Notes and Comment

An application of the gradient space representation

R. A. CLEMENT

Aston University, Birmingham, England

Many examples of outline drawings, such as the Necker cube and Mach book figures, contain no classical depth cues but are seen in depth. In order to investigate the apparent shape of these figures, it is helpful to have a method of describing the apparent orientations in depth of the surfaces of the figures. A suitable method is provided by the gradient space representation, in which the orientation of a planar surface in space is represented by a single point in gradient space. The gradient space representation has been reviewed by Marr (1982). The utility of the gradient space approach can be appreciated by considering alternative hypotheses concerning the apparent shape of the Mach book figure.

Mach (1866/1959) himself proposed two hypotheses concerning the apparent shape of the figure, namely that "all angles exhibit the tendency to become right angles" and that each half of the figure appears to be rotated about its long diagonal in the picture. The set of all possible gradients where the planar surfaces of the apparent shape of the Mach book are rectangular and project orthographically into the two-dimensional outline of the figure determine a locus of points in gradient space, as shown for various picture angles in Figure 1. Details of the calculations of these curves are given in the Appendix; for clarity, only the gradient of the right half of the Mach book is shown. The location of the long diagonal is determined by the length ratio of the sides of the Mach book; to investigate the loci in gradient space determined by Mach's hypotheses, the length ratio of the sides of the Mach book was held fixed and the picture angle was varied. In Figure 1, the loci with length ratios of 3:1, 1:1, and 1:3 are shown as dashed lines. The procedure used to calculate these curves is described in the Appendix. For a given picture angle and length ratio, there are two points in gradient space such that the right half of the Mach book appears both rectangular and rotated about its long diagonal in the picture. The book appears to have a convex or concave appearance, according to which point is selected.

Alternative quantitative hypotheses concerning the shape of outline drawings come from computational studies that exploit the constraint that planar surfaces must intersect along the appropriate edges in space. Mackworth (1973) implemented an approach to the interpretation of

line drawings of polyhedra in which the gradient of one surface is guessed at and the geometric constraints at the edges are used to determine the possible gradients of the other surfaces. Perkins (1976) criticized this scheme with respect to human vision because it does not lead to apparent shapes with symmetry or a preponderance of right angles, and Mackworth (1976) proposed modifying his scheme so that objects are assumed to be rectangular whenever possible.

If it is assumed that all the angles in the Mach book, including those between the two halves of the book, are seen as right angles, then the predicted gradients correspond to the chained line in Figure 1. The procedure used to calculate the points on this line is described in the Appendix. The intersection of the line with the $\partial z/\partial x$ axis occurs in the limiting case of a picture angle of 90° when the spine of the book is vertical and each half of the Mach book is at an angle of 45° to the picture plane. The drawback of this hypothesis is that it is not possible for all three angles to be right angles if the picture angle is less than 45° , and Perkins (1972) has shown that human observers are adept at discerning this geometric constraint.

Another set of predictions can be derived from the more recent work of Kanade (1981), who made use of the concept of skewed symmetry, which arises when the points on a two-dimensional projection are symmetrical about a line at a fixed angle to the axis of symmetry of a three-dimensional shape. Kanade implemented a scheme which exploits the fact that a skewed symmetry is produced when a real symmetry is viewed from an oblique direction. In the case of rectangular shapes, the gradient selected according to his scheme is identical to that selected with a 1:1 length ratio according to Mach's hypotheses. This set of gradients has the property that relative lengths in the apparent shape are equal to the corresponding relative lengths in the picture. These gradients also have the property that the rectangular surfaces associated with them have the least possible slant, with respect to the picture plane.

The main difference between the computational theories and Mach's hypotheses is that the predicted orientations according to the computational theories depend only on the picture angles, whereas Mach's predictions depend on the lengths of the sides of the figure as well. In view of the experimental findings of Attneave and Frost (1969), who discovered that changes in the lengths of the sides of a projection of a parallelepiped altered the apparent tilt of the figure, it seems likely that the lengths of the sides of the figure will affect the appearance of the figure.

Representation of the alternative theories in gradient space highlights just how remarkable it is that only one convex and one concave version of the figure is seen. Even

The author's mailing address is: Department of Vision Sciences, Aston University, Aston Triangle, Birmingham B4 7ET, England.

Figure 1. Gradient space representations of the hypothesized orientations of the three-dimensional shape of the Mach book. The continuous lines constitute the loci where the three-dimensional shape is rectangular, for the specified picture angles. The dashed lines constitute the loci that satisfy Mach's hypotheses, for the specified ratios of the length of the vertical arm to the length of the lateral arm ($v:l$) of the picture angle. The chained line constitutes the locus where the three-dimensional shape is rectangular and the angle between the two halves of the shape is a right angle.

with the constraint that all angles should appear to be right angles, there is still an infinite number of possible apparent shapes, and an additional constraint must be operating. It is also interesting that the figure is often described as appearing to lie in the plane of the picture. Inspection of the continuous lines in Figure 1 reveals that it is not geometrically possible for the spine of the book to lie in the picture plane and for all the angles to appear as right angles, except when the picture angle is 90° .

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APPENDIX

1. *Derivation of the right-angle curves.* Consider a Cartesian system of axes centered on the lowest point of the Mach book, with the x axis horizontal, the y axis vertical, and the positive direction of the z axis directed away from the observer. Let A and B be vectors in the directions of the vertical and lateral arms of the acute angle located at the origin of a system of coordinates. If the actual angle between the arms is a right angle, then $A \cdot B = 0$. If it is further assumed that both $|A_x + A_y| = 1$ and $|B_x + B_y| = 1$, then, for a given picture angle p , this expression can be rewritten as:

$$\cos(p) + A_x B_x = 0. \quad (1)$$

The iso-angle curves can be obtained from this equation by treating A_x as a parameter and solving for B_x . The normal N to the plane of the arms of the angle is given by the vector product:

$$\begin{aligned}
 N = A \wedge B &= (0, 1, A_z) \wedge [\sin(p), \cos(p), B_z] \\
 &= [B_z - \cos(p)A_z, \sin(p)A_z, -\sin(p)], \quad (2)
 \end{aligned}$$

and the corresponding coordinates in gradient space are given by $(N_x/N_z, N_y/N_z)$.

2. *Derivation of the Mach curves.* A vector in the direction of the long diagonal of the right half of the Mach book is given by $R = [l \sin(p), l \cos(p) + v, 0]$, where l and v are the lengths of the lateral and vertical arms, respectively. If the figure appears to be rotated around this axis, then the normal N , defined in part 1 of the appendix, must obey the relation $R \cdot N = 0$, that is, $N_x = -N_y R_y / R_x$. By substituting the expressions for N_x and N_y given in Equation 2, one obtains the equation:

$$B_z = A_z [\cos(p) - \sin(p) R_y / R_x], \quad (3)$$

and substituting for B_z in Equation 1 gives the equation:

$$A_z^2 = -1/[1 - \tan(p) R_y / R_x]. \quad (4)$$

This equation can be solved for A_z , which can be used to calculate B_z from Equation 3, and these values can be used to calculate N from Equation 2.

3. *Derivation of the cubic corner curves.* Let B be the vector defined in part 1 of the appendix, with $|B_x + B_y| = 1$ as before; a vector in the direction of the other lateral arm at the base of the Mach book is then given by $(-B_x, B_y, B_z)$. If the angle between these two vectors is 90° , one obtains the constraint:

$$\cos(2p) = -B_z^2. \quad (5)$$

The value for B_z obtained from this equation can be substituted in Equation 1 to obtain the value of A_z , and these values can be used together to obtain N from Equation 2.

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