An APL function for the asymptotic test of significance for Goodman and Kruskal's gamma statistic

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Goodman and Kruskal's (1954) gamma is a measure of association that is often used to measure the degree of association between two cross-classified ordinal variables. Gamma is a symmetric measure of ordinal association that utilizes information only from concordant and discordant (i.e., untied) object pairs; Somers' (1962) d , on the other hand, is an asymmetric measure of ordinal association that utilizes the information from the concordant and discordant object pairs, but adjusts for additional object pairs that are tied on the dependent variable but not on the independent variable. Although these two measures examine somewhat different kinds of object pairs obtained from the two ordinal variables, they both estimate the degree to which knowledge of order on one variable predicts order on the other variable.

Several APL routines exist to compute Goodman and Kruskal's (1954) symmetric measure of ordinal association, gamma, as well as Somers' (1962) asymmetric measures, d_{rc} , with row variable dependent, and d_{cr} , with column variable dependent (Berry & Mielke, 1986; Brennan, Nitz, & Daly, 1988). Unfortunately, none of these routines provides the probability value, under the null hypothesis, for any of the three measures. The function GAMMA is a noniterative APL function that calculates the three coefficients of ordinal association—gamma, d_{rc} , and d_{cr} —the associated two-sided probability values, and the 95% and 99% confidence limits, under full multinomial sampling.

The probability values and the confidence limits are computed using the asymptotic standard errors developed by Goodman and Kruskal (1963, 1972) and, specifically, do not use the approximate standard errors given in many textbooks. The approximate tests are based on the maximum standard error (Goodman & Kruskal, 1963), which yields a conservative test with a probability of falsely rejecting the null hypothesis known only to be less than some nominal value, or they assume uniform marginals (Somers, 1980), which may yield inaccurate test results for cross classifications with severely skewed marginal distributions. Moreover, because the approximations introduced by Somers (1980) are based on null populations, they preclude the computation of confidence limits.

Procedure. Consider a full multinomial sample of n discrete objects, which are cross classified by two ordi-

nal variables, r and c , with r representing the row variable, c representing the column variable, and n_{ij} indicating the observed frequency of objects in row i and column j , $1 \leq i \leq R$, and $1 \leq j \leq C$. The cell proportions are given by $p_{ij} = n_{ij}/n$, and the row and column marginal proportions are given by

$$p_{i\cdot} = \sum_j p_{ij}$$

and

$$p_{\cdot j} = \sum_i p_{ij},$$

respectively. Then G , the sample estimate of gamma, is defined as

$$G = \frac{\Pi_s - \Pi_d}{\Pi_s + \Pi_d},$$

with the estimated asymptotic variance of G given by

$$\text{VAR}(G) = \frac{16}{n(\Pi_s + \Pi_d)^4} \sum_i \sum_j p_{ij} [\Pi_s D_{ij} - \Pi_d S_{ij}]^2,$$

where

$$S_{ij} = I_{ij} + III_{ij},$$

$$D_{ij} = II_{ij} + IV_{ij},$$

$$\Pi_s = 2 \sum_i \sum_j p_{ij} II_{ij},$$

$$\Pi_d = 2 \sum_i \sum_j p_{ij} IV_{ij},$$

and where

$$I_{ij} = \begin{cases} 0, & \text{if } i = R \text{ and/or } j = C \\ \sum_{a=i+1}^R \sum_{b=j+1}^C p_{ab}, & \text{otherwise,} \end{cases}$$

$$II_{ij} = \begin{cases} 0, & \text{if } i = 1 \text{ and/or } j = C \\ \sum_{a=1}^{i-1} \sum_{b=j+1}^C p_{ab}, & \text{otherwise,} \end{cases}$$

$$III_{ij} = \begin{cases} 0, & \text{if } i = 1 \text{ and/or } j = 1 \\ \sum_{a=1}^{i-1} \sum_{b=1}^{j-1} p_{ab}, & \text{otherwise,} \end{cases}$$

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$$IV_{ij} = \begin{cases} 0, & \text{if } i = R \text{ and/or } j = 1 \\ \sum_{a=i+1}^R \sum_{b=1}^{j-1} p_{ab}, & \text{otherwise.} \end{cases}$$

Somers's d_{rc} is defined as

$$d_{rc} = \frac{\Pi_s - \Pi_d}{1 - \sum_j p_{.j}^2},$$

with the estimated asymptotic variance of d_{rc} given by

$$\text{VAR}(d_{rc}) = \frac{4}{n \left(1 - \sum_j p_{.j}^2\right)^4} \sum_i \sum_j p_{ij} \cdot \left[(\Pi_s - \Pi_d)(1 - p_{.j}) - \left(1 - \sum_j p_{.j}^2\right)(S_{ij} - D_{ij}) \right]^2,$$

and Somers's d_{cr} is defined as

$$d_{cr} = \frac{\Pi_s - \Pi_d}{1 - \sum_i p_i^2},$$

with the estimated asymptotic variance of d_{cr} given by

$$\text{VAR}(d_{cr}) = \frac{4}{n \left(1 - \sum_i p_i^2\right)^4} \sum_i \sum_j p_{ij} \cdot \left[(\Pi_s - \Pi_d)(1 - p_{i.}) - \left(1 - \sum_i p_i^2\right)(S_{ij} - D_{ij}) \right]^2.$$

The distributions of $z_G = G/[\text{VAR}(G)]^{1/2}$, $z_{d_{rc}} = d_{rc}/[\text{VAR}(d_{rc})]^{1/2}$, and $z_{d_{cr}} = d_{cr}/[\text{VAR}(d_{cr})]^{1/2}$ are shown by Goodman and Kruskal (1972) to be asymptotically normal, when the null hypothesis is true.

Function. APL function GAMMA computes Goodman and Kruskal's G , Somers' d_{rc} , and Somers' d_{cr} , the associated two-sided probability values, and the 95% and 99% confidence limits for each measure. Function GAMMA calls function ZTAB (Evans & Gilfillan, 1986) to calculate the three probability values from the com-

puted z scores. The user supplies, as a right argument to function GAMMA, the observed cell frequencies of a cross classification arranged as an $R \times C$ matrix. Because the three measures are designed for ordinal variables, properly signed coefficients will be returned if the matrix is arranged with the high-high values in the upper-left cell and the low-low values in the lower-right cell of the matrix. Two examples are provided in the Appendix; the data in the first example are from Kendall and Stuart (1967, p. 564), and the data in the second example are from Agresti and Finlay (1986, p. 221).

Language. Function GAMMA is written in APL*PLUS/PC, Version 7.0, from STSC, Inc. IBM APL does not accept the diamond separator; enter the expressions separated by the diamonds in order from left to right. The output of function GAMMA was tested and verified against the results from the four empirical data structures referenced in Somers (1980).

Availability. A listing of function GAMMA is available from Kenneth J. Berry, Department of Sociology, Colorado State University, Fort Collins, Colorado 80523.

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APPENDIX

Functions Used in the Calculation of Probability Values

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▽ GAMMA P; N; NU; DEL; L; R; S; D; FIS; PID; COF; GAM; DRC; DCR; ZG; ZR; ZC
[1]  A FUNCTION GAMMA COMPUTES THE COEFFICIENTS AND ASSOCIATED
[2]  PROBABILITY VALUES FOR GOODMAN AND KRUSKAL'S GAMMA,
[3]  SOMERS' D(RC), AND SOMERS' D(CR) TEST STATISTICS
[4]  → (2*ρPF)/ER ◇ ' ' ◇ 'OBSERVED TABLE: ' ◇ ((2*↑1↑ρP)ρ 7 0)⊖P
[5]  N←+/+/P ◇ P←P÷N ◇ S←(ρP)↑(1-ρP)↑Φ⊖+λ⊖+\\ΦP ◇ ' '
[6]  FIS←2*+/+/P×S ◇ S←S+Φ⊖(ρP)↑(1-ρP)↑Φ⊖+λ⊖+\\ΦP
[7]  D←Φ(ρP)↑(1-ρP)↑Φ⊖+λ⊖+\\P ◇ PID←2*+/+/P×D

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APPENDIX (Continued)

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[8] D←D+e(ρP)↑(1-ρP)↑Φe+λe+λΦeP ϕ NU←PIS-PID ϕ DEL←PIS+PID
[9] COF←16÷N×DEL*4 ϕ DCR←NU÷1-÷/(+/P)*2 ϕ DRC←NU÷1-÷/(+/P)*2
[10] GAM←NU÷DEL ϕ ZG←GAM÷(COF×+/+/P×((PIS×D)-PID×S)*2)*0.5
[11] DEL←1-÷/(+/P)*2 ϕ COF←4÷N×DEL*4
[12] ZC←DCR÷(COF×+/+/P×((Q(ΦP)ρNU×1-÷+/P)-DEL×S-D)*2)*0.5
[13] DEL←1-÷/(+/P)*2 ϕ COF←4÷N×DEL*4
[14] ZR←DRC÷(COF×+/+/P×(((ρP)ρNU×1-÷+/P)-DEL×S-D)*2)*0.5
[15] R←(3 17 ρ' -- 2-SIDED P ='),68 3 1 ρ2×ZTAB ~1×IZG,ZR,ZC
[16] (3 7 ρ' GAMMA =D(RC) =D(CR) ='),(68 3 1 ρGAM,DRC,DCR),R
[17] R←~1.96 ~2.576 ×I (GAM,DRC,DCR)÷ZG,ZR,ZC
[18] L←,QR+ 2 3 ρGAM,DRC,DCR ϕ L[(L<~1)/~6]←~1
[19] R←,Q(~1×R)+ 2 3 ρGAM,DRC,DCR ϕ R[(R>1)/~6]←1 ϕ ' '
[20] P←' < GAMMA < < D(RC) < < D(CR) < '
[21] P←6 12 ρ,(6 12 ρP), 6 12 ρP ϕ D←' } = 0.95 } = 0.99'
[22] (6 2 ρ'P('),(68 6 1 ρL),P,(68 6 1 ρR), 6 10 ρD ϕ →0
[23] ER:'ERROR: RIGHT ARGUMENT SHOULD BE TWO DIMENSIONAL ARRAY'

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▽

▽ P←ZTAB Z;T

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[1] A FUNCTION ZTAB RETURNS THE NORMAL PROBABILITY OF VALUES ≤ Z
[2] A AND IS FROM S. EVANS AND L. GILFILLAN, BEHAVIOR RESEARCH
[3] A METHODS, INSTRUMENTS, & COMPUTERS 18 (1986): 337-338.
[4] P←1+0.2316419×IZ+Z ϕ T←0.3989422804×*-0.5×Z*2 ϕ P←P×.*~5
[5] P←P×.× 0.31938153 ~0.356563782 1.781477937 ~1.821255978 1.330274429
[6] P←1-P×T ϕ T←(Z<0)/~PZ ϕ P[T]←1-P[T]

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▽

EXAMPLE 1:

- (1) Enter the cell frequencies as a vector.
- (2) Reshape the vector to an R by C matrix.
- (3) Call and execute APL function GAMMA.

GAMMA 4 4p821 112 85 35 116 494 145 27 72 151 583 87 43 34 106 331

OBSERVED TABLE:

821	112	85	35
116	494	145	27
72	151	583	87
43	34	106	331

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GAMMA = .775704 -- 2-SIDED P = .000000
D(RC) = .643945 -- 2-SIDED P = .000000
D(CR) = .641831 -- 2-SIDED P = .000000

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P( .751599 < GAMMA < .799809 ) = 0.95
P( .744024 < GAMMA < .807384 ) = 0.99
P( .620003 < D(RC) < .667886 ) = 0.95
P( .612478 < D(RC) < .675411 ) = 0.99
P( .617843 < D(CR) < .665819 ) = 0.95
P( .610303 < D(CR) < .673359 ) = 0.99

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EXAMPLE 2:

- (1) Enter the cell frequencies as a vector.
- (2) Reshape the vector to an R by C matrix.
- (3) Call and execute APL function GAMMA.

GAMMA 3 3p65 14 54 14 2 6 26 3 3

APPENDIX (Continued)

OBSERVED TABLE:

65	14	54
14	2	6
26	3	3

GAMMA = $-.481345$ -- 2-SIDED P = $.000041$

D(RC) = $-.213113$ -- 2-SIDED P = $.000103$

D(CR) = $-.265027$ -- 2-SIDED P = $.000048$

P($-.711189$ < GAMMA < $-.251501$) = 0.95

P($-.783426$ < GAMMA < $-.179264$) = 0.99

P($-.320661$ < D(RC) < $-.105565$) = 0.95

P($-.354462$ < D(RC) < $-.071764$) = 0.99

P($-.392812$ < D(CR) < $-.137241$) = 0.95

P($-.432973$ < D(CR) < $-.097080$) = 0.99

(Manuscript received February 24, 1989;
revision accepted for publication May 24, 1989.)