Notes and Comment

A note on estimation in the power law

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By now it is common practice to fit psychophysical functions with power functions of the form

$$\psi_i = \alpha I_i^c, \quad (1)$$

where the ψ_i stand for perceived intensities or magnitude estimates at amounts I_i on some physical scale and α and β denote constants. This simple model may be linearized by taking logarithms and obtaining

$$y_i = \beta x_i + A, \quad (2)$$

where $y_i = \ln(\psi_i)$, $x_i = \ln(I_i)$, and $A = \ln(\alpha)$.

Normally, estimates $\hat{A}$ and $\hat{\beta}$ may be obtained by least squares regression or analysis of variance procedures (e.g., Bonet & Bentler, 1983; Coleman, Graf, & Alf, 1981; Thomas, 1981). An unbiased estimate of A is then easily obtained by

$$\tilde{A} = \hat{A} + S^2/2$$

with

$$S^2 = \sum_i (y_i - \hat{\beta}x_i - \hat{A})^2 / (n-2).$$

Unfortunately, the problem is that neither $e^{\tilde{A}}$ nor $e^{\hat{A}}$ are unbiased estimates of α .

This problem may be solved if we are willing to assume that the measurements of ψ_i contain an error component that is log-normally distributed. This assumption seems justified in numerous situations where the variance of the ψ_i at each level of I_i seems to increase with increasing I_i .

Let us then assume that the measures ψ_i represent samples from a log-normal distribution. Under this assumption, the original model (1) is rewritten as

$$\psi_i = \alpha V_i I_i^c, \quad (1')$$

where the V_i denote elements from a log-normal distribution (Johnson & Kotz, 1970) with expectation $-\sigma^2/2$ and

$$y_i = \beta x_i + A + z_i, \quad (2')$$

where the $z_i = \ln(V_i)$ are elements from a normal $(0, \sigma^2)$ distribution.

Now the results obtained by Finney (1941) and Heien (1968) lead to a uniform, minimum-variance, unbiased estimate of $\hat{\alpha}$ of α :

$$\hat{\alpha} = e^{\hat{A}} g(cS^2), \quad (3)$$

with

$$g(t) = 1 + \frac{n-2}{n-1}t + \sum_{j=2}^{\infty} \frac{(n-2)^{2j-1}t^j}{(n-1)^j n(n+2) \dots (n+2j-4)j!} \quad (4)$$

$$c = \frac{1}{2} - (n-1) \sum_i x_i^2 / [n(n-2) \sum_i (x_i - \bar{x})^2].$$

The above formulas may look awkward. However, if the fit of the model (1') is not too bad, cS^2 will be rather small, so the series in Equation 4 rapidly converges and an approximation up to an error less than $|10^{-8}|$ will present no computational problems (Elzinga, 1985).

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