Additivity of risk in portfolios*

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A collection of gambles constituting a portfolio may itself be represented as a single gamble played once. A portfolio may be constructed by requiring each component gamble to be played once, and there is a theory of additive risk which requires that the perceived risk of the portfolio be an additive function of the perceived riskiness of the component gambles. Alternatively, a portfolio may be constructed as a probability mixture of the component gambles, and there is a theory of expected risk which requires that the perceived risk of the portfolio be the average of the riskiness of the component gambles. Existing evidence suggests that both of these theories cannot be true. A test of additivity of portfolios under both kinds of composition functions was made under two forms of display, the components displayed separately and fused as a single gamble. This experiment supports expected risk and rejects additive risk, especially under fused display.

Risk is an almost universal property of the options available for choice in everyday behavior, and our understanding of the higher level cognitive processes involved in decision making would be enhanced by an understanding of the psychological nature of risk and the tendencies of individuals toward risk-taking. Risk, however, tends to be an idiosyncratic concept and not well defined, so the development of a descriptive theory of perceived risk has been slow to develop.

A portfolio theory of risk preference, which separates the theory of preference for risk from the theory of the nature of perceived risk itself, was recently advanced (Coombs & Meyer, 1970; Coombs & Huang, 1970b). These experimental studies, however, involved the manipulation of perceived risk on the basis of some ad hoc assumptions. The results of these experiments stimulated the elaboration of these assumptions in the form of a simple polynomial model for perceived risk which was tested in another experiment by conjoint measurement methods (Coombs & Huang, 1970a).

The context of these experiments has been 2-outcome gambles of the form (y, p, z) , in which an individual has a probability, p , of winning y , otherwise a probability, $1 - p$, of winning z . These experiments involved the perceived risk of and preference for portfolios of such gambles in which rather weak and general assumptions were made about the nature of risk.

Pollatsek and Tversky (1970) built a

stronger theory of perceived risk in axiomatic form and derived some very substantial implications, one of which the experiment reported here is designed to test, along with an implication drawn from another independent theory of perceived risk (Huang¹).

Each of these two theories, the Pollatsek-Tversky theory and the Huang theory, is concerned with the riskiness of a portfolio as a function of the riskiness of its component gambles, but the two theories differ in the manner of composition of the portfolio as follows. A portfolio of two 2-outcome gambles may be made up in either of two natural ways which we shall call compound composition and contingent composition. By compound composition, we mean the addition of two gambles in the sense of playing both. By contingent composition, we mean the addition of two gambles in the sense of playing one with probability t , otherwise playing the other.

Formally, if $g = (y, 1/2, z)$, $g' = (v, 1/2, w)$, and $\oplus$ designates the operation of compound composition, then the compound composition of g and g' is defined as follows:

$$(g \oplus g') = (y + v, 1/4, y + w, 1/4, z + v, 1/4, z + w, 1/4) \quad (1)$$

and the contingent composition of g and g' is defined as follows (for $t = 1/2$):

$$(g, 1/2, g') = (y, 1/4, z, 1/4, v, 1/4, w, 1/4) \quad (2)$$

In each case we note that the portfolio consisting of two 2-outcome gambles under either of these forms of composition reduces to a single

4-outcome gamble which, of course, under appropriate values for y, z, v, w , could be fewer than four outcomes.

Pollatsek and Tversky (1970) have offered a set of axioms for risk implying the existence of a measure of risk which is additive under compound composition. That is, that the riskiness of a portfolio under compound composition is equal to the sum of the riskinesses of its component gambles.

Huang, in her thesis, has provided another set of axioms for risk, implying that the riskiness of a portfolio under contingent composition is the weighted average of the riskiness of the component gambles.

So we have a theory of additive risk under compound composition and a theory of expected risk under contingent composition.

Huang has two other results of interest here. Her axioms imply (1) that the riskiness of a 2-outcome gamble is a weighted combination of functions of the two outcomes and (2) that if both the theory of additive risk and the theory of expected risk are true then the functions of the two outcomes must be linear functions.

Another way of expressing these results is that if additivity holds in the conjoint measurement sense under compound composition and under contingent composition with t fixed, then the riskiness of a gamble must be a linear function of its expected value; this is an implication that is certainly violated. Any study that purports to show, for example, that variance is a significant variable in perceived risk or risk preference (e.g., Coombs & Pruitt, 1960, 1961) would contradict the above implication because variance is a quadratic function of the outcomes.

Hence we anticipate that additivity of risk under compound and under contingent composition cannot be true in general, and it is of considerable interest to observe for which, if either, it does obtain.

It is worthwhile to point out that additivity of risk under compound composition has a strong implication. A 4-outcome game with outcomes x_1, x_2, x_3, x_4 with their respective probabilities a_1, a_2, a_3, a_4 can be decomposed into two 2-outcome games under compound composition if it meets the following two necessary and sufficient conditions: (1) a cross-product rule for the probabilities:

$$a_1 a_4 = a_2 a_3;$$

(2) a difference rule for the outcomes:

$$x_1 - x_3 = x_2 - x_4.$$

If these conditions are met then the

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Fig. 1. Portfolios generated by compound composition. The meaning of the cross-hatching is explained in the text.

Fig. 2. Portfolios generated by contingent composition. The meaning of the cross-hatching is explained in the text.

decomposition is unique with respect to probabilities, but there is an infinite number of equivalent decompositions as follows: if $g = (y, p, z)$ and $g' = (v, q, w)$ then $g \oplus g' = (y + b, p, z + b) \oplus (v - b, q, w - b)$, where b is an arbitrary real number.

That is, the compound composition of two games is invariant if an amount b is subtracted from the outcomes of one game and added to the outcomes of the other.

If risk is additive under compound composition, then the changes in the riskiness of one game induced by adding the constant, b , to its outcomes must be exactly compensated for by the change in the riskiness of the other

game induced by subtracting the same constant from its outcomes. Hence, risk must be linear with the value of b , which is itself linear with expected value. So, if risk is additive under compound composition, then it is linear with expected value for fixed variance.

The following experiment, then, has been designed to test whether or not the riskiness of a portfolio under compound composition could be an additive function of the riskiness of its component gambles and whether or not, under contingent composition, it could be an average of the riskiness of its component gambles. Both of these propositions can be tested by additive

conjoint measurement procedures.

A problem of considerable methodological concern in the design of an appropriate experiment is that of display: Should the components of a portfolio be displayed separately or should the result of their combination be presented as a single gamble (fused)? The result of the composition, whether compound or contingent, given by Eqs. 1 or 2, respectively, is, of course, the true portfolio, and any difference in the S's response to the portfolio when the components are displayed separately might be interpreted as an error in his internal computer or imputed to some psychological variable not considered here.

The present experiment is an attempt to test additivity under both kinds of displays.

METHOD

Subjects

The Ss were 27 men and 21 women, paid volunteers from the student body of the University of Michigan.

Stimuli and Display

Seven 2-outcome gambles, designating the three rows and four columns of Fig. 1 (and Fig. 2) were used to generate the portfolios used as stimuli. The cells of Fig. 1 contain the portfolios generated under compound composition (Eq. 1), and the cells of Fig. 2 contain the portfolios generated under contingent composition (Eq. 2).

The display variable has two levels: (1) One displays each portfolio as a single game in the manner shown in the cells of Figs. 1 and 2, drawn on a 3 x 5 card and embedded in laminated plastic; this kind of display is called fused. (2) The other displays the portfolio in terms of its components and manner of composition, drawn on a 3 x 5 card and embedded in laminated plastic; this kind of display is called separate and is illustrated in Fig. 3 for compound and for contingent composition.

Design

The two kinds of composition and the two forms of display yield the four cells of a 2 by 2 design, each cell of the design consisting of a set of 12 portfolios.

Procedure

Each S came for two sessions, usually with 2 days intervening, but in several instances there was 1 day or 3 days intervening. In each session, S rank-ordered in perceived risk the set of 12 gambles in each of the four cells of the design. For 28 Ss the sets in each session were presented in the following order: (1) compound-fused, (2) contingent-separate,

COMPOUND

CONTINGENT

Fig. 3. Illustrating separate display under two kinds of composition.

(3) compound-separate, (4) contingent-fused. For the other 20 Ss, the sets were presented in the reverse order. The Ss given the first presentation order are numbered 1 to 28, with Nos. 1-15 being males and Nos. 16-28, females. The Ss given the second presentation order are numbered 29 to 48, with Nos. 29-40 being males and Nos. 41-48, females.

The following version of a method of accumulation was used to obtain the rank order: S was presented with two gambles (the shaded cells in Figs. 1 and 2) and asked which, in S's judgment, appeared riskier. The riskier was then placed to the right of the other. A third gamble, randomly selected, was presented and S was told to place it either to the left, between them, or to the right, depending on how risky it appeared relative to the first two. The fourth gamble was then placed relative to the first three, etc., until all 12 were ordered from least to most risky. The S was then asked to check the final ordering and to correct any errors of judgment he might have made.

RESULTS

Consistency Analysis

Kendall's τ coefficients were calculated for each S between the two replications in each cell of the design. This gave four measures for each S of the consistency of his rank ordering on

Table 1

Average τ s Between Sessions 1 and 2

Composition Variable	Display Variable	
	Separate	Fused
Compound	.86	.85
Contingent	.87	.91

Table 2
Number of Cyclic Permutations from Additivity

S	τ	Cp-F	Ct-S	Cp-S	Ct-F	Total	Cp-F	Ct-S	Cp-S	Ct-F	Total	Total
9	.98	0	0	0	0	0	0	0	0	0	0	0
1	.95	0	0	0	0	0	2	0	0	1	3	3
14	.95	0	0	0	0	0	1	0	0	0	1	1
17	.95	0	0	0	0	0	2	0	0	0	2	2
28	.94	2	0	0	0	2	2	0	0	0	2	4
13	.93	3	0	0	0	3	0	1	0	0	1	4
29	.93	2	0	0	0	2	3	0	0	0	3	5
31	.93	0	0	0	0	0	0	0	0	0	0	0
2	.92	0	0	0	0	0	0	0	0	0	0	0
5	.91	0	0	0	2	2	0	0	1	0	1	3
10	.91	1	0	1	0	2	1	0	0	0	1	3
15	.91	2	0	2	0	4	2	0	6	0	8	12
18	.91	2	1	1	0	4	4	0	2	0	6	10
30	.91	1	2	0	0	3	0	0	0	0	0	3
39	.91	1	0	0	0	1	0	0	0	0	0	1
22	.90	4	0	0	0	4	3	0	0	0	3	7
23	.90	0	0	1	2	3	0	0	0	0	0	3
33	.90	1	0	1	0	2	2	2	1	0	5	7
44	.90	2	0	0	0	2	3	0	0	0	3	5
45	.90	2	0	0	0	2	2	0	0	0	2	4
8	.89	1	1	1	0	3	3	0	0	2	5	8
26	.89	0	0	0	0	0	2	1	0	1	4	4
35	.89	0	0	1	1	2	1	0	0	0	1	3
4	.88	1	0	0	0	1	0	0	0	0	0	1
7	.88	0	2	0	0	2	0	0	0	0	0	2
32	.88	2	0	0	0	2	1	0	0	0	1	3
42	.88	0	7	6	0	13	0	0	0	0	0	13
36	.88	0	0	0	0	0	0	0	1	0	1	1
41	.88	2	3	0	0	5	1	1	0	0	2	7
37	.88	2	2	0	0	4	2	0	0	0	2	6
38	.88	0	0	0	0	0	0	0	0	0	0	0
21	.87	1	0	2	0	3	1	0	0	0	1	4
24	.87	2	1	2	6	11	3	0	0	0	3	14
27	.87	1	0	0	0	1	0	0	0	0	0	1
20	.84	1	2	4	1	8	0	0	0	0	0	8
25	.84	5	0	0	0	5	3	0	1	3	7	12
43	.83	2	0	2	0	4	0	0	1	1	2	6
48	.83	0	0	0	1	1	1	0	2	2	5	6
34	.82	12	1	1	0	14	2	1	0	0	3	17
16	.81	0	1	1	0	2	1	0	0	0	1	3
19	.81	2	2	1	1	6	2	2	0	1	5	11
40	.81	3	1	4	1	9	2	1	2	1	6	15
46	.79	0	7	7	0	14	0	0	0	2	2	16
47	.79	0	2	1	0	3	1	0	1	0	2	5
11	.78	1	1	1	0	3	1	0	0	0	1	4
12	.77	3	8	0	0	11	0	2	1	1	4	15
6	.76	2	5	6	2	15	0	0	1	0	1	16
3	.67	14	1	0	0	15	1	4	1	0	6	21
		80	50	46	17	193	55	15	21	15	106	299

12 gambles over the interval between Session 1 and Session 2. The mean τ s (across 48 Ss) within cells of the design are reported in Table 1.

Additivity Analysis

For each of the eight data matrices for each S, a count was made of the number of pairwise cyclic permutations necessary to satisfy additivity. The data are reported in detail in Table 2, with the Ss listed in order of consistency as measured by the average τ between replications. Tables 3 and 4 summarize the data reported in Table 2.

A two-tailed sign test for matched pairs was applied to the difference between compound and contingent proportions for both the separate and the fused display. The z values are reported in the bottom row of Tables 3 and 4. Under separate display, we

see there are no significant differences, and under fused display, all differences were significant. It should be noted that in going from Table 3 to Table 4, the number of Ss is cut in half selectively, rejecting the most inconsistent Ss, yet the reduction in violations is 87% for the contingent-separate cell (combined) but 63% at most in the other three cells. This suggests that inconsistency was highly related to the violations of additivity in the contingent-separate cell and much less so in the others.

The analyses reported in Tables 3 and 4 pool the data matrices and neglect any differences between Ss. Another analysis may be made in which we ask if there is any difference in the number of Ss who satisfy or violate additivity in each of the four cells of the design.

Table 3
Number of Cyclic Permutations Required for Additivity for All Ss

	First Session		Second Session		Combined	
	Separate	Fused	Separate	Fused	Separate	Fused
Compound	46	80	21	55	67	135
Contingent	50	17	15	15	65	32
<i>z</i>	0.05	3.87	0.06	4.0	0.35	4.47

Table 4
Number of Cyclic Permutations Required for Additivity for the Most Consistent Half of Ss

	First Session		Second Session		Combined	
	Separate	Fused	Separate	Fused	Separate	Fused
Compound	8	25	10	33	18	58
Contingent	4	5	4	4	8	9
<i>z</i>	1.50	2.44	0.00	3.61	0.67	3.60

Table 5
Summary of Additivity Analysis by Ss

	Number of Ss Who Satisfied Additivity Perfectly in Both Replications		Number of Ss Who Violated Additivity in Both Replications	
	Separate	Fused	Separate	Fused
Compound	22	10	7	21
Contingent	26	32	6	3
<i>z</i>	.25	4.00	0.00	3.80

In Table 5 a summary of the additivity analysis by Ss is provided. The total number of Ss in each cell is, of course, 48. Again, *z* values for a two-tailed sign test for matched pairs are reported on the bottom of each table. Under separate display, we see there were no significant differences, and under fused display, all differences were significant.

DISCUSSION

As may be seen from Table 1, portfolios formed under contingent composition were in general more consistently ranked than were portfolios under compound composition, and this is especially true when the manner of display was fused. Analysis of the within-S variance in consistency of risk ordering revealed that this difference was the only significant one ($p < .005$). It should be pointed out that contingent portfolios in the same row or in the same column of Fig. 2 have two outcomes in common which should make comparison easier and thus account for higher consistency. The use of 2-outcome gambles that have an outcome in common can avoid this difficulty.²

The additivity analysis shows that additivity was clearly satisfied under compound composition and fused

display by 21% of the Ss and under contingent composition and fused display by 67%. Supporting this difference is the further finding for fused display that under compound composition 44% clearly violated additivity and under contingent composition only 6% clearly violated additivity. So, in the case of fused display, compound composition generally violated additivity and contingent composition did not.

Under separate display there was little difference between the two kinds of composition in terms of the proportion of Ss who satisfied or violated additivity, but across data matrices for the most consistent half of the Ss again contingent composition tended to satisfy additivity and compound composition not.

It would seem that componentwise display obscures the composition function and increases the salience of the components per se. This was certainly the impression the E got in watching the Ss make their judgments. Another possible explanation of a difference between fused and separate display is that a fused display says judgments should depend on final outcomes and not on the intermediate steps by which they are obtained. Separate display emphasizes the intermediate steps in reaching the final

result, and if these make a difference or merely if the S's internal computer is in error, which is perhaps more probable for contingent composition than for compound composition, then there can be a difference between separate and fused display.

The results of this experiment appear incompatible with the Coombs-Huang (1970a) polynomial model for perceived risk. In the latter model the multiple-play operation on a gamble corresponds, of course, to the construction of a portfolio by compound composition (convolution) of a gamble with itself, a weaker condition than that assumed by Pollatsek and Tversky. According to the Coombs-Huang polynomial model, the convolution of a game *with itself* is additive. According to this experiment, the convolution of *any* two games is not. Another difference between the two experiments is that in the Coombs-Huang experiment the display was not fused—the 2-outcome gamble was displayed with the indication that it was to be played once or exactly five times. We may note in the experiment reported here that under separate display, compound composition satisfied additivity about as well as contingent composition did across Ss. It is possible, then, that the polynomial might not hold under fused display.

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NOTES

1. L. C. Huang. Mathematical structure of a theory of expected risk. In preparation.
2. Suggested by David Krantz, personal communication.

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