METHODS & DESIGNS

Manipulation of two-dimensionally periodic stimulus patterns

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An increasing number of visual studies require precise knowledge of the two-dimensional Fourier components of periodic spatial patterns. For patterns that are more complex than a simple sine-wave grating, however, the frequencies, magnitudes, phases, and even the orientations of these components are not always obvious from inspection of the pattern. Although the fast Fourier transform can provide a finite, sampled approximation to any desired spectrum, blind use of a packaged computer program sacrifices the intuitive insight of the analytic transform, which is particularly important for conceiving new experiments. This paper draws attention to some analytic techniques involving arrays of Dirac delta functions, which the author has found helpful in designing and understanding both real and *Gedanken* experiments in pattern perception.

Early studies with periodic spatial targets (Campbell & Robson, 1968; Schade, 1948; Westheimer, 1960) used rectilinear gratings that were periodic in one dimension of the visual field (usually horizontal), but aperiodic in the other. We will refer to such patterns as "one-dimensional," although any real visual stimulus necessarily stimulates a two-dimensional region of the retina. Properly designed, these one-dimensional targets can isolate individual frequencies for the measurement of contrast sensitivity, the study of orientation effects, and various other purposes.

Recently, however, it has been discovered that patterns requiring two dimensions to define their spatial periodicities produce certain visual effects not found with one-dimensional, rectilinear gratings. Examples of such two-dimensional patterns are checkerboards (Green, Corwin, & Zemon, 1976; Kelly, 1976; May & Matteson, 1976), grid patterns (Hermann, 1870; Schachar, 1976), and crossed sine-wave gratings (Carlson, Cohen, & Gorog, 1977). Circular gratings (Kelly, 1960) also require a two-dimensional description in Cartesian spatial frequency coordinates, but their properties have been discussed elsewhere (Kelly & Magnuski, 1975); this report is confined to rectilinear periodicities.

The researcher who is not familiar with multiple-integral transforms is likely to encounter pitfalls in working with such two-dimensional patterns, partly because some of their properties have no one-dimensional analogs; for example, rotation, anamorphic scale changes, and off-axis translation. Appropriate textbooks

are available (e.g., Bracewell, 1965; Papoulis, 1962; Sneddon, 1951), but are seldom consulted by those who work in pattern perception.

The purpose of this note is not to substitute for such consultation, but merely (1) to point out that some very useful analytic shortcuts are available for designing two-dimensional stimuli and calculating their Fourier spectra, and (2) to motivate the reader to delve into this subject deeply enough to understand all the tautologous properties of his stimuli before doing visual experiments. The following techniques, involving arrays of Dirac delta functions, should illustrate the point.¹

Let $q(x,y)$ be any transformable function of the two space coordinates x and y (in, e.g., degrees of visual angle). Then, if we define the Fourier spectrum of q as

$$Q(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q(x,y) e^{-j2\pi(xu+yv)} dx dy, \quad (1)$$

where u and v are the horizontal and vertical frequency coordinates (in, e.g., cycles per degree) and $j^2 = -1$, we can recover $q(x,y)$ from $Q(u,v)$ by the inverse transform,

$$q(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(u,v) e^{j2\pi(xu+yv)} du dv. \quad (2)$$

[By absorbing the 2π factor into the kernel of the transform, this modified Campbell and Foster (1931) notation simplifies most of the expressions that follow.] As a shorthand expression for the forward and inverse transforms in Equations 1 and 2, we will represent transform pairs as $q(x,y) \rightarrow Q(u,v)$ or $Q(u,v) \leftarrow q(x,y)$.

First, we recall a few useful theorems for Fourier transforms. Translation of a two-dimensional pattern shifts the phases of its spectrum according to the rule

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$q(x-a, y-b) \rightarrow Q(u,v)e^{-j2\pi(au+bv)}$. Scale changes are transformed according to $q(x/a, y/b) \rightarrow abQ(au, bv)$. Rotation of the pattern about the origin rotates the spectrum through the same angle. (The translation rule can be invoked for rotations about other points.) If both members of a transform pair are real, then a new pair can be obtained by interchanging the two functions. If $q(x,y)$ is a real, even function [i.e., $q(x,y) = q(-x, -y)$], then $Q(u,v)$ is also real and even. Finally, the transform of the convolution of two patterns is the product of their individual transforms; that is,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q(x,y)p(x'-x, y'-y) dx dy \rightarrow Q(u,v)P(u,v).$$

Next we introduce the two-dimensional Dirac delta function, $\delta(x,y)$, which has the property that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x,y) dx dy = 1,$$

although $\delta(x,y) = 0$ everywhere except at the origin. When this function occurs in a definite integral, the value of the integral is simply the value of the integrand where the arguments of δ are both zero. Thus, convolving any pattern, $q(x,y)$, with a δ function merely reproduces $q(x,y)$. Note that

$$\delta(x-x_0, y-y_0) \rightarrow e^{-j2\pi(x_0u+y_0v)}, \quad (3)$$

and, as a special case, $\delta(x,y) \rightarrow 1$. Among other things, this notation permits us to calculate Fourier series directly from the Fourier integral.

THE ON-AXIS ARRAY

Extending Woodward's (1953) rep function to two dimensions, we now define a regular orthogonal array of spatial frequency components as

$$\text{rep } \delta(u,v) = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \delta(u-n, v-m),$$

where n and m are integers. This array represents the coefficients of a two-dimensional Fourier series, in which every symmetric pair of delta functions is the transform of a sinusoidal pattern in (x,y) space. For example,

$$\delta(u+n, v) + \delta(u-n, v) \leftarrow 2 \cos 2\pi nx.$$

The sum of all these component waves approaches a regular array of delta functions in space, so that in the limit,

$$\text{rep } \delta(x,y) \rightarrow \text{rep } \delta(u,v). \quad (4)$$

Thus $\text{rep } \delta(x,y)$ is a self-reciprocal function; that is, it is its own Fourier transform, as illustrated in Figure 1. This relation can be used in various ways. For example,

Figure 1. Schematic diagram showing the central region of a two-dimensional array of Dirac delta functions (on the left), which transforms into the same array in the spatial frequency domain (on the right). Both members of this Fourier pair actually extend over infinite planes, and every plus sign represents a delta function of unit strength. (Pair 4 in the text.)

let $q(x,y)$ represent one period of any two-dimensionally periodic function, where $q(x,y) = 0$ if $|x| > \frac{1}{2}$ or $|y| > \frac{1}{2}$ and $q(x,y) \rightarrow Q(u,v)$ as usual. Then if we convolve $q(x,y)$ with $\text{rep } \delta(x,y)$, the convolution theorem yields

$$\text{rep } q(x,y) \rightarrow Q(u,v) \text{rep } \delta(u,v). \quad (5)$$

Note that the right-hand member of this pair is simply $Q(n,m)$; it represents the coefficients of the two-dimensional Fourier series for $\text{rep } q(x,y)$. The Fourier coefficients for periodic stimulus patterns can thus be obtained from the transform of a single period.

For example, let q be a square with sides equal to s (and $|s| < 1$), so that

$$q(x,y) = 1 \text{ for } |x| < s/2 \text{ and } |y| < s/2; \\ = 0 \text{ elsewhere.}$$

Then $\text{rep } q(x,y)$ becomes a regular array of squares. By varying s , we obtain a Hermann grid with variable spacing. [By giving $q(x,y)$ a pincushion shape, the Schachar pattern could be represented in the same way.]

Substituting the single square into Equation 1 and performing the indicated integration, we find $Q(u,v)$, the envelope of the Fourier coefficients in Equation 5. The discrete coefficients are:

$$Q(n,m) = \frac{\sin \pi ns}{\pi n} \frac{\sin \pi ms}{\pi m}. \quad (6)$$

Thus, a Hermann grid may contain not only diagonal frequency components ($n = \pm m$), but *all possible off-axis frequencies* ($n, m \neq 0$). Generally, some of these components are negative. If we set $s = \frac{1}{2}$, the even terms vanish (as in a one-dimensional square wave). At $s = 1$, neighboring squares touch, and all terms vanish; $\text{rep } q(x,y)$ is then a uniform field.

Of course, Pair 5 still holds when $q(x,y)$ is not uniform or even square. It can be used to find the Fourier coefficients for any small pattern that is duplicated as a regular array in two dimensions.

The Hermann-grid illusion is best obtained with $\frac{1}{2} < s < 1$, where most frequencies are present. How-

ever, the lowest (and strongest) frequencies of such a grid (e.g., its fundamental frequencies at $u = \pm 1$ and $v = \pm 1$) are always on the u and v axes of the spatial frequency domain.

THE OFF-AXIS ARRAY

On the other hand, the self-reciprocal regular array of Figure 1 is not quite suitable for analyzing patterns that have no horizontal or vertical frequency components, such as the checkerboard. Its axial component could be eliminated by a 45-deg rotation, but then it would contain twice the number of harmonics required, as we shall see later.

Another way of eliminating all axial components is simply to translate $\delta(u,v)$ by half an interval in both directions. From Pairs 3 and 4, we then find the inverse transform of this modified spectrum:

$$\text{rep } \delta(u - \frac{1}{2}, v - \frac{1}{2}) \leftarrow \cos \pi x \cos \pi y \text{ rep } \delta(x,y). \quad (7)$$

The resulting spatial pattern is an array of delta functions that alternate in sign along any row or column (but not along the diagonals), as shown in Figure 2. This alternating spatial array has no zero-frequency component, but its fundamental frequencies are lower than those of the all-positive array, and they are located on the diagonals of the frequency domain. All their even harmonics are missing.

Since Pair 7 is not self-reciprocal, a new pair is obtained by interchanging its spatial and frequency arrays, namely,

$$\text{rep } \delta(x - \frac{1}{2}, y - \frac{1}{2}) \rightarrow \cos \pi u \cos \pi v \text{ rep } \delta(u,v).$$

Here the spatial array is a translated version of the array in Figure 1. Therefore, only the phases of the corresponding frequency components are shifted, in accord with the translation theorem. Multiplying the frequency array by $\cos \pi u \cos \pi v$ changes the sign of every odd component (i.e., shifts its phase by 180 deg), but does not change the amplitudes.

A checkerboard can be constructed from Figure 2 in the same way that we constructed Hermann grids from Figure 1. Convolving the alternating spatial array of Pair 7 with a square of side s , we find the Fourier coefficients,

$$Q(n,m) = (2/\pi)^2 \frac{\cos \pi n s}{2n+1} \frac{\cos \pi m s}{2m+1}. \quad (8)$$

In our (n,m) notation, the fundamental frequencies now fall at $(-1,0)$, $(0,0)$, $(-1,-1)$, and $(0,-1)$, but every pair of terms that is symmetric in u and v corresponds to one spatial sinusoid, as usual. With $s = 1$, Equation 8 represents the same frequency spectrum that Kelly (1976) obtained for an infinite checkerboard by conventional Fourier-series methods (e.g., the horizontal and vertical factors vary as $1, -1/3, 1/5, -1/7$, etc.).

Figure 2. Here, every harmonic in Figure 1 has been translated half an interval in both u and v directions, which reverses the sign of alternate delta functions in the space domain (without changing their positions). Negative delta functions are shown by minus signs. (Pair 7 in the text.)

If $s < 1$ in Equation 8, then the light and dark squares of the checkerboard are separated by grid lines of intermediate tone. That changes the amplitudes of the Fourier components, *but not their frequencies*. Thus we must not regard this separated-square checkerboard as "a Hermann grid with alternating light and dark squares." The frequency spectrum of any such pattern is still an off-axis array (as in Figure 2), which has different properties from the Hermann-grid array of Figure 1, illustrating the fact that verbal descriptions of stimuli can be misleading.

Other variations on the checkerboard array can also be constructed. If we let $q(x,y)$ represent some other small pattern (instead of a uniform square), the same technique can be used to find the Fourier coefficients for any such pattern that alternates in sign on a regular array.

Since radiant energy is a nonnegative quantity, all realizable stimulus patterns must of course contain a zero-frequency component. According to Figures 1 and 2, however, this component plays an essential role in the Hermann grid, but not in the checkerboard. The same conclusion can be drawn from Kelly's (1976) derivation of the checkerboard spectrum. In order to calculate its Fourier coefficients, he treated his checkerboard as the product of one-dimensional, crossed square-wave gratings, alternating between $+1$ and -1 , with no zero-frequency component. When two patterns are combined by multiplication, new spatial frequencies are generated, making it difficult to visualize the formation of the resulting spectrum. But if we start with nonnegative, square-wave gratings, alternating between 0 and 2, for instance, their product is not a checkerboard, but a Hermann grid (with $s = 1/2$). Merely adding a zero-frequency component to the one-dimensional gratings changes their two-dimensional product from a checkerboard to a grid, bringing in all the on-axis components.

THE DIAGONAL ARRAY

A third class of patterns can be obtained simply by adding together the on-axis and off-axis spectra (and corresponding spatial arrays) from Figures 1 and 2. As shown in Figure 3, the resulting frequency

Figure 3. Fourier pair obtained by adding Pairs 4 and 7, which doubles the number of harmonics in the frequency domain and cancels the negative delta functions shown in Figure 2. The resulting diagonal arrays differ from Pair 4 by a 45-deg rotation and a $\sqrt{2}$ scale change. Here, dot size represents delta function strength (as in Kelly, 1976).

components are twice as dense, but the new spatial array is half as dense, because the positive and negative delta functions of Figure 2 alternately cancel and reinforce the all-positive ones of Figure 1.

Inspection of Figure 3 reveals that the same results must also be derivable from Pair 4 alone, by a scale change and a 45-deg rotation. Noting that the scale change affects the strength of the delta functions, we can write

$$\text{rep } \delta(u, v) + \text{rep } \delta(u - \frac{1}{2}, v - \frac{1}{2}) = 2 \text{ rep } \delta(u - v, u + v);$$

hence, the pair illustrated in Figure 3 is

$$\text{rep } \delta[(x - y)/2, (x + y)/2] \rightarrow 2 \text{ rep } \delta(u - v, u + v).$$

A change of scale converts this pair to the self-reciprocal pair,

$$\begin{aligned} \text{rep } \delta[(x - y)/\sqrt{2}, (x + y)/\sqrt{2}] \rightarrow \\ \text{rep } \delta[(u - v)/\sqrt{2}, (u + v)/\sqrt{2}], \end{aligned} \quad (9)$$

which is simply Pair 4 rotated 45 deg about the origin. The $\sqrt{2}$ factor in Pair 9 keeps the rotated delta functions at the same distances from the origin as in Pair 4.

Clearly, Pair 9 can be applied in the same ways as Pairs 4 and 7. For example, convolving $q(x, y)$ with the diagonal array, we obtain

$$\begin{aligned} \text{rep } q[(x - y)/\sqrt{2}, (x + y)/\sqrt{2}] \rightarrow \\ Q(u, v) \text{ rep } \delta[(u - v)/\sqrt{2}, (u + v)/\sqrt{2}]. \end{aligned}$$

By reinterpreting n and m , this pair could be used to find the Fourier coefficients for any small pattern repeated diagonally. These coefficients can also be found by rotating the results obtained with Pair 5, so Pair 9 is not essential for that purpose. However, it is required in the derivation of a more important pair, given below.

THE CLOSE-PACKED ARRAY

So far we have considered only orthogonal arrays, in which lines connecting the nearest neighbor delta functions would meet at 90-deg angles. This orthogonality is preserved under anamorphic scale changes for Pairs 4 and 7, but not for Pair 9. For example, if we stretch $\text{rep } \delta(u - v, u + v)$ in the vertical direction in Figure 3, the scale-change theorem requires that the corresponding spatial array be compressed in the vertical direction; neither array remains orthogonal. Lines connecting neighboring points would then form diamonds instead of squares, vertically elongated in the frequency domain and horizontally in the space domain.

If the v scale is increased (and the y scale decreased) by exactly $\sqrt{3}$, these diamonds assume the shape of two contiguous, equilateral triangles; each delta function then has six nearest neighbors instead of four. Thus the pair of hexagonal arrays shown in Figure 4 can be readily derived by applying the required scale change to Pair 9. The result is:

$$\begin{aligned} \text{rep } \delta[(\sqrt{3}x - y)/(\sqrt[4]{3}\sqrt{2}), (\sqrt{3}x + y)/(\sqrt[4]{3}\sqrt{2})] \rightarrow \\ \text{rep } \delta[(u - \sqrt{3}v)/(\sqrt[4]{3}\sqrt{2}), (u + \sqrt{3}v)/(\sqrt[4]{3}\sqrt{2})]. \end{aligned} \quad (10)$$

In spite of its simple relation to Pair 9, this Fourier pair has some important properties that are lacking in orthogonal arrays. It is not strictly self-reciprocal, but one of its members can be obtained from the other by a 30-deg rotation. It maintains the even symmetry of Pair 9 about the horizontal and vertical axes, of course, but it is also symmetric about four other meridians, 30 deg apart.

By analogy with the square in Pair 5, $q(x, y)$ should be a hexagon in order to exactly fill all of the space in Figure 4, thus creating a uniform field. When that happens, $Q(u, v)$ must go through zero at every delta function in the frequency domain except the origin, in order to suppress all the Fourier components. Since these components are hexagonally symmetric, $Q(u, v)$ must be hexagonally symmetric; it must also oscillate

Figure 4. Fourier pair of close-packed arrays, obtained from Figure 3 by an anamorphic scale change ($\sqrt{3}$ vertical magnification). Each delta function has six, equispaced, nearest neighbors, but the arrays in the frequency and space domains are not identical; they differ by a 30-deg rotation. (Pair 10 in the text.)

with decreasing amplitude along all of its six axes of symmetry.

Convolving a smaller hexagon with the spatial array of Figure 4 creates a honeycomb pattern in space. This also increases the scale of $Q(u,v)$ so that its zero crossings no longer coincide with the Fourier components, but its form is unchanged. Therefore, the Fourier coefficients of the honeycomb pattern are also symmetric about the same six axes, with changes of sign (and generally decreasing magnitudes) as their spatial frequencies increase. For example, suppose $q(x,y)$ is exactly half its space-filling size. Then the resulting pattern will contain square-wave profiles in three directions, 60 deg apart. To find the transform, we must double the scale of $Q(u,v)$; therefore, its zero crossings cancel the even harmonics along the corresponding three axes of symmetry in the frequency domain.

An intriguing hierarchy of spatial patterns can be formed on the basis of increasing symmetry of their fundamental frequencies. Thus one symmetric pair of Fourier components forms the fundamental of a one-dimensional grating; two symmetric, orthogonal pairs, the fundamental of a checkerboard; and three symmetric, equispaced pairs, the fundamental of a hexagonal array. Also, the magnitude of each component decreases as more components are required to form the desired pattern at constant contrast. Kelly (1976) made use of this property in comparing the thresholds for bar gratings and checkerboards; appropriate hexagonal patterns should produce greater effects.

However, hexagonal symmetry probably represents the point of diminishing returns; it seems unlikely that four, five, or six pairs of fundamental components would reveal any properties not seen with two or three. In the limit, a pattern with many fundamental components approaches the transform of a continuous delta function sheet with rotational symmetry, that is, a zero-order Bessel function in polar coordinates (Kelly, 1960). Such patterns have already been tried. Kelly and Magnuski (1975) did obtain greater effects with circular periodic targets than Kelly (1976) did with checkerboards, but only at high spatial frequencies.

Perhaps the most important property of Pair 10 is that it represents the well known close-packing array for round objects in two dimensions. In this sense, the frequency domain of Figure 4 is as densely filled with Fourier components as it can be. As a sampling function in the space domain, a hexagonal array permits smoother interpolation than an orthogonal one (Petersen & Middleton, 1962), which is advantageous in image-processing studies,² including retinal modeling (Ochs & Peinado, 1975). In fact, since the retinal mosaic of receptor cells is close-packed (locally, at least), this is the spatial sampling function used by the visual process.

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NOTES

1. An engineering course that emphasizes the applications of two-dimensional delta function arrays is given at Purdue University, its text will shortly be published (Huang, in press).
2. In 1962, W. F. Schreiber and his students at the Massachusetts Institute of Technology produced demonstrations of hexagonally de-sampled pictorial material. Even with matched sampling densities, the hexagonal structure of these pictures was much less visible than the comparable orthogonal structure.

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