	0.28	0.36
MFDFA _{ma}	0.10	0.02*	0.09	0.17	0.25	0.32	0.40†
MFDFA _{emd}	0.22	0.14	0.24†	0.29†	0.35†	0.31	0.29
Entropy	0.11	0.06	0.05	0.05*	0.09	0.15	0.21
LWT	0.09*	0.07	0.05	0.07	0.12	0.16	0.21
GMWP	0.23	0.19†	0.15	0.09	0.04*	0.04*	0.09*
DFA _{loc}	0.23	0.19†	0.15	0.09	0.04*	0.04*	0.09*

†, * represents a significantly larger and smaller mean error E_D (paired t -test; $p < 0.05$)

Table 7 Comparison of the error $E_{\bar{h}}$ of the position of the central tendency of MFDFA_{emd} and GMWP for $\Delta B_H(A(t))$ with sample size 4,096 for five different H s and two different signal-to-noise ratios (upper part) and two different signal-to-trend ratios (lower part)

	$H = 0.5$	$H = 0.6$	$H = 0.7$	$H = 0.8$	$H = 0.9$
Signal-to-noise = 0					
MFDFA _{emd}	0.04	0.04	0.04	0.04	0.04
GMWP	0.01	0.01	0.01	0.02	0.02
Signal-to-noise = 0.33					
MFDFA _{emd}	0.03	0.04	0.04	0.05	0.06
GMWP	0.02	0.02	0.03	0.04	0.05
Signal-to-noise = 1					
MFDFA _{emd}	0.06	0.09	0.13	0.16	0.19
GMWP	0.06	0.03	0.01	0.01	0.03
Signal-to-trend = 0.33					
MFDFA _{emd}	0.29	0.22	0.16	0.10	0.07
GMWP	0.34	0.24	0.14	0.05	0.02
Signal-to-trend = 1					
MFDFA _{emd}	0.44	0.36	0.29	0.20	0.12
GMWP	0.41	0.31	0.22	0.15	0.10

analyses for multiharmonic trends with the larger signal-to-trend ratio of 1. In this case, the MFDFA, LWT, and entropy analyses underestimated the multifractal spectrum width, especially for the larger spectra indicated by a larger estimation error for $h_{\max} - h_{\min} = 0.5$ and 0.6 in the lower part of Table 6. This increased estimation error was caused by a truncation of the right side (i.e., negative q s) of the multifractal

spectra (see red spectra in Fig. 7). MFDFA_{emd} had an improved performance, as compared with the other analyses based on q -order statistics, for $H = 0.5$ (see red trace in Fig. 7), but this result was not consistent across other H s. In contrast, GMWP and DFA_{loc} were quite robust to the presence of large multiharmonic trends across H s because of a narrow scaling range close to the limit $s \rightarrow 0$ (see the scaling range in Table 1).

Fig. 7 Mean D_h and ± 1 SD (bars) estimated by the seven multifractal analyses in Table 1 for the ensemble of $\Delta B_H(A(t))$ with multiharmonic trends. The estimated D_h represented for no multiharmonic trend (blue

bars), a signal-to-trend ratio of 0.33 (green bars), and a signal-to-trend ratio of 1 (red bars). The parameter settings for $\Delta B_H(A(t))$ are $h_{\max} - h_{\min} = 0.6$ and $H = 0.9$, and the sample size is 4,096

Table 8 Mean error E_D for six different multifractal spectrum widths $h_{\max}-h_{\min}$ for $\Delta B_H(A(t))$ with $H = 0.5$ and with signal-to-trend ratio = 0.33, but with no noise present

	0	0.1	0.2	$h_{\max}-h_{\min}$		0.5	0.6
				0.3	0.4		
N = 1024, Signal-to-trend = 0.33							
MF DFA	0.08*	0.07*	0.09	0.08	0.09	0.11	0.15
MF DFA _{ma}	0.11	0.09	0.07*	0.07*	0.09	0.13	0.17
MF DFA _{emd}	0.11	0.17	0.21	0.22†	0.22†	0.18†	0.19
Entropy	0.11	0.12	0.13	0.10	0.08	0.06	0.06
LWT	0.12	0.13	0.16	0.14	0.11	0.09	0.08
GMWP	0.22†	0.22†	0.20	0.14	0.09	0.04*	0.04*
DFA _{loc}	0.22†	0.22†	0.20	0.14	0.09	0.04*	0.04*
N = 4096, Signal-to-trend = 0.33							
MF DFA	0.09	0.05*	0.02*	0.05	0.08	0.13†	0.16
MF DFA _{ma}	0.11	0.05*	0.02*	0.06	0.09	0.14†	0.19†
MF DFA _{emd}	0.05*	0.09	0.11	0.13†	0.11†	0.11	0.10
Entropy	0.08	0.07	0.06	0.04*	0.03	0.04	0.08
LWT	0.10	0.10	0.10	0.09	0.05	0.04	0.05*
GMWP	0.20†	0.17†	0.13†	0.08	0.02*	0.04	0.09
DFA _{loc}	0.20†	0.17†	0.13†	0.08	0.02*	0.04	0.09

†, * represents a significantly larger and smaller mean error E_D (paired t -test; $p < 0.05$)

The central tendencies of the multifractal spectra were highly sensitive to the presence of multiharmonic trends with the low and high signal-to-trend ratios. The central tendency $\bar{h}$ for all H of $\Delta B_H(A(t))$ was estimated close to $\bar{h} = 1$, as indicated by large estimation errors $E_{\bar{h}}$ in the lower part of Table 7 for $H = 0.5$ and 0.6 (see also the top of the red spectra in Fig. 7). Thus, multifractal analyses were not able to detect differences in the central tendency $\bar{h}$ when strong multiharmonic trends were present. Furthermore, the detrending procedures of both MFDFAs and wavelet-based analyses were not able to remove the influence of the multiharmonic trends. As an example, MF DFA_{emd}, which contains the most adaptive detrending procedure, performed less precisely and less consistently than MF DFA with other detrending procedures when multiharmonic trends were present for the small sample size of $N = 1,024$ (see crucifix in the upper part of Table 8 and lower part of Table 7).

Discussion

The purpose of the present study was to compare the performance of seven multifractal analyses to provide a guide for the choice of multifractal analyses of response time series and other behavioral variables. In the discussion, the present comparative study of multifractal analyses is compared with the results in previous comparative studies. In addition, the implications of the sensitivity of the multifractal analyses to additive noise and multiharmonic trends are discussed in the last subsection.

MF DFA_{lin} versus MF DFA_{ma} and MF DFA_{emd}

The performance of MF DFA was found in previous studies to be dependent on the detrending procedures. MF DFA_{ma}

and MF DFA_{emd} have been shown to outperform the original MF DFA_{lin} when employed to a simple binominal cascade (Gu & Zhou, 2010; Manimaran, Panigrahi, & Parikh, 2009; Qian, Zhou, & Gu, 2009). In the present study, MF DFA_{ma} and MF DFA_{emd} had similar or larger estimation errors, as compared with the original MF DFA_{lin}. The deviation of the performance of the detrending procedures in the present study, as compared with previous studies, is probably caused by smaller sample sizes and a more advanced multiplicative cascading model used in the present study.

The poor performance of the detrending procedures of the MF DFA and wavelet-based analyses indicates that the detrending procedure has to be exactly matched to the shape of a complex multiharmonic trend in order for these procedures to remove the influence of trends on the estimated D_h . The exact tuning of detrending procedures will be extremely difficult in multifractal analyses of response time series and other behavioral variables where a priori theoretical argumentations for a particular shape of a complex trend do not exist. Thus, the comparison of the performances of the original MF DFA_{lin} versus MF DFA_{ma} and MF DFA_{emd} suggests that the original MF DFA_{lin} should be used unless theoretical argumentation exists for the exact shape of the trend.

LWT versus MF DFA

Serrano and Figliola (2009) concluded that LWT gave an improved estimation of D_h , as compared with MF DFA, when employed to binominal and random wavelet cascades. In contrast, X. Y. Huang et al. (2011) found that MF DFA performed better than LWT considering only positive q s when employed on canonical Mandelbrot cascades. The present study confirms the findings by X. Y. Huang et al. that MF DFA had better

performance than LWT for $\Delta B_H(A(t))$ with $H = 0.5$. However, LWT had a more consistent error E_D than the MFDFA and entropy analyses across the range of H s of $\Delta B_H(A(t))$ and had significantly better performance for $H = 0.8$ and 0.9 . Furthermore, LWT seems to have a more consistent performance, as compared with both the MFDFA and entropy analyses, when estimating the left side (i.e., positive q s) of the multifractal spectrum D_h . Thus, LWT is a good choice of multifractal analyses of response time series, especially for long sample sizes of 4,096 and when positive q s are considered.

Entropy analysis versus MFDFA and LWT

Entropy analysis has recently been suggested as a superior estimation procedure of D_h of response time series, as compared with the multifractal analyses based on the Legendre transformation in Eq. 6 (Stephen & Dixon, 2011). This suggestion is based on the original argumentation by Chhabra and Jensen (1989) in their development of entropy analysis. However, Chhabra and Jensen compared entropy analysis with a basic histogram analysis based on Eqs. 8 and 9. The present study shows only minor differences between the entropy analysis of Chhabra and Jensen and MFDFA and LWT. Furthermore, the probability measure used in entropy analysis (i.e., Eq. 2) can be constructed by dividing other choices of scale-dependent measure $\mu_{s,v}$ by the total sum of μ_{s,t_0} . Substituting the original scale-dependent measure in entropy analysis (i.e., Eq. 7a) with these probability measures will yield the exact same D_h as the Legendre transformation of ζ_q in Eq. 6. Thus, the performance of multifractal analyses with similar $\mu_{s,v}$ is insensitive to the choice of estimating D_h directly by entropy functions or estimating D_h indirectly by the Legendre transformation of ζ_q .

GMWP and DFA_{loc} versus MFDFA, LWT, and entropy analysis

The benchmark test performed by Turiel et al. (2008) concluded superior performance by GMWP, especially for multifractal signals with large spectrum width $h_{\max} - h_{\min}$ and small sample sizes (i.e., $N = 1,024$). The present study also found that both GMWP and DFA_{loc} performed better than the multifractal analyses based on q -order statistics (i.e., MFDFA, LWT, and entropy analysis) for the larger spectrum width. However, GMWP and DFA_{loc} both had a larger overestimation of the spectrum width of monofractal signals, as compared with MFDFA, LWT, and entropy analysis. A shortcoming of the benchmark test conducted by Turiel et al. (2008) was the inclusion of other multifractal analyses based on q -order statistics, which have inferior performance, as compared with analyses included in the present study (Jaffard, Lashermes, & Abry, 2006; Oświęcimka, Kwapien, & Drozd, 2006; Serrano &

Figliola, 2009). Another shortcoming of the benchmark test conducted by Turiel et al. (2008) was the use of random cascades that are not appropriate candidates to mimic the intermittent variation in behavioral variables by their scale-discrete nature and strictly positive values. Furthermore, Pont, Turiel, and Perez-Vicente (2006) have shown that the GMWP overestimates the spectrum widths for fractional Gaussian noise (i.e., $\Delta B_H(A(t))$ with $h_{\max} - h_{\min} = 0$) confirmed in the present result. However, the same authors showed that the performance of both GMWP and DFA_{loc} can be improved by modifications of the scale-dependent measure μ_{s,t_0} when employed to monofractal $\Delta B_H(A(t))$. In summary, the present study is less conclusive regarding the performance of both GMWP and DFA_{loc}, as compared with the benchmark test conducted by Turiel et al. (2008). The obtained D_h from GMWP and DFA_{loc} of response time series should be compared with D_h of monofractal surrogate series to prevent false positive presence of multifractality.

Sensitivity to additive noise and multiharmonic trends

The presence of both trends and noise yields a false negative result for the presence of multifractal response time variation, a false positive result for the presence of long-range trial dependency when trends dominate, and a false negative result for long-range trial dependency when noise dominates. The sensitivity of multifractal analyses to noise and trends has two important implications for the employment of multifractal analyses to response time series and the interpretation of the estimated D_h .

First, a narrow D_h reflects the lack of a multiplicative interaction or phase coupling between temporal scales of response time series when no noise or trends are present. Ihlen and Vereijken (2010) reported multifractal spectra with smaller widths and central tendencies closer to 1 for the response series of the interval estimation task, as compared with other cognitive tasks. Furthermore, Kuznetsov and Wallot (2011) reported that the local singularity exponent h_t of a response series in an interval estimation task had less variation (i.e., smaller spectrum width) and an average value (i.e., central tendency) closer to 0.5 in the presence of performance feedback. The sensitivity of multifractal analyses to noise and trends suggests that both these results can be created by task- and feedback-dependent modulation of additive noise and trends. Thus, noise and trends will disguise the task and feedback modulation of multifractal spectra caused by modulation of the intrinsic multiplicative interactions between temporal scales. However, both the Ihlen and Vereijken (2010) and Kuznetsov and Wallot studies found significant differences between the multifractal spectrum of the response time series and multifractal spectra of synthesis surrogate series that eliminate multiplicative interactions between the temporal scales within the response time series. Furthermore, Kuznetsov and Wallot found no correlation between the

Table 9 Summary of all pros and cons of the seven multifractal analyses in Table 1

	Pros	Cons
	Estimation of D_h indirectly from the q -order statistics of $\mu_{s,v}$	
MF DFA	<ul style="list-style-type: none"> • Superior performance for $H \rightarrow 0.5$ for small sample size • Polynomial detrending 	<ul style="list-style-type: none"> • Inferior performance when $H \rightarrow 1$ for small sample size • Inconsistent performance across H for small sample size • Trend and noise sensitive
MF DFA _{ma}	<ul style="list-style-type: none"> • Superior performance for $H \rightarrow 0.5$ for small sample size • Moving average detrending 	<ul style="list-style-type: none"> • Inferior performance when $H \rightarrow 1$ for small sample size • Inconsistent performance across H for small sample size • Trend and noise sensitive
MF DFA _{emd}	<ul style="list-style-type: none"> • Consistent performance with the other MF DFA for large sample size • Detrending by empirical mode decomposition • Superior detrending for $H \rightarrow 0.5$ 	<ul style="list-style-type: none"> • Inferior performance for small sample size • Inconsistent performance across H for small sample size • Trend and noise sensitive
Entropy	<ul style="list-style-type: none"> • Superior performance for large sample size • Consistent performance across H 	<ul style="list-style-type: none"> • Inferior performance for $H \rightarrow 0.5$ for small sample size • No detrending • Trend and noise sensitive
LWT	<ul style="list-style-type: none"> • Superior performance for large sample size • Consistent performance across H • More consistent estimation error for positive q's • Wavelet detrending 	<ul style="list-style-type: none"> • Inferior performance for $H \rightarrow 0.5$ for small sample size • Trend and noise sensitive • Inferior estimation of the central tendency of D_h
	Estimation of D_h directly from μ_{s,t_0}	
GMWP	<ul style="list-style-type: none"> • Superior estimation of D_h with large widths • Superior estimation of the central tendency of D_h • Wavelet detrending • Relatively trend resistant 	<ul style="list-style-type: none"> • Overestimation of D_h widths for monofractal signals • Inconsistent performance across H for small sample size • Less consistent estimation error of D_h and central tendency of D_h • Noise sensitive
DFA _{loc}	<ul style="list-style-type: none"> • Superior estimation of D_h with large widths • Consistent estimation error for large sample size • More consistent estimation error of central tendency of D_h • Polynomial detrending • Relatively trend resistant 	<ul style="list-style-type: none"> • Overestimation of D_h widths for monofractal signals • Inconsistent performance across H for small sample size • Noise sensitive

multifractal spectrum width and the central tendency of the spectrum that would have been expected in the presence of a high level of noise or strong trends. In summary, surrogate tests and correlation analyses of multifractal spectrum widths and the central tendencies should be conducted together with the multifractal analyses to control for the presence of additive noise and trends.

Second, the presence of noise and trends leads to a degeneration of D_h and deviation from linearity in the log-scaling plot of the q -order statistics for the multifractal analyses based on the q -order statistics. The degeneration of D_h appeared in the right end of D_h and was caused by a nonmonotonic decreasing h_q for negative q s (see red trace estimated by LWT in Fig. 7). The deviation from linearity in the log-scaling plot was

also most apparent for negative q s and suggests that noise and trends with large magnitudes will influence the scale invariance of response time series, especially for the intermittent periods with response time variation of small magnitudes. Consequently, the employment of multifractal analyses based on q -order statistics should interpret the right end of the estimated D_h with caution when noise and trends are expected.

Conclusion

Table 9 summarizes the pros and cons of the seven multifractal analyses compared in the present study. All multifractal analyses were able to differentiate between

monofractal and multifractal time series, but GMWP and DFA_{loc} showed a larger overestimation of the multifractal spectrum size for the monofractal time series. However, GMWP and DFA_{loc} were more robust to multiharmonic trends of large magnitudes, and GMWP had the most precise estimation of the central tendency of the multifractal spectrum. MFDFA_{lin} and MFDFA_{ma} had a superior performance for $H = 0.5$, as compared with both LWT and entropy analysis. However, the LWT and entropy analyses had more consistent performances across H and had superior performances, as compared with MFDFA, for $H = 0.9$. The detrending procedures of MFDFA and the wavelet-based analyses were unable to remove the influence of a multiharmonic trend on the estimated multifractal spectra. The present study concludes that the estimated multifractal spectrum from two or more multifractal analyses should be compared with response time series and other behavioral data when these analyses are employed. In addition, the estimated multifractal spectra should be compared with the estimated spectra from monofractal surrogate series to reveal a false positive presence of multifractality.

Acknowledgement The multiplicative cascading noise $\Delta B_{\text{eff}}(A(t))$ are synthesized by MATLAB functions available at <http://www.isima.fr/~pchainai/PUB/software.html>. The EMD is available at <http://perso.ens-lyon.fr/patrick.flandrin/software2.html>. The MATLAB codes of the multifractal analyses in Table 1 are available at www.ntnu.edu/inm/geri/software. The MATLAB codes for LWT are developed by Wendt (2008) and are available at <http://perso.ens-lyon.fr/herwig.wendt/index.php?page=codes>.

Appendix 1 Detrending procedures in MFDFA

Linear detrending (MFDFA_{lin})

The scale-dependent measure $\mu_{s,v}$ of the original MFDFA_{lin} is defined by the following equation (Kantelhardt et al., 2002):

$$\mu_{s,v} = \sqrt{\frac{1}{s} \sum_{t=(v-1)s+1}^{vs} [y_t - \hat{y}_{t,m}]^2} \quad (13)$$

The integrated profile $y_t = \sum_{i=1}^t (x_i - \bar{x})$ of the response time series x_i is divided into $N_s = N/s$ nonoverlapping segments with segment number $v = 1, \dots, N_s$ where each segment contains the response times of s trials. The polynomial trend of order m is least square fitted to y_t within each

segment v and was chosen to be $m = 1$ (i.e., linear) in the present study.

Moving average detrending (MFDFA_{ma})

An alternative way to provide a scale-dependent detrending in the computation of $\mu_{s,v}$ is to use a moving average detrending procedure (Gu & Zhou, 2010):

$$\mu_{s,v} = \sqrt{\frac{1}{s} \sum_{t=(v-1)s+1}^{vs} [y_t - \hat{y}_{s,t}]^2} \quad (14)$$

where the scale-dependent trend $\hat{y}_{s,t}$ is the moving average of y_t with a floating trial interval $[t - s/2, t + s/2]$ that contains the response time of s trials:

$$\hat{y}_{s,t} = \frac{1}{s} \sum_{i=t-s/2}^{t+s/2} y_i \quad (15)$$

Empirical mode decomposition detrending (MFDFA_{emd})

The scale-dependent trend $\hat{y}_{s,t}$ in Eq. 13 can also be defined by the intrinsic mode functions $d_{s,t}$ defined by empirical mode decomposition (EMD) of y_t (N. E. Huang et al., 1998):

$$\hat{y}_{s,t} = \sum_{[s]} d_{s,t} \quad (16)$$

where $[s]$ indicates that the sum is taken over all intrinsic mode functions $d_{s,t}$ with a scale larger than s . The scale of each $d_{s,t}$ was defined by the inverse of their mean instant frequency $f_{s,t}$ (i.e., $s = 1/\langle f_{s,t} \rangle$), where $f_{s,t}$ was computed by differentiating the phase angle of the Hilbert transform of $d_{s,t}$ (i.e., $f_{s,t} = d\theta_{s,t}/dt$) by an algorithm suggested by Boashash (1992). It is important to notify that this detrending procedure defines an EMD modification of the MFDFA suggested by Manimiran et al. (2009) and is, therefore, different from the EMD modification of MFDFA developed by Qian et al. (2009). MATLAB codes for MFDFA_{lin} and MFDFA_{emd} are available at www.ntnu.edu/inm/geri/software, whereas the MATLAB code for MFDFA_{ma} is available in the Appendix in Gu and Zhou (2010).

Appendix 2 Estimation of the local singularity exponent h_t

The local singularity exponent was estimated by a method developed by Struzik (2000). The method assumes that the time series is generated by a multiplicative cascading process where the temporal variation of the scale-dependent measure μ_{s,t_0} decays with increasing scale s . The local singularity exponent h_{t_0} in Eq. 8 is estimated by the coarse-grained singularity exponents h_{s,t_0} in the limit $s \rightarrow 0$. The coarse-grained singularity exponent is defined by the following equation (Ihlen, 2012; Struzik, 2000):

$$h_{s,t_0} = \frac{\log(\mu_{s,t_0}) - (\bar{h} \log(s) + C)}{\log(s) - \log(s_{\max})} + \bar{h} \quad s \rightarrow 0 \quad (17)$$

$\bar{h} \log(s) + C$ is the regression line at scale s with the slope $\bar{h}$ and estimated by a q -order based analysis for $q = 0$. The scale-dependent measure μ_{s,t_0} was defined by Eq. 3 for GMWP and by Eq. 4 for DFA_{loc}. The maximum scale $s_{\max}$ was set to the length of the time series (Struzik, 2000). The MATLAB codes for both GMWP and DFA_{loc} are available at www.ntnu.edu/inm/geri/software.

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