Invariance tests for bisection and fractionation scaling¹

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If we are to accept the power function as the psychophysical law, then estimates of the exponent of a particular form of the power law should be independent of changes in basic independent variables. In the present study, various power law hypotheses were tested for the bisection and fractionation scaling of brightness. The results indicated that for bisection, estimates of exponents were dependent on the particular interval bisected, suggesting rejection of the simple power law for bisection. For fractionation, two additional forms of the power law were tested, each form involving a threshold parameter. One was the Φ -law (involving a translation on the intensity axis), and the other was the Ψ -law (involving a translation on the psychological axis). The Φ -law provided a poor fit to the data, whereas the Ψ -law appeared to fit well for at least one S. Analysis for individuals showed that for all five Ss, the variance due to standards was appreciably larger for the Φ -law.

Stevens and Galanter (1957) have presented psychophysical data for a dozen perceptual continua, which have generally been interpreted as giving strong support to the simple power law (without additive constant) as a good first approximation to the psychophysical law in the case of ratio scaling. However, this function appears to deviate from data near threshold values. Stevens (1959), Galanter and Messick (1961), Scharf and Stevens (1959), and Ekman (1958) have attempted to account for such discrepancies by introducing a threshold parameter into the intensity domain, whereas McGill (1960), Fagot (1963), Lochner and Burger (1961), and Zwislocki and Hellman (1961) have introduced a threshold parameter which effects a translation on the psychological axis. In the case of the simple power law, goodness of fit tests have been made from the linearity of the log-log plots. Such an approach is far from simple when the more complex forms of the power law are investigated, since it involves the estimation of additive constants (including the threshold parameter). An alternative approach, advocated by Fagot (1963), is to estimate the exponent based on a minimal subset of stimuli, and test for possible non-invariance of these independent estimates. Obviously, if the power law holds, the estimates should be independent of the particular subset used. For example, in the case of bisection, the exponent of the simple power law can be estimated from the bisection of a single interval; i.e., a single pair of stimuli. Similarly, if the threshold is known or assumed zero (as in the simple power law), then the exponent can be estimated from the fractionation

of a single standard. This method has been applied previously by Fagot, Eskildsen, and Stewart (1966) in studying the effect of rate of change in luminance on bisection and fractionation judgments of brightness.

In the present study, we shall test the power law hypotheses for (a) brightness scaling using bisection, by testing the equality of exponents relative to each interval bisected; and (b) brightness scaling using fractionation, by testing the equality of exponents relative to each standard for which a half-judgment is made. For the fractionation data, a separate test will be made for each form of the power function.

METHOD

Subjects

Five Ss were used, two males and three females, all in their 20's, and all were naive with respect to the bisection and fractionation procedures. Ss were paid 1 dollar per hour for their time.

Apparatus

The apparatus, described in detail elsewhere (Eskildsen, 1963), provided two (for fractionation) or three (for bisection) luminous white circular targets 13 mm in diameter, placed horizontally and separated by 90 mm, center to center. The luminance (Φ) was continuously variable in each of the targets from 0.1 ft-L to 600 ft-L. When Φ s of less than 0.1 ft-L were required, Kodak Wratten neutral density filters were fitted over the target apertures.

In fractionation procedures, one target—to S's right—was held constant in luminance as the standard (Φ_H), while S could vary the luminance of the other target continuously over the entire stimulus range by means of a control switch. For bisection, a third target was introduced, and the middle target was under the control of S. With threshold measurements, a single light was presented, and E rather than S controlled the luminance of the variable stimulus.

In procedures where S varied the luminance of the stimulus, his control switch activated a cam mechanism which varied the luminance of the light at a rate which depended on the shape of the particular cam being used. In the present experiment a "0.3 exponent" cam was used, which means the cam was cut to provide a uniform rate of change in brightness (not luminance) given the psychophysical relationship is a power function with exponent equal 0.3 (see Eskildsen, 1963; and Fagot, Eskildsen, & Stewart, 1966).

Procedure

For fractionation, three standards were used, the Φ s being 500, 5, .05 ft-L. Ss were tested on three successive days, each day's session lasting about 1 h. Each of the three sessions was broken up into six blocks, and each standard was used exactly once in each block, and therefore six times in each session. Three observations were made for each standard in each block, a total of 18 observations for each standard in each session. The order of standards was determined by a table of random numbers independently for each block. For the three sessions combined there were a total of 18 blocks and a total of 54 observations with each standard.

Table 1 exhibits the eight conditions used for bisection, where Φ_H denotes the higher stimulus defining the interval to be bisected, Φ_L the lower stimulus, and $L = \Phi_L / \Phi_H$. There were a total of nine sessions for bisection, each session lasting about 1 h, and occurring on nine different days. Each of the nine sessions was divided into two blocks, each of the eight conditions being used exactly once in each block. Three observations were made under each condition in each block, making a total of six observations under each condition per day, and a total of 54 observations under each condition over the nine-day period. The order of the eight conditions was determined by a table of random numbers independently for each block.

A final session was used in the measurement of the absolute threshold using the method of minimal change (Guilford, 1954, pp. 101-107).

Since a previous study in our laboratory (Fagot, Eskildsen, & Stewart, in press) showed that fractionation was affected by prior experience with bisection, but not vice-versa, all Ss fractionated first. Thus for all Ss the order was three sessions of fractionation, followed by nine sessions of bisection, then one session of threshold measurement.

For fractionation, the following instructions were given:

You will be presented with two lights. The light on your right will be of a fixed, constant brightness, while the light on your left will be variable in brightness. Your task will be to adjust the brightness of the left hand light such that its brightness is one-half that of the right hand light.

For bisection, the following instructions were given:

You will be presented with three lights. The light on your left will be somewhat dimmer than

the light on the right. The light in the middle will be a brightness somewhere in between the right and left hand lights. Your task will be to adjust the brightness of the middle light in such a way that its brightness is half way in distance between the bright and dim lights. In other words, the interval of brightness between the dim and middle lights should be equal to the interval of brightness between the middle and bright lights.

All Ss were cautioned against making their judgments on the basis of variables which might be related to luminance, such as the apparent size of the light or apparent distance. The starting points from which each judgment was begun were randomly distributed over the entire range of Φ , and Ss were also cautioned against using time as an aid in making their judgments.

Each session of fractionation or bisection was preceded by a brief dark adaptation period of 10 to 15 min. Ss were seated at a distance of 12 ft from the targets in an ordinary classroom desk seat. Adjustments in the intensity of the variable light were made by means of a control switch, and at the conclusion of each observation S pressed a button which closed a shutter over the stimulus lights and signaled to E that the observation was completed. After recording, E reset the cam device at a starting point previously determined by a randomizing procedure, and opened the shutter which signaled S to make the next observation.

A 30 min dark adaptation period preceded the threshold measurement sessions. S was instructed to increase the luminance of the variable light from invisible to "just visible" and from a point where the light was clearly visible to "just invisible." The procedure was repeated twice to give E the general range in which the observations should be made. The method of minimal changes was then used to make 10 observations of the threshold. No fixation point was provided for S; however, instructions were given to fixate only above or below the threshold stimulus target, and never to the right or left.

RESULTS

For both bisection and fractionation, the analysis was first carried out for the simple power laws (without introducing a threshold parameter), but involving an arbitrary additive constant in the case of bisection. In both cases, the dependent variable was the exponent of the assumed power law, and was estimated by the method outlined by Fagot (1963, 1966). An iterative solution to the equation

$$G = \left[1/2(\hat{\Phi}_L^k + \hat{\Phi}_H^k) \right]^{1/k}, \quad (1)$$

gave estimates of the exponents k for bisection. In Eq. (1) Φ_L and Φ_H are the luminances in ft-L of the lower and upper stimuli, respectively, of the interval

Table 1. Values of Φ_L for the experimental conditions of bisection

L	$\Phi_H=5$	$\Phi_H=500$
.100	.500	50.00
.010	.050	5.00
.001	.005	.50
.000	.000	.00

Table 2.
Analysis of variance summary table for bisection exponents

Source	df	F
Φ_H	1	12.29*
L	3	16.27**
Day	8	1.25
Subj.	4	41.25**
$\Phi_H \times L$	3	1.67
$\Phi_H \times \text{Day}$	8	1.33
$\Phi_H \times \text{Subj.}$	4	12.96**
L $\times$ Day	24	< 1
L $\times$ Subj.	12	8.43**
Day $\times$ Subj.	32	1.88*
$\Phi_H \times L \times D$	24	1.11
$\Phi_H \times L \times S$	12	7.62**
$\Phi_H \times D \times S$	32	1.38
L $\times$ D $\times$ S	96	1.49
$\Phi_H \times L \times D \times S$	96	

* $p < .05$, ** $p < .01$

bisected, and G is the geometric mean³ of the observations. For fractionation, the exponents were estimated directly from the following equation:

$$\hat{k} = \frac{\log 1/2}{\log (G/\Phi_H)} \quad (2)$$

where Φ_H is the standard in ft-L.

The bisection data were analyzed by means of a four-way (upper standard (Φ_H) by L by Day by S) factorial design (one observation per cell) with the S factor treated as a random effect, and the remaining factors fixed effects. The analysis of variance summary results are given in Table 2. It will be noted that all main effects except Day were significant ($p < .05$), as well as several of the interactions.

Since the Day factor was not significant, the observations were combined over the nine days of the bisection experiment to obtain a more stable estimate of the exponent. Table 3 shows the simple power law exponents, each estimate based on 54 observations.

Similarly, the fractionation data were analyzed by means of a three-way (standard (Φ_H) by Day by S) factorial design (one observation per cell), the dependent variable being the exponent as estimated from Eq. (2), with S treated as a random effect, and the other two factors as fixed effects. The analysis of variance summary results are given in Table 4. The main

effects Ss and standards were significant ($p < .01$), but (as with bisection) the effect of Day was not significant. As might be expected, the S factor generated significant interactions: both the S by Φ_H and the S by Day interactions were significant ($p < .01$).

As in the case of bisection, since Day was not a significant factor, the observations for the fractionation conditions were combined over days, and estimates of the exponents computed for the simple power law. The results are presented in Table 5.

For the fractionation data, the next step was to test the more complex forms of the power law obtained by introducing a threshold parameter in the following ways:

$$\Psi = a(\Phi^k - \Phi_0^k), \text{ and} \quad (3)$$

$$\Psi = a(\Phi - \Phi_0)^k, \quad (4)$$

where Φ denotes intensity, Ψ the corresponding value on the psychological scale, and Φ_0 the threshold obtained by the method of minimal change. Equation (3) involves translation on the psychological axis, and is designated the Ψ -Law, whereas Eq. (4) involves translation on the intensity axis, and is designated the Φ -Law.

For the Ψ -Law, exponents were estimated using Eq. (1), setting $\Phi_L = \Phi_0$, and letting Φ_H denote the standard. In the case of the Φ -Law, estimates of the exponent were obtained by means of Eq. (5):

$$\hat{k} = \frac{\log 1/2}{\log \frac{(G - \Phi_0)}{(\Phi_H - \Phi_0)}} \quad (5)$$

Estimated threshold values (Φ_0) used in the estimation equations are given in the last column of Table 5.

The Φ -Law was tested by using as dependent variable exponents estimated from Eq. (5) in the same factorial design used to test the simple power law. The results of the significance tests were qualitatively the same as in the tests of the simple power law for fractionation; i.e., the same effects were significant at the same levels (see Table 4).

The exponents for the Ψ -Law were also used as the dependent variable in the same type factorial design, but in this case the effect of standard was found to be significant only at the .05 level. The same two-way

Table 3. Bisection exponents*

L	$\Phi_H = 500$				$\Phi_H = 5$			
	.1	.01	.001	0	.1	.01	.001	0
Φ_L	50	5	.5	0	.5	.05	.005	0
Mike	.2224	.1399	.1214	.1966	.4185	.1956	.2458	.3284
M.E.	.0736	.0025	-.0232	.1238	.4467	.1127	.0545	.1913
J.O.	-.0605	.0415	-.0020	.1477	.2850	.0178	.0908	.2161
Beth	.4892	.0933	.0770	.1510	.3550	.1219	.0985	.2177
Ray	.1022	.0185	-.0161	.1679	.3058	.0679	.1185	.2538
Mean	.1654	.0591	.0314	.1574	.3622	.1032	.1216	.2415

* Note: Negative exponents imply a bisecting stimulus below the geometric mean of the standards ($G < \sqrt{\Phi_L \Phi_H}$) bisected by the subject (see Fagot, 1963).

Table 4.

Analysis of variance summary table for fractionation exponents

Source	df	F (simple law)	F (Φ -law)	F (Ψ -law)
Φ_H	2	30.13**	31.13**	6.74*
Day	2	< 1	2.06	2.17
Subj.	4	128.90**	202.73**	66.65**
Φ_H X Day	4	< 1	< 1	1.48
Φ_H X Subj.	8	18.95**	29.88**	10.68**
Day X Subj.	8	7.42**	16.77**	11.29**
Φ_H X Day X Subj.	16			

* $p < .05$, ** $p < .01$

interactions were significant for the Ψ -Law as for the simple law (S by Φ_H and S by Day).

It is difficult to interpret the finding (for either law) that the main effect due to standard is significant since the averaging over Ss (in view of significant interactions) could obscure real differences in individual psychophysical functions. For this reason, the two alternative forms of the power function were further compared by testing the effect due to standard separately for each S. Table 6 contains the Fs for both the Φ -Law and Ψ -Law for the effect of standard, along with ω^2 —the proportion of the total variance accounted for by the standard (Hays, 1963, pp. 381-384). Estimates of the different variances were obtained by appropriate linear combinations of the mean squares, and require the assumption that the triple interaction is zero. The appropriate error term for testing the effect of standard for each S, as can be seen from an examination of the expected mean squares, is the overall triple interaction, with the necessary assumption that the triple interaction is zero.

Table 6 shows that the effect of standard for the Φ -Law was significant for all Ss ($p < .01$), whereas for the Ψ -Law, the effect of standard was not significant for one of the Ss. For all five Ss, ω^2 was appreciably smaller for the Ψ -Law.

Although not germane to the main issue of this study, it is worth noting that for both bisection and fractionation the plot of standard deviation as a function of the bisecting and ratio stimulus, respectively, is approximately linear, giving indirect support to a generalized Weber function.

DISCUSSION

It is interesting to consider the results of the present study in connection with several previous

studies. Hanes (1949a, b), in two studies of brightness scaling using the method of fractionation, presented data which showed the standard deviation as an approximately linear function of the mean of the judgments (Hanes did not mention this point), and hence our results with both bisection and fractionation are consistent with his finding, and both give indirect support to a generalized Weber function. Hanes also found the ratio of the fractionation stimulus to the standard decreased with increases in the standard, which is consistent with our finding that the estimates of the exponent decrease with increases in the luminance of the standard.

Galanter and Messick (1961) used a controversial principle of theory construction by Luce (1959) to argue theoretically that the Φ -Law is the appropriate form, but suggest also that one cannot hope to look to the data to choose between the alternative forms of the power function. Luce and Galanter (1963, p. 281) also suggest the variability of magnitude estimation judgments is so large that both functions fit equally well, and that a choice between the two forms of the law will have to be reached indirectly. It is certainly true that if judgments are restricted to medium and high intensities, the problem of discrimination is extremely difficult. However, as Fagot (1966) has shown, it should be quite feasible to choose between the laws based on observations near the threshold, and it appears the data of the present study provide a fair basis for choosing between the laws.

Several investigators (Marks & Stevens, 1966; Scharf & Stevens, 1959; Stevens & Stevens, 1960, 1963; and Stevens, 1959, 1961, 1966a, b) take for granted the Φ -Law as the appropriate form of the power law, and fail to do the kind of analysis that would allow comparison of the fit for the two laws. This is all the more surprising in view of the studies by McGill (1960) which showed the Ψ -Law was the rule for individual observers for judgments of loudness, and by Hellman and Zwislowski (1961) who studied the form of the loudness function near threshold intensities. Lochner and Burger (1961), in analyzing the data of the Hellman-Zwislowski study, conclude the data is better fit by the Ψ -Law. In addition, Zwislowski and Hellman (1960) conclude the Ψ -Law may provide a better fit to various sense modalities than the Φ -Law.

The data of the present study do not support the tacit assumption that the psychophysical function con-

Table 5. Fractionation exponents

Subject	500	Φ_H		
		5.0	.05	Φ_0
Mike	.2707	.3948	.6416	.0020
M.E.	.1725	.2283	.2867	.0001
J.O.	.2037	.2474	.4157	.0004
Beth	.2233	.3409	.4676	.0008
Ray	.2055	.2885	.5530	.0011
Mean	.2151	.3000	.4729	.0009

Table 6. F-test and ω^2 for the effect of Standard

Subject	Φ -law		Ψ -law	
	F	ω^2	F	ω^2
Mike	344.00**	.926	37.19**	.639
M.E.	40.74**	.934	2.30	.350
J.O.	147.64**	.965	10.77**	.698
Beth	167.76**	.966	13.90**	.684
Ray	350.94**	.833	50.42**	.507

** $p < .01$

forms to the Φ -Law, but rather leads conclusively to the rejection of both the simple power law and the Φ -Law.⁴ Not only was the effect of standard for the Φ -Law highly significant ($p < .001$) in the overall design, but further analysis leads to the rejection of the Φ -Law for each individual S. In fact, as can be seen from Table 6, over 90% of the total variance is accounted for by the effect of standard for four of the five Ss, and the mean ω^2 is .925.

The status of the Ψ -Law is less clear. Although the effect of standard was again significant, the level of significance was only .05. Also, separate analysis by individuals showed the effect of standard was not significant for one of the five Ss. In the overall design, in all cases where an effect was significant, the given effect was more significant in the case of the Φ -Law than the Ψ -Law. Furthermore, for all five Ss, ω^2 was appreciably larger for the Φ -Law, with a mean ω^2 of .567 for the Ψ -Law, compared to .925 for the Φ -Law. Thus, although these data do not give strong support for the Ψ -Law considered in isolation, a comparison of the three alternative forms clearly favors the Ψ -Law over the other two.

We note also that the simple law did not provide an adequate description of the bisection data. This result is support for a point made recently by Adams and Fagot (1966), in which they suggest that what is called for—at least for bisection—is a redirection away from studies of parameters of a poorly fitting power law towards an attempt to account for those features of the psychophysical function which are well supported by data; e.g., why it should be convex up or down.

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Notes

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2. Now at San Francisco State College.
3. In Eq. 1, as well as Eqs. 2 and 5 to follow, the geometric mean of the observations was taken as the customary rule-of-thumb estimator. However, of the cases discussed, if the observations have a log-normal distribution, then the use of the geometric mean as an average in the estimation equation provides a maximum likelihood estimate of the exponent. For a fuller discussion, and development of estimation procedures for magnitude and ratio estimation scaling, see Fagot (1966, pp. 210-213).
4. Note that since an independent estimate of the threshold was used in the Φ -law and the Ψ -law, it would be possible for the simple power law to provide a better fit to the data than either threshold form. Inspection of Table 4 shows that use of the Φ -Law did indeed worsen the fit to the data, but the situation was improved somewhat by using the Ψ -Law.

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