Notes and Comment

An area theorem for the same-different experiment

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If observers in a same-different experiment base their decisions on the absolute difference between observations on a trial, the area under the receiver operating characteristic equals the maximum proportion of correct decisions that an unbiased independent-observations observer could attain. Even though the differencing strategy is suboptimal, the area measure yields an index of optimal performance.

A well-known theorem in psychophysics states that the area under the receiver operating characteristic (ROC) from the yes-no experiment equals the proportion of correct decisions of an unbiased observer in a two-alternative forced-choice experiment (Green, 1964; Green & Swets, 1966). We now present an area theorem for the same-different experiment. We show that the area under the ROC for observers who base their decision on the absolute difference between observations in a same-different experiment is equal to the proportion of correct decisions of an unbiased observer who makes optimal independent judgments of each observation in the experiment. Basing a decision on the difference between observations is often a suboptimal strategy, but nevertheless, the area measure yields an index of optimal performance.

Although procedurally simple, the same-different experiment is theoretically more complex than the single-interval or forced-choice experiment. The equality we wish to prove involves that complication. The complication arises from the fact that at least two different decision strategies are open to an observer in deciding whether two events are the same or different (see, e.g., Macmillan & Creelman, 1991). One strategy that an observer may adopt is to base a decision on the absolute difference between the two observations of a trial. Unless that difference exceeds a criterion value, the observer will decide that the observations stemmed from the same event. Another strategy open to an observer is to make independent observations of the two events presented on a trial. If the ratio of

the likelihoods that the observations stemmed from the same and different events exceeds a criterion value, the observer will decide that the observations stemmed from the same event.

Different experimental designs (fixed or roving) and different kinds of events (simple or complex) promote one or other of these strategies (see Dai, Versfeld, & Green, 1996; Francis & Irwin, 1995; Hautus, Irwin, & Sutherland, 1994; Macmillan & Creelman, 1991). Because there are two decision strategies at the observer's disposal, the same-different experiment can give rise to two different ROCs, one for each strategy. Similarly, the proportion of correct decisions depends on which decision strategy is adopted.

We first present the area under the ROC for the same-different experiment based on the differencing strategy and then the proportion of correct decisions of an unbiased independent-observations observer. Then, we show that these two quantities are identical. We then compare this area theorem with other psychophysical area theorems and briefly consider its possible use.

Area Under the Same-Different Receiver Operating Characteristic

The parametric equations for the ROCs for each of the strategies have been formulated and reported in the literature (Dai et al., 1996; Hautus et al., 1994; Irwin & Hautus, 1997; Macmillan, Kaplan, & Creelman, 1977). These equations are derived from the standard theory of signal detection, in which each event is assumed to give rise to Gaussian distributions of equal variance. Knowing the hit rate, $\Pr(\text{"Same"} | \text{same})$, and the false alarm rate, $\Pr(\text{"Same"} | \text{different})$, the area under the ROC can be computed (see Egan, 1975, pp. 46-48). For the differencing strategy, the equation for the hit rate is known to be

$$H(x) = \text{erf}\left(\frac{x}{2}\right), \quad (1)$$

where $H(\cdot)$ is the hit rate, x is the criterion, and $\text{erf}(\cdot)$ is the error function that is related to the normal distribution function, $\Phi(\cdot)$, by the equation (see Abramowitz & Stegun, 1965, p. 934)

$$\text{erf}(x) = 2\left(\Phi\left(x\sqrt{2}\right) - 1\right). \quad (2)$$

The equation for the false alarm rate is

$$F(x, d') = \frac{1}{2} \left(\text{erf}\left(\frac{x-d'}{2}\right) + \text{erf}\left(\frac{x+d'}{2}\right) \right), \quad (3)$$

where d' is the detection theory index of discriminability. For an arbitrary paradigm, the area under an ROC can be found from the integral over all criteria of the product of

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the hit rate and the partial derivative of the false alarm rate with respect to the criterion (Egan, 1975, Equation 2.18). For the *same-different* experiment, the derivative of the false alarm rate is

$$\frac{dF(x, d')}{dx} = \frac{1}{2\sqrt{\pi}} e^{-\left(\frac{x+d'}{2}\right)^2} (e^{xd'} + 1). \quad (4)$$

The area under the ROC, therefore, is

$$A_{\text{sdds}} = \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-\left(\frac{x+d'}{2}\right)^2} (e^{xd'} + 1) \text{erf}\left(\frac{x}{2}\right) dx, \quad (5)$$

where the subscript, sdds, refers to the *same-different* differencing strategy.

Proportion Correct of Unbiased Independent-Observations Observer

The maximum proportion correct in a *same-different* experiment is attained by an independent-observations observer. Several investigators have derived the relation between proportion correct and discriminability for such an observer (Macmillan & Creelman, 1991; Noreen, 1981). Macmillan and Creelman show how the *same-different* experiment can be decomposed into two separate yes-no components. Proportion correct for an unbiased observer in the experiment is, then,

$$p(c)_{\text{sdiOS}} = (p(c)_{\text{yn}})^2 + (1 - p(c)_{\text{yn}})^2, \quad (6)$$

where $p(c)_{\text{sdiOS}}$ is proportion correct in the *same-different* experiment for the independent-observations strategy, and $p(c)_{\text{yn}}$ is that in the yes-no experiment. Expressing the result in terms of d' and the error function yields

$$p(c)_{\text{sdiOS}} = \frac{1}{4} \left(\text{erf}\left(\frac{\sqrt{2}d'}{4}\right) + 1 \right)^2 + \frac{1}{4} \left(\text{erf}\left(\frac{\sqrt{2}d'}{4}\right) - 1 \right)^2. \quad (7)$$

Equality of Area and Maximum Proportion Correct

We wish to show that the difference between Equations 5 and 7 is zero. Let this difference be $G(d')$. It will be sufficient to prove that $G(0) = 0$ and that $dG(d')/dd' = 0$. We therefore break the proof into these two parts.

To simplify the proof, Equation 5 can be expressed in a form that is similar to that of Equation 7:

$$A_{\text{sdds}} = \int_0^\infty \frac{e^{-\left(\frac{x-d'}{2}\right)^2}}{2\sqrt{\pi}} \text{erf}\left(\frac{x}{2}\right) dx + \int_0^\infty \frac{e^{-\left(\frac{x+d'}{2}\right)^2}}{2\sqrt{\pi}} \text{erf}\left(\frac{x}{2}\right) dx. \quad (8)$$

The plan is to show that the difference between the left-hand parts of Equations 7 and 8 is zero and, by similar reasoning, that the difference between the right-hand parts of those equations is also zero.

The difference between the left-hand parts of Equations 7 and 8, which we call $\varphi(d')$, is

$$\varphi(d') = \int_0^\infty \frac{e^{-\left(\frac{x-d'}{2}\right)^2}}{2\sqrt{\pi}} \text{erf}\left(\frac{x}{2}\right) dx - \frac{1}{4} \left(\text{erf}\left(\frac{\sqrt{2}d'}{4}\right) + 1 \right)^2. \quad (9)$$

We first show that $\varphi(0) = 0$. Setting $d' = 0$ in Equation 9 yields

$$\begin{aligned} \varphi(0) &= \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-\frac{x^2}{4}} \text{erf}\left(\frac{x}{2}\right) dx - \frac{1}{4} \\ &= \frac{1}{\pi} \int_0^\infty e^{-\frac{x^2}{4}} \int_0^{\frac{x}{2}} e^{-t^2} dt dx - \frac{1}{4}. \end{aligned} \quad (10)$$

The substitutions $x = \sqrt{2}u$, $t = v/\sqrt{2}$, ($dx = \sqrt{2} du$, $dt = dv/\sqrt{2}$) provide a more familiar form to Equation 10:

$$\varphi(0) = \frac{1}{\pi} \int_0^\infty \int_0^u e^{-\frac{u^2}{2} - \frac{v^2}{2}} dv du - \frac{1}{4}. \quad (11)$$

It is clear that the double integral in Equation 11 is one quarter of the standard bivariate normal distribution and is, therefore, equal to $1/4$.

The second part of the proof requires that the derivative of the difference be zero—that is,

$$\frac{d\varphi(d')}{dd'} = 0. \quad (12)$$

Thus,

$$\begin{aligned} \frac{d\varphi(d')}{dd'} &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \left(\frac{x-d'}{2} \right) e^{-\left(\frac{x-d'}{2}\right)^2} \text{erf}\left(\frac{x}{2}\right) dx \\ &\quad - \frac{\sqrt{2}}{4\sqrt{\pi}} e^{-\frac{d'^2}{8}} \left(\text{erf}\left(\frac{\sqrt{2}d'}{4}\right) + 1 \right). \end{aligned} \quad (13)$$

If we let $u = \text{erf}(\frac{1}{2}x)$ and $v = \frac{1}{2}(x - d') \exp(-\frac{1}{2}(x - d')^2)$, then integration by parts of Equation 13 yields

$$\begin{aligned} \frac{d\varphi(d')}{dd'} &= \frac{e^{-\frac{d'^2}{8}}}{2\pi} \int_0^\infty e^{-\frac{1}{2}\left(\frac{x-d'}{2}\right)^2} dx \\ &\quad - \frac{\sqrt{2}}{4\sqrt{\pi}} e^{-\frac{d'^2}{8}} \left(\text{erf}\left(\frac{\sqrt{2}d'}{4}\right) + 1 \right). \end{aligned} \quad (14)$$

Substitute $x = \frac{1}{2}d' + y$ ($dx = dy$):

$$\begin{aligned} \frac{d\varphi(d')}{dd'} &= \frac{\sqrt{2}}{4\sqrt{\pi}} e^{-\frac{d'^2}{8}} \left[\frac{2}{\sqrt{2\pi}} \int_{-\frac{d'}{2}}^\infty e^{-\frac{y^2}{2}} dy \right] \\ &\quad - \frac{\sqrt{2}}{4\sqrt{\pi}} e^{-\frac{d'^2}{8}} \left[\text{erf}\left(\frac{\sqrt{2}d'}{4}\right) + 1 \right]. \end{aligned} \quad (15)$$

The terms in the two sets of square brackets in Equation 15 are equivalent, and hence, $d\phi(d')/dd' = 0$. This completes the proof that the left-hand parts of Equations 7 and 8 are equal. Similar reasoning can show that the right-hand parts of Equations 7 and 8 are also equal. In other words, the area under the ROC for observers who base their decision on the difference between observations in a *same-different* experiment is equal to the proportion of correct decisions of an unbiased independent-observations observer in that experiment.

The identity of these two measures of performance, one based on the difference between observations in a *same-different* experiment and the other on independent observations, provides further insight, in addition to that offered by Dai et al. (1996), into the theoretical connection between these decision strategies. Even though adopting a differencing strategy in the *same-different* experiment may be suboptimal, the area under the ROC for that strategy is equivalent to the maximum proportion of correct decisions attainable.

Comparison With Other Psychophysical Area Theorems

Green's area theorem. The *same-different* area theorem describes an equality between indices arising from different strategies for solving the task, whereas Green's theorem relates indices from two different experiments, yes-no and forced-choice. However, Macmillan (personal communication, March 3, 1998) has drawn our attention to an interpretation of Green's equality that allows it to be rephrased as one relating two strategies. Because in the two-alternative forced-choice experiment, each of the two events occurs in one or other of the two observation intervals, the task can be performed by ignoring one interval, since the other interval always contains the complementary event. An observer who ignores one of the intervals in that experiment can be said to have adopted a yes-no strategy. That strategy for the two-alternative forced-choice experiment is suboptimal, because it ignores information available in one of the observation intervals. When the Gaussian assumptions are satisfied, the area under the ROC for this suboptimal strategy is $\Phi(d'/\sqrt{2})$, the area under the yes-no ROC. This quantity is identical to the maximum proportion of correct decisions of an unbiased observer who adopts an optimal strategy and makes use of the information available in both intervals (see Macmillan & Creelman, 1991, Equation 5.8). In this respect, Green's area theorem could be construed as showing an equality between different indices, area and proportion correct, arising from different strategies.

The reminder experiment. Macmillan (1998) has pointed out a similar analysis of the reminder experiment. In the two-interval reminder experiment (see Macmillan & Creelman, 1991), the first observation interval always contains the same stimulus, a stimulus that serves to remind the observer of the magnitude of the standard; the second interval contains either the reminder stimulus or a comparison stimulus. The observer must decide whether

the second interval contains the standard or the comparison stimulus, or equivalently, whether the two stimuli were the same or different. One strategy is to ignore the reminder stimulus, in which case the experiment is identical to a single-interval yes-no experiment. When the Gaussian assumptions are satisfied, the proportion correct of an unbiased observer adopting the yes-no strategy is equal to $\Phi(d'/2)$ (Macmillan & Creelman, 1991, Equation 2.17), and the area under its ROC is $\Phi(d'/\sqrt{2})$. An alternative strategy for the reminder experiment is to base one's decision on the difference between the two observations. As Macmillan and Creelman show, when this strategy is adopted, the variance in the reminder stimulus reduces the estimate of d' by $\sqrt{2}$. The area under the reminder ROC for the differencing strategy is, therefore, $\Phi((d'/\sqrt{2})/\sqrt{2})$ —that is, $\Phi(d'/2)$ —which is equal to the proportion correct of an unbiased observer adopting the independent-observations strategy in that experiment. Once again, the area under the ROC for a suboptimal strategy equals the proportion correct of an optimal strategy.

Four-interval *same-different* experiment. Our proof concerns the area under the *same-different* ROC when an observer adopts a differencing strategy. But more than one strategy is available in the *same-different* experiment, and different strategies give rise to different ROCs. It can be shown that, for any pair of stimuli, the area under the ROC for an observer who adopts an independent-observations strategy is always greater than the area under the ROC of an observer who adopts a differencing strategy. One might wonder whether the larger independent-observations area has an equivalent proportion correct. Indeed it has.

Noreen (1981) has shown how the area under the independent-observations ROC for the *same-different* experiment is equal to the proportion of correct decisions of an unbiased observer in a four-interval *same-different* experiment. In a four-interval *same-different* experiment (see, e.g., Pisoni & Lazarus, 1974), two stimuli are presented in four observation intervals: The stimuli in either the first pair of intervals or the second pair of intervals are the same; the stimuli in the other pair of intervals are different. The observer must decide whether the first pair or the second pair of stimuli are the same.

Noreen (1981) has observed how the pairs of stimuli in the four-interval experiment may be likened to the single stimuli in the two-interval forced-choice experiment: The observer must decide whether the same pair was presented in the first two intervals or in the second two intervals. By analogous reasoning, the pairs of stimuli (either same or different) presented in the standard two-interval *same-different* experiment may be likened to the single stimulus presented in the one-interval yes-no experiment. By adopting a yes-no strategy, the observer decides on any trial whether the same pair or the different pair was presented. It follows from Green's area theorem that the area under the ROC (analogous to the area under the yes-no ROC) equals the proportion of correct decisions in the four-interval *same-different* experiment (analogous to the proportion correct in a two-interval

forced-choice experiment). Furthermore, because of their correspondence to the two-alternative forced-choice and yes-no experiments, the identity of indices from the four-interval and the standard *same-different* experiments can similarly be interpreted as an equality between indices arising from different decision strategies. All the area theorems we have reviewed may, therefore, share a common origin, in that they equate indices based on different decision strategies.

Implications for Practice

Although we think the *same-different* area theorem presented here is mainly of theoretical interest, it may have some practical implications. The *same-different* experiment is prone to response bias; as a consequence, unadorned proportion correct will not usually provide a satisfactory index of accuracy. ROC analysis can salvage a bias-prone method, and confidence rating with the appropriate analytical software offers a straightforward route to such an analysis. (Note that the standard single-interval programs for analyzing rating data are not suitable for this design.¹) Yet, proportion correct has the appealing characteristic of being a readily understood and seemingly natural index of performance. The area theorem we have presented allows an investigator to interpret the area measure for an observer adopting the differencing strategy as a maximum proportion correct for the experiment. Proportion correct for an observer who adopts a differencing strategy can also be determined, but that value will always be less than the corresponding proportion correct for an independent-observations observer. The corresponding values for an unbiased observer are tabulated in Noreen (1981), and, for that observer, the largest difference in the table is 0.08.

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NOTE

1. Software suitable for the analysis of rating data, collected from a *same-different* experiment, is available on request from the second author.

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