The analysis of multifactor experiments designed to contrast the within- and between-subjects manipulation of the independent variables

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Experiments are frequently performed in which the major question of interest is whether the effect of the substantive independent variable depends on its manipulation within as opposed to between subjects. Erlebacher (1977) discussed the design of such experiments in which one independent variable is manipulated, and presented a method of analysis by an analysis of variance technique. This paper discusses the analysis of three different experimental designs in which the within-subjects/between-subjects design type is one factor and *two* substantive independent variables are manipulated. A rule is also presented that can be used to generate statistical tests in a generalized multifactor design.

Within- and between-subjects designs of experiments have often been used interchangeably in psychology. Many experiments, though, have shown that the two design types sometimes yield different results. Erlebacher (1977) listed a number of experiments in several different areas of psychology that were designed specifically to investigate the design-type effect. Typically, in such an experiment, one has a substantive independent variable which is varied between groups. One additional group of subjects receives all the levels of the independent variable. The major question of concern is whether the difference in means among the conditions in the between-subjects design is equal to the difference in means among the conditions in the within-subjects design. If we explicitly designate design type as a second independent variable, we may phrase the major question of concern as whether there exists an interaction between the substantive independent variable and the design type.

Erlebacher (1977) presented an analysis of variance procedure for analyzing the results of experiments in which the effect of *one* substantive independent variable and its interaction with design type is investigated. This paper presents the appropriate analysis of variance procedure for three experimental design cases in which there exist *two* substantive independent variables in addition to the design variable. The pro-

cedures may be generalized to the case of three or more substantive independent variables.

THE THREE CASES

In all three cases there are two substantive independent variables, A and B. Let A have I levels and let there be J levels of variable B. The design factor is designated D. In each case there are two levels of the design factor. These are designated D_1 and D_2 .¹ In the first case, variable B is manipulated between subjects. Variable A, however, is a within-subjects variable at D_1 but a between-subjects variable at D_2 . We designate this as Case AwBb-AbBb. In Case AwBw-AbBw, variable B is always manipulated within subjects, but variable A is a within-subjects variable at D_1 and a between-subjects variable at D_2 . Finally, in Case AwBw-AbBb, both variables are within-subjects variables at D_1 but between-subjects at D_2 . Let there be n subjects in each of the independent groups.

Case 1: AwBb-AbBb

In the first case, the experiment consists of $J(I+1)$ independent groups and, therefore, $nJ(I+1)$ subjects. At D_1 , there are J groups. These groups are treated according to the different levels of variable B. Each of the nJ subjects at D_1 , however, receives all I treatments of variable A. At D_2 , there are IJ independent groups and, therefore, nIJ subjects. Each of these groups receives one of the IJ treatments resulting from the $I \times J$ crossing of variables A and B.

The additive score model has X_{ijk} s, the score of

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subject s in design type k and in levels i and j of variables A and B , respectively. We have

$$\begin{aligned} X_{ijks} = & \mu_{\dots} + \alpha_{i\dots} + \beta_{\dots j} + \delta_{\dots k} \\ & + (\alpha\beta)_{ij\dots} + (\alpha\delta)_{i\dots k} + (\beta\delta)_{\dots jk} \\ & + (\alpha\beta\delta)_{ijk\dots} + \pi_{ijks} + (\alpha\pi)_{ijks} \\ & + \epsilon_{ijks}, \end{aligned}$$

where $\mu_{\dots}$ is a common value that is part of all scores; $\alpha_{i\dots}$ is the contribution of variable A in its i th level (let $i = 1, 2, \dots, I$); $\beta_{\dots j}$ is the contribution of variable B in the j th level (let $j = 1, 2, \dots, J$); $\delta_{\dots k}$ is the contribution of the design type ($k = 1, 2$); the next four terms represent the contributions of the first- and second-order interactions among the three independent variables; π_{ijks} is the contribution due to subject s in condition ijk (there are n subjects in each of the conditions, i.e., $s = 1, 2, \dots, n$); $(\alpha\pi)_{ijks}$ represents the interaction of subjects with independent variable A ; ϵ_{ijks} is random error.

The design type, D , and the substantive independent variables, A and B , are considered to be fixed variables. *Subjects* is a random variable. The usual

analysis of variance assumptions are made, with the exception that, in design-type D_1 ,

$$\sum_i (\alpha\pi)_{ij1s} = 0,$$

while, in design-type D_2 ,

$$E(\alpha\pi)_{ij2s} = 0$$

and

$$E(\alpha\pi)_{ijks}^2 = \sigma_{As}^2,$$

E indicating expected value.

The total sum of squares should be partitioned according to the sources of variation presented in Table 1. The table also indicates the degrees of freedom and the definitional formulas for the sums of squares.

The means needed for the sums of squares can be expressed in terms of the effects from the model. This is done in Table 2. For purposes of arriving at the expected values of mean squares, note that in D_1 , since the *same* subjects are measured in all levels of A , $\pi_{\dots j1s}$ is based on but one score, $\pi_{\dots j1}$ is based on n scores and $\pi_{\dots 1}$ is based on Jn scores. This

Table 1
Partitioning of the Total Sum of Squares for Case 1

Source of Variation	df	Sum of Squares
A	$I - 1$	$2Jn\sum_i (\bar{X}_{i\dots} - \bar{X}_{\dots})^2$
B	$J - 1$	$2In\sum_j (\bar{X}_{\dots j} - \bar{X}_{\dots})^2$
$A \times B$	$(I - 1)(J - 1)$	$2n\sum_{ij} (\bar{X}_{ij\dots} - \bar{X}_{i\dots} - \bar{X}_{\dots j} + \bar{X}_{\dots})^2$
Design Type (D)	1	$IJn\sum_k (\bar{X}_{\dots k} - \bar{X}_{\dots})^2$
$A \times D$	$I - 1$	$Jn\sum_{ik} (\bar{X}_{ik\dots} - \bar{X}_{i\dots} - \bar{X}_{\dots k} + \bar{X}_{\dots})^2$
$B \times D$	$J - 1$	$In\sum_{jk} (\bar{X}_{\dots jk} - \bar{X}_{\dots j} - \bar{X}_{\dots k} + \bar{X}_{\dots})^2$
$A \times B \times D$	$(I - 1)(J - 1)$	$n\sum_{ijk} (\bar{X}_{ijk\dots} - \bar{X}_{ij\dots} - \bar{X}_{i\dots k} - \bar{X}_{\dots jk} + \bar{X}_{i\dots} + \bar{X}_{\dots j} + \bar{X}_{\dots k} - \bar{X}_{\dots})^2$
Subjects/ $B(D_1 \text{ only}) [s/B(D_1)]$	$J(n - 1)$	$I\sum_{js} (\bar{X}_{\dots j1s} - \bar{X}_{\dots j1})^2$
$A \times s/B(D_1)$	$J(I - 1)(n - 1)$	$\sum_{ijs} (\bar{X}_{ij1s} - \bar{X}_{ij1} - \bar{X}_{\dots j1s} + \bar{X}_{\dots j1})^2$
$s/AB(D_2)$	$IJ(n - 1)$	$\sum_{ijs} (\bar{X}_{ij2s} - \bar{X}_{ij2})^2$
Total	$2IJn - 1$	$\sum_{ijk} (\bar{X}_{ijks} - \bar{X}_{\dots})^2$

Table 2
Means Expressed in Terms of Model Effects: Case 1

Mean	Expression in Effects
$\bar{X}_{\dots}$	$\mu_{\dots} + \frac{\pi_{\dots 1.}}{2} + \frac{\pi_{\dots 2.}}{2} + \frac{(\bar{\alpha}\bar{\pi})_{\dots 2.}}{2} + \bar{\epsilon}_{\dots}$
$\bar{X}_{i\dots}$	$\mu_{\dots} + \alpha_{i\dots} + \frac{\pi_{\dots 1.}}{2} + \frac{\pi_{i\dots 2.}}{2} + \frac{(\bar{\alpha}\bar{\pi})_{i\dots 1.}}{2} + \frac{(\bar{\alpha}\bar{\pi})_{i\dots 2.}}{2} + \bar{\epsilon}_{i\dots}$
$\bar{X}_{\dots j}$	$\mu_{\dots} + \beta_{\dots j} + \frac{\pi_{\dots j1.}}{2} + \frac{\pi_{\dots j2.}}{2} + \frac{(\bar{\alpha}\bar{\pi})_{\dots j2.}}{2} + \bar{\epsilon}_{\dots j}$
$\bar{X}_{ij\dots}$	$\mu_{\dots} + \alpha_{i\dots} + \beta_{\dots j} + (\alpha\beta)_{ij\dots} + \frac{\pi_{\dots j1.}}{2} + \frac{\pi_{ij2.}}{2} + \frac{(\bar{\alpha}\bar{\pi})_{ij1.}}{2} + \frac{(\bar{\alpha}\bar{\pi})_{ij2.}}{2} + \bar{\epsilon}_{ij\dots}$
$\bar{X}_{\dots 1.}$	$\mu_{\dots} + \delta_{\dots 1.} + \bar{\pi}_{\dots 1.} + \bar{\epsilon}_{\dots 1.}$
$\bar{X}_{\dots 2.}$	$\mu_{\dots} + \delta_{\dots 2.} + \bar{\pi}_{\dots 2.} + (\bar{\alpha}\bar{\pi})_{\dots 2.} + \bar{\epsilon}_{\dots 2.}$
$\bar{X}_{i\dots 1.}$	$\mu_{\dots} + \alpha_{i\dots} + \delta_{\dots 1.} + (\alpha\delta)_{i\dots 1.} + \bar{\pi}_{\dots 1.} + (\bar{\alpha}\bar{\pi})_{i\dots 1.} + \bar{\epsilon}_{i\dots 1.}$
$\bar{X}_{i\dots 2.}$	$\mu_{\dots} + \alpha_{i\dots} + \delta_{\dots 2.} + (\alpha\delta)_{i\dots 2.} + \bar{\pi}_{\dots 2.} + (\bar{\alpha}\bar{\pi})_{i\dots 2.} + \bar{\epsilon}_{i\dots 2.}$
$\bar{X}_{\dots j1.}$	$\mu_{\dots} + \beta_{\dots j} + \delta_{\dots 1.} + (\beta\delta)_{\dots j1.} + \bar{\pi}_{\dots j1.} + \bar{\epsilon}_{\dots j1.}$
$\bar{X}_{\dots j2.}$	$\mu_{\dots} + \beta_{\dots j} + \delta_{\dots 2.} + (\beta\delta)_{\dots j2.} + \bar{\pi}_{\dots j2.} + (\bar{\alpha}\bar{\pi})_{\dots j2.} + \bar{\epsilon}_{\dots j2.}$
$\bar{X}_{ij1\dots}$	$\mu_{\dots} + \alpha_{i\dots} + \beta_{\dots j} + \delta_{\dots 1.} + (\alpha\beta)_{ij\dots} + (\alpha\delta)_{i\dots 1.} + (\beta\delta)_{\dots j1.} + (\alpha\beta\delta)_{ij1\dots} + \bar{\pi}_{\dots j1.} + (\bar{\alpha}\bar{\pi})_{ij1\dots} + \bar{\epsilon}_{ij1\dots}$
$\bar{X}_{ij2\dots}$	$\mu_{\dots} + \alpha_{i\dots} + \beta_{\dots j} + \delta_{\dots 2.} + (\alpha\beta)_{ij\dots} + (\alpha\delta)_{i\dots 2.} + (\beta\delta)_{\dots j2.} + (\alpha\beta\delta)_{ij2\dots} + \bar{\pi}_{ij2.} + (\bar{\alpha}\bar{\pi})_{ij2.} + \bar{\epsilon}_{ij2.}$
$\bar{X}_{\dots j1s}$	$\mu_{\dots} + \beta_{\dots j} + \delta_{\dots 1.} + (\beta\delta)_{\dots j1.} + \bar{\pi}_{\dots j1s} + \bar{\epsilon}_{\dots j1s}$
X_{ij1s}	$\mu_{\dots} + \alpha_{i\dots} + \beta_{\dots j} + \delta_{\dots 1.} + (\alpha\beta)_{ij\dots} + (\alpha\delta)_{i\dots 1.} + (\beta\delta)_{\dots j1.} + (\alpha\beta\delta)_{ij1\dots} + \pi_{\dots j1s} + (\alpha\pi)_{ij1s} + \epsilon_{ij1s}$

differs from the analogous values in D_2 , which are all based on I times that many scores. This fact has implications for the values of variances based on these terms.

The expected values of the mean squares associated with each of the sources of variation are given in Table 3. The S^2 refer to "variances" for mixed and fixed effects, while the σ^2 are true variances for purely random effects. The subscripts refer to the effects due to the sources of variation. Double or triple subscripts indicate a variance due to the effect of an interaction. The final value in each line, σ_e^2 , refers to error variance.

The expected values of the mean squares indicate that, in order to make a statistical test of the effectiveness of any potential source of variation, one must form "quasi-F" ratios by dividing the mean square for the effect being tested by a linear combination of mean squares. Two different denominators need to be used. To test the effect of A, $A \times B$, $A \times D$, and $A \times B \times D$, the denominator for the quasi-F ratio is

$$\text{Denominator 1a} = [MS_{A \times B / B(D_1)} + MS_{S / AB(D_2)}] / 2.$$

For testing the effect of B, D, and $B \times D$, one uses

$$\text{Denominator 1b} = [MS_{S / B(D_1)} + MS_{S / AB(D_2)}] / 2.$$

The last column of Table 3 indicates which of the two denominators is to be used for each of the ratios. The degrees of freedom associated with each of the denominators must be approximated. Satterthwaite's (1946) formula can be used. His procedure determines that the number of degrees of freedom, ν , associated with the sum (or mean) of two mean squares, MS_1 and MS_2 , with associated degrees of freedom ν_1 and ν_2 , respectively, is

$$\nu = (MS_1 + MS_2)^2 / [MS_1^2 / \nu_1 + (MS_2^2 / \nu_2)].$$

Case 2: AwBw-AbBw

In the second case, the experiment has $I + 1$ independent groups and, hence, $n(I + 1)$ subjects. In design D_1 , there is but a single group. This group receives all IJ treatments resulting from the $I \times J$ crossing of variables A and B. In design D_2 , there are I independent groups. Each is treated according to

Table 3
Expected Values of Mean Squares for Case 1

Source of Variation	Expected Value of Mean Square	Denominator
A	$2JnS_A^2 + \frac{1}{2}\sigma_s^2 + \frac{(2I-1)}{2I}S_{As}^2 + \sigma_\epsilon^2$	1a
B	$2InS_B^2 + \frac{(I+1)}{2}\sigma_s^2 + \frac{(I-1)}{2I}S_{As}^2 + \sigma_\epsilon^2$	1b
A × B	$2nS_{AB}^2 + \frac{1}{2}\sigma_s^2 + \frac{(2I-1)}{2I}S_{As}^2 + \sigma_\epsilon^2$	1a
D	$IJnS_D^2 + \frac{(I+1)}{2}\sigma_s^2 + \frac{(I-1)}{2I}S_{As}^2 + \sigma_\epsilon^2$	1b
A × D	$JnS_{AD}^2 + \frac{1}{2}\sigma_s^2 + \frac{(2I-1)}{2I}S_{As}^2 + \sigma_\epsilon^2$	1a
B × D	$InS_{BD}^2 + \frac{(I+1)}{2}\sigma_s^2 + \frac{(I-1)}{2I}S_{As}^2 + \sigma_\epsilon^2$	1b
A × B × D	$nS_{ABD}^2 + \frac{1}{2}\sigma_s^2 + \frac{(2I-1)}{2I}S_{As}^2 + \sigma_\epsilon^2$	1a
s/B(D ₁)	$I\sigma_s^2 + \sigma_\epsilon^2$	
Axs/B(D ₁)	$S_{As}^2 + \sigma_\epsilon^2$	
s/AB(D ₂)	$\sigma_s^2 + \frac{(I-1)}{I}S_{As}^2 + \sigma_\epsilon^2$	

one of the I levels of variable A. All subjects, however, receive all of the J treatments of variable B. and

The model is slightly different from Case 1. Here we have

$$E(\alpha\pi)_{i \cdot ks}^2 = \sigma_{As}^2$$

$$E(\alpha\beta\pi)_{ijk s}^2 = \sigma_{ABs}^2$$

$$\begin{aligned} X_{ijks} = & \mu \dots + \alpha_{i \dots} + \beta_{\cdot j \dots} + \delta_{\cdot \cdot k} \cdot \\ & + (\alpha\beta)_{ij \cdot \cdot} + (\alpha\delta)_{i \cdot k \cdot} + (\beta\delta)_{\cdot jk} \cdot \\ & + (\alpha\beta\delta)_{ijk \cdot} + \pi_{i \cdot ks} + (\alpha\pi)_{i \cdot ks} \\ & + (\beta\pi)_{ijk s} + (\alpha\beta\pi)_{ijk s} + \epsilon_{ijk s} \cdot \end{aligned}$$

The first eight terms on the right are exactly the same as those in Case 1. The next two terms have a dot replacing the j. The next two terms are new and are due to the B × Subject and the A × B × Subject interactions. The last term is due to random error. Other than the common analysis of variance assumptions, here we assume that in D₁

$$\sum_i (\alpha\pi)_{i \cdot 1s} = \sum_{ij} (\alpha\beta\pi)_{ij1s} = 0,$$

while in D₂

$$E(\alpha\pi)_{i \cdot 2s} = E(\alpha\beta\pi)_{ij2s} = 0$$

Table 4 indicates the Case 2 sources of variation in the partitioning of the total sum of squares, the associated degrees of freedom, and the definitional formulas for the sums of squares. Note that the first seven sources of variation and their sums of squares are unchanged from Case 1.

Table 5 indicates the expression of the means needed for the sums of squares in terms of the model effects. Means with expressions unchanged from Case 1 are not listed. In order to arrive at expected values of mean squares, we must note differences in the number of independent scores going into the means of the two design types. In D₁, $\pi_{\cdot \cdot 1s}$ is based on one score, $\bar{\pi}_{\cdot \cdot 1}$ is based on n scores, $(\beta\pi)_{\cdot j1s}$ is based on one score, and $(\bar{\beta}\pi)_{\cdot j1}$ is based on n scores, whereas the analogous terms in D₂ are based on I times that many scores.

Table 6 shows the expected values of the mean squares associated with each source of variation. Again, quasi-F ratios are needed in order to make statistical tests. Four different denominators are

Table 4
Partitioning of the Total Sum of Squares for Case 2

Source of Variation	df	Sum of Squares
A, B, A × B, D, A × D, B × D, A × B × D	Same as Case 1	Same as Case 1
Subjects (D ₁ only) [s(D ₁)]	n - 1	$IJ\Sigma(\bar{X}_{...1s} - \bar{X}_{...1.})^2$
Axs(D ₁)	(I - 1)(n - 1)	$J\Sigma\Sigma(\bar{X}_{i.1s} - \bar{X}_{i.1.} - \bar{X}_{...1s} + \bar{X}_{...1.})^2$
Bxs(D ₁)	(J - 1)(n - 1)	$I\Sigma\Sigma(\bar{X}_{.j1s} - \bar{X}_{.j1.} - \bar{X}_{...1s} + \bar{X}_{...1.})^2$
AxBxs(D ₁)	(I - 1)(J - 1)(n - 1)	$\Sigma\Sigma\Sigma(X_{ij1s} - \bar{X}_{ij1.} - \bar{X}_{i.1s} - \bar{X}_{.j1s} + \bar{X}_{i.1.} + \bar{X}_{.j1.} + \bar{X}_{...1s} - \bar{X}_{...1.})^2$
s/A(D ₂ only) [s/A(D ₂)]	I(n - 1)	$J\Sigma\Sigma(\bar{X}_{i.2s} - \bar{X}_{i.2.})^2$
Bxs/A(D ₂)	I(J - 1)(n - 1)	$\Sigma\Sigma\Sigma(X_{ij2s} - \bar{X}_{i.2s} - \bar{X}_{.j2.} + \bar{X}_{i.2.})^2$
Total	2IJn - 1	$\Sigma\Sigma\Sigma\Sigma(X_{ijk s} - \bar{X}_{...})^2$

Table 5
Means Expressed in Terms of Model Effects: Case 2

Mean	Expression in Effects
$\bar{X}_{j..}$	$\mu_{...} + \beta_{j..} + \frac{\bar{\pi}_{...1.}}{2} + \frac{\bar{\pi}_{...2.}}{2} + \frac{(\bar{\alpha}\bar{\pi})_{...2.}}{2} + \frac{(\bar{\beta}\bar{\pi})_{...j1.}}{2} + \frac{(\bar{\beta}\bar{\pi})_{...j2.}}{2} + \frac{(\bar{\alpha}\bar{\beta}\bar{\pi})_{...j2.}}{2} + \bar{\epsilon}_{j..}$
$\bar{X}_{ij..}$	$\mu_{...} + \alpha_{i...} + \beta_{j..} + (\alpha\beta)_{ij..} + \frac{\bar{\pi}_{...1.}}{2} + \frac{\bar{\pi}_{i.2.}}{2} + \frac{(\bar{\alpha}\bar{\pi})_{i.1.}}{2} + \frac{(\bar{\alpha}\bar{\pi})_{i.2.}}{2} + \frac{(\bar{\beta}\bar{\pi})_{.j1.}}{2} + \frac{(\bar{\beta}\bar{\pi})_{ij2.}}{2} + \frac{(\alpha\beta\bar{\pi})_{ij1.}}{2} + \frac{(\alpha\beta\bar{\pi})_{ij2.}}{2} + \bar{\epsilon}_{ij..}$
$\bar{X}_{.j1.}$	$\mu_{...} + \beta_{j..} + \delta_{...1.} + (\beta\delta)_{.j1.} + \bar{\pi}_{...1.} + (\bar{\beta}\bar{\pi})_{.j1.} + \bar{\epsilon}_{.j1.}$
$\bar{X}_{.j2.}$	$\mu_{...} + \beta_{j..} + \delta_{...2.} + (\beta\delta)_{.j2.} + \bar{\pi}_{...2.} + (\bar{\alpha}\bar{\pi})_{...2.} + (\bar{\beta}\bar{\pi})_{.j2.} + (\bar{\alpha}\bar{\beta}\bar{\pi})_{.j2.} + \bar{\epsilon}_{.j2.}$
$\bar{X}_{ij1.}$	$\mu_{...} + \alpha_{i...} + \beta_{j..} + \delta_{...1.} + (\alpha\beta)_{ij..} + (\alpha\delta)_{i.1.} + (\beta\delta)_{.j1.} + (\alpha\beta\delta)_{ij1.} + \bar{\pi}_{...1.} + (\bar{\alpha}\bar{\pi})_{i.1.} + (\bar{\beta}\bar{\pi})_{.j1.} + (\bar{\alpha}\bar{\beta}\bar{\pi})_{ij1.} + \bar{\epsilon}_{ij1.}$
$\bar{X}_{ij2.}$	$\mu_{...} + \alpha_{i...} + \beta_{j..} + \delta_{...2.} + (\alpha\beta)_{ij..} + (\alpha\delta)_{i.2.} + (\beta\delta)_{.j2.} + (\alpha\beta\delta)_{ij2.} + \bar{\pi}_{i.2.} + (\bar{\alpha}\bar{\pi})_{i.2.} + (\bar{\beta}\bar{\pi})_{ij2.} + (\bar{\alpha}\bar{\beta}\bar{\pi})_{ij2.} + \bar{\epsilon}_{ij2.}$
$\bar{X}_{...1s}$	$\mu_{...} + \delta_{...1.} + \pi_{...1s} + \bar{\epsilon}_{...1s}$
$\bar{X}_{i.1s}$	$\mu_{...} + \alpha_{i...} + \delta_{...1.} + (\alpha\delta)_{i.1.} + \pi_{...1s} + (\alpha\pi)_{i.1s} + \bar{\epsilon}_{i.1s}$
$\bar{X}_{.j1s}$	$\mu_{...} + \beta_{j..} + \delta_{...1.} + (\beta\delta)_{.j1.} + \pi_{...1s} + (\beta\pi)_{.j1s} + \bar{\epsilon}_{.j1s}$
X_{ij1s}	$\mu_{...} + \alpha_{i...} + \beta_{j..} + \delta_{...1.} + (\alpha\beta)_{ij..} + (\alpha\delta)_{i.1.} + (\beta\delta)_{.j1.} + (\alpha\beta\delta)_{ij1.} + \pi_{...1s} + (\alpha\pi)_{i.1s} + (\beta\pi)_{.j1s} + (\alpha\beta\pi)_{ij1s} + \epsilon_{ij1s}$

Table 5 Continued

$\bar{X}_{1..2s}$	$\mu_{...} + \alpha_{i...} + \delta_{...2.} + (\alpha\delta)_{i..2.} + \pi_{1..2s} + (\alpha\pi)_{i..2s} + \epsilon_{1..2s}$
X_{ij2s}	$\mu_{...} + \alpha_{i...} + \beta_{.j..} + \delta_{...2.} + (\alpha\beta)_{ij..} + (\alpha\delta)_{i..2.} + (\beta\delta)_{.j2.} + (\alpha\beta\delta)_{ij2.} + \pi_{1..2s} + (\alpha\pi)_{i..2s} + (\beta\pi)_{.j2s} + (\alpha\beta\pi)_{ij2s} + \epsilon_{ij2s}$

NOTE—Means with expressions unchanged from Case 1 are not listed.

Table 6
Expected Values of Mean Squares for Case 2

Source of Variation	Expected Value of Mean Square	Denominator
A	$2JnS_A^2 + \frac{J}{2}\sigma_s^2 + \frac{J(2I-1)}{2I}S_{As}^2 + \sigma_\epsilon^2$	2a
B	$2InS_B^2 + \frac{(I+1)}{2}S_{Bs}^2 + \frac{(I-1)}{2I}S_{ABs}^2 + \sigma_\epsilon^2$	2b
A × B	$2nS_{AB}^2 + \frac{1}{2}S_{Bs}^2 + \frac{(2I-1)}{2I}S_{ABs}^2 + \sigma_\epsilon^2$	2c
D	$IJnS_D^2 + \frac{J(I+1)}{2}\sigma_s^2 + \frac{J(I-1)}{2I}S_{As}^2 + \sigma_\epsilon^2$	2d
A × D	$JnS_{AD}^2 + \frac{J}{2}\sigma_s^2 + \frac{J(2I-1)}{2I}S_{As}^2 + \sigma_\epsilon^2$	2a
B × D	$InS_{BD}^2 + \frac{(I+1)}{2}S_{Bs}^2 + \frac{(I-1)}{2I}S_{ABs}^2 + \sigma_\epsilon^2$	2b
A × B × D	$nS_{ABD}^2 + \frac{1}{2}S_{Bs}^2 + \frac{(2I-1)}{2I}S_{ABs}^2 + \sigma_\epsilon^2$	2c
s(D ₁)	$IJ\sigma_s^2 + \sigma_\epsilon^2$	
Axs(D ₁)	$JS_{As}^2 + \sigma_\epsilon^2$	
Bxs(D ₁)	$IS_{Bs}^2 + \sigma_\epsilon^2$	
AxBxs(D ₁)	$S_{ABs}^2 + \sigma_\epsilon^2$	
s/A(D ₂)	$J\sigma_s^2 + \frac{J(I-1)}{I}S_{As}^2 + \sigma_\epsilon^2$	
Bxs/A(D ₂)	$S_{Bs}^2 + \frac{(I-1)}{I}S_{ABs}^2 + \sigma_\epsilon^2$	

needed to test all effects. Each is determined by taking the arithmetic mean of two mean squares. First,

$$\text{Denominator 2a} = [MS_{Axs(D_1)} + MS_{s/A(D_2)}]/2$$

is to be used to test the effect of A and the A × D interaction. To test the effect of B and the B × D interaction, one uses

$$\text{Denominator 2b} = [M_{Bxs(D_1)} + MS_{Bxs/A(D_2)}]/2.$$

To test the A × B and the A × B × D interactions, one uses

$$\text{Denominator 2c} = [MS_{AxBxs(D_1)} + MS_{Bxs/A(D_2)}]/2.$$

Finally,

$$\text{Denominator 2d} = [MS_{s(D_1)} + MS_{s/A(D_2)}]/2$$

is used to test the main effect of D. The last column of Table 6 indicates the different denominators to be used. The number of degrees of freedom associated

with each of the denominators can again be estimated using Satterthwaite's (1946) formula.

Case 3: AwBw-AbBb

In the last case, the within-subjects between-subjects comparison is of interest for both substantive independent variables. The experiment consists of $IJ + 1$ independent groups requiring $n(IJ + 1)$ subjects. At D_1 there is a single group of subjects. Each of the subjects in this group receives all of the IJ treatments stemming from the $I \times J$ crossing of variables A and B. This is exactly equivalent to design D_1 in the last case, AwBw-AbBw. At D_2 there are IJ independent groups. Each of these groups receives one of the IJ treatments resulting from the $I \times J$ crossing of variables A and B. This D_2 is the same D_2 design as in the first case, AwBb-AbBb.

The model for Case 3 has

$$\begin{aligned} X_{ijks} = & \mu_{\dots} + \alpha_{i\dots} + \beta_{\dots j\dots} + \delta_{\dots k\dots} \\ & + (\alpha\beta)_{ij\dots} + (\alpha\delta)_{i\dots k\dots} + (\beta\delta)_{\dots jk\dots} \\ & + (\alpha\beta\delta)_{ijk\dots} + \pi_{ijks} + (\alpha\pi)_{ijks} + (\beta\pi)_{ijks} \\ & + (\alpha\beta\pi)_{ijks} + \varepsilon_{ijks}. \end{aligned}$$

The terms here are the same as those for Case 2, except that the π and $(\alpha\pi)$ terms are differentiated according to j in this model; in the Case 2 model the j subscript on these two terms can be replaced by a dot. The normal analysis of variance assumptions are modified here by differentiating some terms between the two design types. In D_1

$$\sum_i (\alpha\pi)_{i\dots 1s} = \sum_j (\beta\pi)_{\dots j1s} = \sum_i (\alpha\beta\pi)_{ij1s} = \sum_j (\alpha\beta\pi)_{ij1s} = 0,$$

while only the expected values of those terms equal zero in D_2 :

$$E(\alpha\pi)_{ij2s} = E(\beta\pi)_{ij2s} = E(\alpha\beta\pi)_{ij2s} = 0.$$

The partitioning of the total sum of squares is the same for Cases 2 and 3, with the exception that the

last two terms in the Case 2 partitioning, $s/A(D_2)$ and $Bxs/A(D_2)$, are replaced by a single source of variation, subjects/ $AB(D_2)$ only, $[s/AB(D_2)]$, which has degrees of freedom and sum of squares:

$$df = IJ(n - 1)$$

$$SS = \sum_i \sum_j \sum_s (X_{ij2s} - \bar{X}_{ij2.})^2.$$

Table 7 gives the expression of the means appearing in the sums of squares in terms of the model effects. Means whose expressions are unchanged from Case 2 do not appear in the table. Furthermore, because the changes in the expressions for $\bar{X}_{\dots j\dots}$, $\bar{X}_{ij\dots}$, $\bar{X}_{\dots j2.}$, $\bar{X}_{ij2.}$, and X_{ij2s} are trivial, the means do not appear in the table. The only changes that occur for these means is that $\bar{\pi}_{\dots 2.}$, $(\bar{\alpha\pi})_{\dots 2.}$, $\bar{\pi}_{i\dots 2.}$, $(\bar{\alpha\pi})_{i\dots 2.}$, $\bar{\pi}_{\dots j2s}$, and $(\bar{\alpha\pi})_{\dots j2s}$ appearing in Case 2 expressions are replaced by $\bar{\pi}_{\dots j2.}$, $(\bar{\alpha\pi})_{\dots j2.}$, $\bar{\pi}_{ij2.}$, $(\bar{\alpha\pi})_{ij2.}$, π_{ij2s} , and $(\alpha\pi)_{ij2s}$, respectively. Again, we must note differences in the number of independent scores on which some means in the two design types are based. In D_1 , $(\beta\pi)_{\dots j1.}$ and $(\beta\pi)_{\dots j1s}$ are based on n scores and one score, respectively, while their counterparts in D_2 are based on I times that many scores; $(\bar{\alpha\pi})_{i\dots 1.}$ and $(\alpha\pi)_{i\dots 1s}$ are based on n and one score(s), respectively, while their counterparts are based on J times as many scores; $\bar{\pi}_{\dots 1.}$ and $\pi_{\dots 1s}$ are based on n and one score(s), respectively, and their counterparts are based on IJ times as many scores.

Table 8 shows the expected values of the mean squares associated with each source of variation. Four different denominators are required for hypothesis tests; each will be an arithmetic mean of two mean squares. To test the A and $A \times D$ effects, we need

$$\text{Denominator 3a} = [MS_{Axs(D_1)} + MS_{s/AB(D_2)}]/2.$$

To test the B and $B \times D$ effects, we use

$$\text{Denominator 3b} = [MS_{Bxs(D_1)} + MS_{s/AB(D_2)}]/2.$$

The $A \times B$ and $A \times B \times D$ interactions are tested with

Table 7
Means Expressed in Terms of Model Effects: Case 3

Mean	Expression in Effects
$\bar{X}_{\dots\dots}$	$\mu_{\dots\dots} + \frac{\bar{\pi}_{\dots 1.}}{2} + \frac{\bar{\pi}_{\dots 2.}}{2} + \frac{(\bar{\alpha\pi})_{\dots 2.}}{2} + \frac{(\bar{\beta\pi})_{\dots 2.}}{2} + \frac{(\bar{\alpha\beta\pi})_{\dots 2.}}{2} + \bar{\varepsilon}_{\dots\dots}$
$\bar{X}_{i\dots\dots}$	$\mu_{\dots\dots} + \alpha_{i\dots\dots} + \frac{\bar{\pi}_{\dots 1.}}{2} + \frac{\bar{\pi}_{i\dots 2.}}{2} + \frac{(\bar{\alpha\pi})_{i\dots 1.}}{2} + \frac{(\bar{\alpha\pi})_{i\dots 2.}}{2} + \frac{(\bar{\beta\pi})_{i\dots 2.}}{2} + \frac{(\bar{\alpha\beta\pi})_{i\dots 2.}}{2} + \bar{\varepsilon}_{i\dots\dots}$
$\bar{X}_{\dots j\dots\dots}$	$\mu_{\dots\dots} + \delta_{\dots j\dots\dots} + \bar{\pi}_{\dots j2.} + (\bar{\alpha\pi})_{\dots j2.} + (\bar{\beta\pi})_{\dots j2.} + (\bar{\alpha\beta\pi})_{\dots j2.} + \bar{\varepsilon}_{\dots j\dots\dots}$
$\bar{X}_{i\dots j\dots\dots}$	$\mu_{\dots\dots} + \alpha_{i\dots\dots} + \delta_{\dots j\dots\dots} + (\alpha\delta)_{i\dots j\dots\dots} + \bar{\pi}_{i\dots j2.} + (\bar{\alpha\pi})_{i\dots j2.} + (\bar{\beta\pi})_{i\dots j2.} + (\bar{\alpha\beta\pi})_{i\dots j2.} + \bar{\varepsilon}_{i\dots j\dots\dots}$

NOTE—Means with expressions unchanged or trivially changed from Case 2 are omitted.

Table 8
Expected Values of Mean Squares for Case 3

Source of Variation	Expected Value of Mean Square	Denominator
A	$2JnS_A^2 + \frac{1}{2}\sigma_s^2 + \frac{(IJ+I-1)}{2I}S_{As}^2 + \frac{(J-1)}{2J}S_{Bs}^2 + \frac{(IJ-I-J+1)}{2IJ}S_{ABs}^2 + \sigma_\epsilon^2$	3a
B	$2InS_B^2 + \frac{1}{2}\sigma_s^2 + \frac{(I-1)}{2I}S_{As}^2 + \frac{(IJ+J-1)}{2J}S_{Bs}^2 + \frac{(IJ-I-J+1)}{2IJ}S_{ABs}^2 + \sigma_\epsilon^2$	3b
A × B	$2nS_{AB}^2 + \frac{1}{2}\sigma_s^2 + \frac{(I-1)}{2I}S_{As}^2 + \frac{(J-1)}{2J}S_{Bs}^2 + \frac{(2IJ-I-J+1)}{2IJ}S_{ABs}^2 + \sigma_\epsilon^2$	3c
D	$IJnS_D^2 + \frac{(IJ+1)}{2}\sigma_s^2 + \frac{(I-1)}{2I}S_{As}^2 + \frac{(J-1)}{2J}S_{Bs}^2 + \frac{(IJ-I-J+1)}{2IJ}S_{ABs}^2 + \sigma_\epsilon^2$	3d
A × D	$JnS_{AD}^2 + \frac{1}{2}\sigma_s^2 + \frac{(IJ+I-1)}{2I}S_{As}^2 + \frac{(J-1)}{2J}S_{Bs}^2 + \frac{(IJ-I-J+1)}{2IJ}S_{ABs}^2 + \sigma_\epsilon^2$	3a
B × D	$InS_{BD}^2 + \frac{1}{2}\sigma_s^2 + \frac{(I-1)}{2I}S_{As}^2 + \frac{(IJ+J-1)}{2J}S_{Bs}^2 + \frac{(IJ-I-J+1)}{2IJ}S_{ABs}^2 + \sigma_\epsilon^2$	3b
A × B × D	$nS_{ABD}^2 + \frac{1}{2}\sigma_s^2 + \frac{(I-1)}{2I}S_{As}^2 + \frac{(J-1)}{2J}S_{Bs}^2 + \frac{(2IJ-I-J+1)}{2IJ}S_{ABs}^2 + \sigma_\epsilon^2$	3c
s(D ₁)	$IJo_s^2 + \sigma_\epsilon^2$	
Axs(D ₁)	$JS_{As}^2 + \sigma_\epsilon^2$	
Bxs(D ₁)	$IS_{Bs}^2 + \sigma_\epsilon^2$	
AxBxs(D ₁)	$S_{ABs}^2 + \sigma_\epsilon^2$	
s/AB(D ₂)	$\sigma_s^2 + \frac{(I-1)}{I}S_{As}^2 + \frac{(J-1)}{J}S_{Bs}^2 + \frac{(IJ-I-J+1)}{IJ}S_{ABs}^2 + \sigma_\epsilon^2$	

Denominator 3c = $[MS_{AxBxs(D_1)} + MS_{s/AB(D_2)}]/2$.

Finally, the effect of variable D is tested using

Denominator 3d = $[MS_{s(D_1)} + MS_{s/AB(D_2)}]/2$.

The final column of Table 3 indicates the denominators to be used to test the effect of the various sources of variation. Satterthwaite's formula for estimating the number of degrees of freedom associated with each of the denominators is to be employed.

A GENERAL RULE

Although the expected values of mean squares differ for the three cases, a general rule emerges for determining the appropriate linear combination of mean squares for the testing of a specific effect. (1) For the test of any effect due to substantive independent variables (A, B, A × B), one obtains the arithmetic mean of the mean square that would be the appropriate denominator were the entire experiment done as D₁ and the mean square that would be the appropriate denominator were the entire experiment done as D₂. (2) The denominator for testing the main effect of D is the arithmetic mean of the two subjects-within-groups mean squares. (3) The de-

nominator for the test of any interaction with D (A × D, B × D, A × B × D) is the same denominator as that employed for the test of the effect interacting with D (A, B, and A × B, respectively). From the fact that an interaction between two variables is not intrinsically different from a main effect, it follows that the rule need not be restricted to experiments with two substantive independent variables. The rule can be used for experiments of any size.

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NOTE

1. The D₁ and D₂ designation differs from usage in the earlier paper. In that paper, the symbols W (for "within") and B (for "between") were used to designate the two levels of D. Since here neither level of D represents a purely within- or between-subjects design, the earlier symbols become inappropriate. In this paper, the letter B is used to designate one of the two substantive independent variables.

(Received for publication March 21, 1978;
revision accepted June 28, 1978.)