Notes and Comment

The importance of the convex hull for human performance on the traveling salesman problem: A comment on MacGregor and Ormerod (1996)

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MacGregor and Ormerod (1996) have presented results purporting to show that human performance on visually presented traveling salesman problems, as indexed by a measure of response uncertainty, is strongly determined by the number of points in the stimulus array falling inside the convex hull, as distinct from the total number of points. It is argued that this conclusion is artifactually determined by their constrained procedure for stimulus construction, and, even if true, would be limited to arrays with fewer than around 50 points.

MacGregor and Ormerod (1996) present a study of human performance on the notorious combinatorial optimization problem known as the *traveling salesman problem* (TSP). The general TSP may be posed as follows: Given a set of cities, and a specified cost incurred in moving between any given pair of cities, devise an itinerary, starting and finishing at the same city, that will permit one to visit each city once only and will minimize the total accumulated cost. Although the problem can be stated succinctly and represented simply, it is one of an important class of so-called NP-complete problems, for which it is believed that there is no algorithm that can be guaranteed to arrive at an optimal solution within a reasonable (polynomial) time (Lawler, 1985; Wilf, 1986).

In the version of the TSP examined by MacGregor and Ormerod, the cities are represented as points on a flat two-dimensional page, and the cost is equated with the standard (Euclidean) distance between pairs of points. In this form, MacGregor and Ormerod raise the possibility that the TSP “may be fundamentally a perceptual problem-

solving task,” in which human subjects “may be capable of achieving reasonable solutions with minimal cognitive load” (1996, p. 528). Along with the earlier studies of Polivanova (1974), their work is an important first step toward an understanding of this achievement.

To attain such an understanding, researchers will need to relate measures of subjects’ performance in reaching solutions with objective measures of problem difficulty. For example, an obvious measure of performance is the total length of the path, or circuit, delineated by the subject, since this is the variable that the subject is explicitly instructed to minimize. In addition, MacGregor and Ormerod consider a number of other measures. In particular, they suggest that a useful (performance) measure of the “problem complexity” of different TSP configurations is provided by an information-theoretic measure of the uncertainty in subjects’ responses. This is a succinct summary of all the individual subjects’ solutions, based on the probabilities (over subjects) of each possible pairwise connection between the points in an array. (Although MacGregor and Ormerod initially refer to this measure as *problem complexity*, this term suggests that what is being measured is an independent, rather than a dependent, variable. A better term, therefore, might be *response uncertainty*, which is used by MacGregor and Ormerod later.)

With respect to objective measures of problem difficulty, the most obvious candidate is the total number of points in the stimulus array, since this determines the combinatorial complexity of the problem. In addition, MacGregor and Ormerod distinguish between the points that fall on the boundary of the convex hull containing the array and the interior, or nonboundary, points that fall inside it. Further, they point to the demonstration by Flood (1956) that any optimal solution to a Euclidean TSP must sequentially connect adjacent points on the convex hull of the configuration, even though it passes through interior points in the process. If human performance makes use of this principle, they go on to argue, there should be no uncertainty about the order in which to connect the boundary points, and performance on TSPs should be a function of the number of nonboundary points, rather than the total number of points.

MacGregor and Ormerod report two experiments in which they examined human performance on a number of 10- and 20-point problems that appear to confirm this last hypothesis. These show that response uncertainty increased as a function of the number of nonboundary points, leading the authors to conclude that “the number of interior points appears to be a relatively powerful determinant of problem complexity” (1996, p. 535).

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Limitations of MacGregor and Ormerod's (1996) Study

There are, however, some significant limitations in MacGregor and Ormerod's (1996) study, and one important qualification to the conclusion above. The limitations stem from the highly constrained procedure for constructing the stimulus configurations, which is biased toward supporting the hypothesis being evaluated. The qualification has to do with the asymptotic behavior of boundary points on convex hulls. We will discuss each in turn.

Constraints on the configurations. The validity of the conclusions drawn by MacGregor and Ormerod may be challenged on the basis of the constraints used in the construction of their stimulus configurations. The 10-point configurations were constructed by evenly spacing a given number of boundary points on a circle of radius 80 mm, then displacing each point along this circle by a random angle (presumably uniformly chosen) within $\pm 5^\circ$. The remaining nonboundary points were then randomly placed within a concentric circle of radius 40 mm. The 20-point configurations were constructed similarly, except that the nonboundary points were randomly placed within a concentric torus having inner radius of 10 mm and outer radius of 50 mm.

This method of construction imposes severe restrictions on the range of configurations that can be generated, and it introduces considerable structure into the stimulus arrays. For example, in the general case of a uniform distribution of points, the nonboundary points should be located with equal probability at any position within the convex hull defined by the boundary points. The requirement that nonboundary points be located within an inner circle or torus sacrifices this generality without any obvious compensatory benefit in terms of increased analytical or other capability.

The severity of the restriction imposed by this method of construction may be quantified by considering the probability that all the nonboundary points, if chosen without constraints, would fall within the circle or torus. If there are b boundary points, the area of the convex hull may be approximated by the area of a regular polygon with b vertices. Consequently, the probability that i nonboundary (interior) points will be located within the circle, assuming that each point within the convex hull is a priori equally likely, is approximately given by

$$p_c(b, i) = \left(\frac{\pi}{2b \cdot \sin\left(\frac{2\pi}{b}\right)} \right)^i, \quad (1)$$

with the corresponding probability in relation to the torus being given by

$$p_t(b, i) = \left(\frac{3}{2} \right)^i p_c(b, i). \quad (2)$$

These two probability functions are evaluated in Table 1 for the values of b and i employed in constructing the con-

Table 1
Probability That a Random Configuration of Nonboundary Points Will Fall Entirely Within the Areas Inside the Convex Hull Prescribed by MacGregor and Ormerod (1996)

Total Points	Nonboundary Points	Probability of Configuration
10	6	3.67×10^{-3}
10	5	3.93×10^{-3}
10	4	8.35×10^{-3}
10	3	2.36×10^{-2}
10	2	7.71×10^{-2}
10	1	2.72×10^{-1}
20	16	2.10×10^{-4}
20	14	1.55×10^{-5}
20	12	2.73×10^{-5}
20	10	1.07×10^{-4}
20	8	5.66×10^{-4}
20	6	3.41×10^{-3}
20	4	2.19×10^{-2}

figurations used by MacGregor and Ormerod. It can be seen that the unconstrained construction of a 10-point configuration with 1 nonboundary point has only a 27% chance of satisfying the inner circle constraint, while that of a 10-point with 2 nonboundary points has about a 7.5% chance. None of the other 10- or 20-point configurations even reaches the 3% probability level. Indeed, the probability that some of the constrained circle and torus configurations would arise from an unconstrained method of nonboundary point location is negligibly small. Clearly, the configurations employed by MacGregor and Ormerod are highly constrained. Moreover, the resultant separation between the interior and the boundary points seems likely to enhance perception of the latter and encourage subjects to conform to the hypothesis under test.

An additional, complementary form of constraint is implied by Figure 1, which shows the distribution of the number of points on the convex hull, for configurations in which each point is independently and uniformly located in the unit square, as a function of the total number of points. These distributions are calculated on the basis of 10,000 simulated random pathways for configurations with 10, 20, 30, ..., and 100 total points. Figure 1 illustrates the point that, in the case of unconstrained 10- and 20-point arrays, the mean number of boundary points is expected to be around 6 and 8, respectively, with an SD of about 1 in both cases. This implies, for example, that 10-point arrays with three, or fewer, interior points (or, conversely, seven, or more, boundary points) are highly improbable. Taken in conjunction with the constraints in stimulus construction above, this means that, in the case of the 10-point arrays, it would be possible to draw a circle that would pass through between 4 and as many as 9 boundary points. In the case of the 20-point arrays, the circle would pass through between 4 and as many as 16 boundary points. It is difficult to believe that subjects would not be sensitive to the high degree of rotational symmetry among the unusually large number of vertices of the convex hull boundary in around half of these arrays.

Figure 1. The number of points falling on the boundary of the convex hull, as a function of the total number of points in an array. The width of the bars varies with the probability of the associated number of boundary points.

Asymptotic behavior of boundary points. As is shown in Figure 1, both the mean number of boundary points and the variance in the distribution of numbers of boundary points exhibit asymptotic behavior, suggesting that the distribution of boundary points is relatively stable for configurations with large numbers of points. A corollary of this observation is that, for these large configurations (with, say, more than 50 points), the concept "total number of points" is diagnostically equivalent to the concept "number of nonboundary points," in the sense that they effectively differ only by a constant. Of course, the particular experimental results obtained by MacGregor and Ormerod are not directly affected by this conclusion, since they involve only 10- and 20-point configurations. However, the more general hypothesis of MacGregor and Ormerod's study, that the "complexity of TSPs for human subjects is a function of the number of interior points rather than of the total number of points" (1996, p. 535), is inconsistent with the evident asymptotic behavior of boundary points in random configurations. Thus, even if MacGregor and Ormerod's conclusion were warranted for configurations with small numbers of points (which remains unlikely, given the constrained nature of the actual configurations that they employ), it is not reasonable to expect it to continue to hold for larger numbers of points.

Conclusions

These results point to two main conclusions. First, MacGregor and Ormerod's (1996) general conclusion concerning the importance of the number of nonboundary points is artifactually determined by their highly con-

strained procedure for constructing the stimulus arrays. Second, it is possible that, with arrays of less than about 50 points, the number of nonboundary points does influence response uncertainty, as these authors conclude. However, this influence is in principle limited to arrays with around 50 points or fewer. As MacGregor and Ormerod (1996, pp. 537-538) suggest, further research might profitably focus on differences in performance between random and constrained arrays. For example, Vickers, Butavicius, Lee, and Medvedev (in press) have found the relative optimality of solutions to random arrays to be independent of the number of boundary points, although both convexity and the total number of points did influence solutions' speed and response uncertainty.

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