Algorithms for fast and precise computation of the normal integral

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Five highly accurate algorithms that evaluate the normal distribution function are presented. The algorithms, which are based on either numerical integration or series expansions, are explicated, and applications of Simpson's rule are discussed. Computer programs that implement the algorithms were compared with respect to accuracy and speed. The programs attain from 11- to 15-decimal-place accuracy in finding one-tailed areas of the normal curve, and most execute very quickly. Recommendations are made for selection of algorithms and programs to approximate the normal distribution.

Highly accurate computation of the value of the normal integral can be achieved with numerical integration (Cyvin, 1964; Smits, 1981; Wood, 1985, 1987) and with polynomial and rational functions, continued fractions, and series expansions (Johnson & Kotz, 1970, chap. 13; Kennedy & Gentle, 1980, chap. 5). In this paper, we review five highly accurate normal curve algorithms, three of which are based on numerical integration, and two on series expansions. We also present, explicate, and compare programs that enact these algorithms, and we describe their performance characteristics. In addition, we explore the efficacy of Simpson's rule in finding normal curve areas with 10- and 15-decimal-place accuracy.

The accuracy of numerical integration is a function of the number of points at which the integral is evaluated. In routines applying Simpson's rule to evaluate the normal integral in the interval $(0, z)$, where z is a standard normal deviate, Smits (1981) used 1,000 segments to attain 10-decimal-place accuracy, and Wood (1985) used 52 segments to attain 6-decimal-place accuracy. In two more versatile programs that compute the integral between any two z values, Wood (1987) used varying numbers of segments to attain from 6- to 12-decimal-place accuracy, and he reported that the programs could be adapted to attain 15-decimal-place accuracy. The number of segments necessary to attain a given level of accuracy can be found empirically or by formula in developing such computer routines. The required number of segments varies with both the desired accuracy and the absolute magnitude of z .

Two methods can be used to determine an appropriate number of segments at run time for the input z and for a predetermined or user-specified accuracy level. The first method, which Cyvin (1964) and Wood (1987) employed,

uses an inequality that provides either the maximum possible error that can occur given a specified number of segments into which $f(z)$ is partitioned, or, following algebraic rearrangement, the number of segments required to mathematically guarantee any specified level of accuracy. The first version of the inequality follows, cited from Anton (1981, p. 534).

$$|\text{error}| \leq \frac{M(b-a)^5}{180N^4}. \quad (1)$$

The second version takes the following form:

$$N \leq \sqrt[4]{\frac{M(b-a)^5}{180|\text{error}|}}. \quad (2)$$

In these inequalities, N is the number of segments into which an interval of $f(z)$ is divided, and M is the maximum absolute value of $f^{iv}(z)$ within that interval; b is the upper bound, and a is the lower bound. Values of $f^{iv}(z)$ can be obtained from Pearson and Hartley (1972, Table 2) or by evaluating the derivative directly as shown below.

$$f^{iv}(z) = (2\pi)^{-1/2} e^{-z^2/2} (z^4 - 6z^2 + 3). \quad (3)$$

The maximum value of $f^{iv}(z)$ is 1.196827, occurring at $z = 0$.

Study of Inequality 2 provides the keys to efficient numerical integration of $f(z)$ by Simpson's rule. Specifically, N is minimized (and a computer program therefore runs most quickly) when the range of numerical integration (i.e., $b-a$) is short and the fourth derivative of $f(z)$ is small. N is especially reduced when $(b-a) < 1$, because fractions become smaller when raised to powers larger than 1, and when $M < 1$. Parenthetically, Simpson's rule requires that N be an even integer; therefore, the value of N is increased to the smallest even integer that is greater than or equal to N as calculated with Inequality 2.

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The second method of determining the number of segments (n) applies Simpson's rule repeatedly, beginning with $n = 2$ and doubling n with each iteration until the difference between two successive results is less than the acceptable error (r). An efficient version of this method reuses previous calculations and newly computes only $(n-1)/2$ of the points otherwise needed at each iteration. Although the method appears relatively crude, it functions with a substantially lower n than the first method when $(b-a)$ is large, thus reducing both program execution time and roundoff error. For example, with $a = 0$, $b = 6$, and $r = 10^{-9}$, the first method, using Inequality 2, establishes $N \leq 478$, but n is only 32 with the second method. In using the first method, Wood (1987) avoided high N when z is large by evaluating the interval (z, ∞) rather than $(0, z)$. (He set ∞ equal to an even larger z value, beyond which the area is negligible relative to the precision of the computations.)

ZSIMPIT

ZSIMPIT (see Appendix A) is a BASIC program that uses the second method to determine n . The program requires input of z and r ; it returns the area (A) in the interval $(0, z)$, the one-tailed probability (p) calculated as $(.5 - A)$, n , the execution time, and an estimate of the error in A based on the difference between the last two approximations of the integral. The estimated error is frequently larger than the true error, but it is not a bound, and it is sometimes smaller than the true error.

ZSIMPIT could be refined by adding a correction involving derivatives at the ends of the integral or by using Richardson's extrapolation. These techniques would result in lower n for a given r , as well as a better estimate of the error. Adaptive programming, which determines an appropriate—and possibly different—number of segments for various subranges within the interval being estimated (Davis & Rabinowitz, 1975, chap. 6), might be used to reduce total n , but that technique, which involves more complex coding, would have little advantage with the well-behaved normal distribution function. In light of the accuracy and speed of the remaining algorithms, replacement rather than refinement of ZSIMPIT seems more appropriate.

ZQUICK

ZQUICK (see Appendix B) is an extension of the Simpson's rule algorithms introduced and subsequently elaborated and expanded by Wood (1985, 1987). It is an improvement over those routines because it is faster by far and because it always returns normal curve areas with 10 accurate decimals. The substantial increases in both speed and accuracy were obtained through dividing the interval $(0, 6.5)$ into 27 subintervals. The area beyond each subinterval, named *AREA*, was determined using an expanded version of NORMLCV.EXT (Wood, 1987) and was written into the program, as was the N value for each subinterval. When run, the algorithm identifies the subinterval into which a user-entered z fits and then employs Simpson's rule to calculate the area between that z and

the upper bound of the subinterval containing it. That area plus *AREA* is the one-tailed p associated with z . This technique restricts the range of numerical integration to a value smaller than 1 and thus enables accuracy to be obtained quickly, as does the fact that $f^{iv}(z) < 1$ when $|z| = 0.26165$. Although the algorithm used in ZQUICK is longer than that cited by Wood (1987), the program still consists of only 58 lines, of which 13 effect the screen display.

When run, ZQUICK requests that z be input; it returns p , A , $2A$, $2p$ (the two-tailed probability), and the execution time.

ZQUICK.EXT

In order to examine the performance of short-range Simpson's rule programming with extremely high precision, we developed an extended version of ZQUICK, named ZQUICK.EXT (see Appendix C), that seeks 15-place accuracy. The program functions as does ZQUICK, but with the following exceptions. When the user inputs z and the program identifies the subinterval into which that z falls, N is determined in one of two ways. In six of the subintervals, the maximum value of $f^{iv}(z)$ is written into the program; in the other subintervals, M is calculated by substituting the user-entered z in Equation 3. M is next substituted in Inequality 2, and N is determined. Then, as in ZQUICK, the Simpson's rule subroutine is engaged, and the results are printed out.

The program does deliver its target accuracy. However, its length is 67 lines, and it is slow, requiring a maximum of 40.7 sec when run in GW-BASICA/D 3.2 on an IBM PC microcomputer with $z = 1E-16$.

ZSTRECOK

ZSTRECOK (see Appendix D) is an algorithm that combines a simple modification of Strecok's (1968) approximation of the error function with an approximation for large values of z from Peizer and Pratt (1968). Strecok's method was the basis of Moran's (1980) approximation of the normal distribution, which led to very accurate, compact computer routines (Brophy, 1983). Strecok's method computes the error function as

$$\begin{aligned} \text{erf}(y) &= 2/\pi^{1/2} \int_0^y e^{-x^2} dx \\ &\approx 2/\pi \left[(y/5) + \sum_{n=1}^{37} n^{-1} e^{-(n/5)^2} \sin(2ny/5) \right]. \quad (4) \end{aligned}$$

Strecok gave another form of the approximation that avoids 37 sine calculations but that requires storage of a like number of coefficients. Because of the availability of intrinsic high-precision, reasonably fast sine functions and the brevity of a computer routine implementing the original form of the approximation, ZSTRECOK uses the original form. Strecok's approximation is adapted to estimate the normal integral by the relationship

$$A(z) = (2\pi)^{-1/2} \int_0^z e^{-x^2/2} dx = \text{erf}(z/2^{1/2})/2. \quad (5)$$

Although Equation 4 uses 37 terms, we found that 29 terms produce equivalent results with 16-digit arithmetic.

ZSTRECOK requires input of z ; it returns A , p , and the execution time.

ZKERCOOK

ZKERCOOK (see Appendix E), another approximation for the normal distribution, is based on a series derived by Kerridge and Cook (1976). As in ZSTRECOK, our algorithm supplements the series with Peizer and Pratt's (1968) approximation for large values of z . The series is

$$\int_0^z e^{-x^2/2} dx = 2e^{-z^2/8} \sum_{n=0}^{\infty} (z/2)^{(2n+1)} \frac{H_{2n}(z/2)}{(2n+1)!}, \quad (6)$$

where $H_n(x)$ is the n th Hermite polynomial. Acknowledging the complicated nature of Equation 6, Kerridge and Cook showed how to simplify its computation by a transformation $[\theta_n(x) = x^n H_n(x)/n!]$ that reduces the magnitude of the numbers involved and by a recurrence relation for $\theta_n(x)$ [viz., $\theta_{n+1} = x^2(\theta_n - \theta_{n-1})/(n+1)$, with $\theta_0(x) = 1$ and $\theta_1(x) = x^2$] that corresponds to a standard recurrence relation for $H_n(x)$. The resulting series,

$$\int_0^z e^{-x^2/2} dx = ze^{-z^2/8} \sum_{n=0}^{\infty} \frac{\theta_{2n}(z/2)}{(2n+1)}, \quad (7)$$

in conjunction with the recurrence relation, affords surprisingly simple coding in a computer routine. An approximation to the normal distribution is obtained through summing enough terms in Equation 7 for convergence to the desired accuracy and then multiplying the result by $(2\pi)^{-1/2}$. According to Moran (1980), the Kerridge and Cook series is not as accurate as his approximations, which have 9-place accuracy, but Kerridge and Cook reported 15-place accuracy with $z \leq 6$ using 30 or fewer terms of their series. ZKERCOOK is written to yield results as accurate as possible on the computer on which it is run. The number of terms needed is a direct function of z . With 16-digit arithmetic, between 4 and 37 terms are used for z values between 0.01 and 6.8. ZKERCOOK requires input of z ; it returns A , p , and the execution time.

The portions of ZSTRECOK and ZKERCOOK that are used with large values of z approximate p as

$$(2\pi)^{-1/2} e^{-z^2/2} m / |z|,$$

where

$$m = 1 - 1/\{z^2 + 3 - 1/[0.22(z^2 + 3.2)]\}.$$

Peizer and Pratt (1968) devised this approximation by adapting Laplace's continued fraction for the normal distribution (Zelen & Severo, 1964, section 26.2.14). We tested a bound from Zelen and Severo (1964, section 26.2.24, P_2) as an alternative approximation for large z and found it to be a little less accurate than the Peizer and Pratt approximation. The crossover points between the two parts of ZSTRECOK and ZKERCOOK were determined empirically. The Peizer and Pratt approximation is used with ZSTRECOK when $z > 6.9$; it is used with ZKERCOOK when $z > 6.8$. In such cases, these programs calculate p directly rather than as $(.5 - A)$ and thus avoid the loss of precision that could otherwise occur.

Tests of the Algorithms

All tests were run on an IBM PC microcomputer with GW-BASICA version 3.2. The /D switch was used in invoking BASIC to enable double-precision evaluation of the exponential function, which is used in all algorithms, and the sine function, which is used in ZSTRECOK. With ZSIMPIT, r was entered as 5E-16. The accuracy of the algorithms was assessed first in comparison with test values of p from the tables of the National Bureau of Standards (1953), which provide 15-decimal-place values of $2A$ for many values of z through infinity, and 7-significant-figure values of $2p$ for $z = 6.00(0.01)10.00$. Simple arithmetic manipulation of the tabled entries yielded the necessary values of p , which should be correct to at least 3 units in the 16th place. When the tables offered a choice of values for a given z , the more precise value was used. ZSIMPIT, ZSTRECOK, and ZKERCOOK were tested with $z = 0.02(0.02)10.00$. ZQUICK and ZQUICK.EXT were tested with the same values, except that the maximum z was 6.5 for ZQUICK and 9.0 for ZQUICK.EXT. The algorithms also were tested with 17-figure z values rounded from White's (1970) tables, which provide 20-decimal-place z values corresponding to $p = .010(.005).500$ and $p = 10^{-m}$, 2.5×10^{-m} , and 5.0×10^{-m} , where m signifies all positive integers from 3 through 20. ZSIMPIT, ZSTRECOK, and ZKERCOOK were tested with all 153 values in White; ZQUICK and ZQUICK.EXT were tested with all but the largest 24 and 4 values, respectively, because those large values exceed the scope of the algorithms.

Table 1 summarizes the results. All of the algorithms except ZQUICK appear a little more accurate in comparison with the more precise test values from White (1970), and those results are likely more valid than the comparisons with the tables of the National Bureau of Standards (1953). ZQUICK.EXT, ZSTRECOK, and ZKERCOOK are accurate to within 1 unit in the 16th decimal place. ZSIMPIT is accurate to within 2 units in the 16th place.

Further analysis of the accuracy of ZQUICK is necessary because the algorithm is most vulnerable in evaluating a z value that is at the lower bound of a subinterval. Therefore, ZQUICK was tested with a number of z values just greater than the lower bound of each subinterval. The maximum absolute error of ZQUICK was determined to be 4.9E-12 (with z values that exceed the lower bound

Table 1
Maximum Absolute Error of Algorithms

Algorithm	Source of Test Values	
	National Bureau of Standards	White
ZSIMPIT*	5.2E-16	2.1E-16
ZQUICK†	3.6E-12	4.1E-12
ZQUICK.EXT	2.8E-16	6E-17
ZSTRECOK	3.1E-16	8E-17
ZKERCOOK	3.1E-16	8E-17

*Tested with "acceptable error" = 5E-16. †Actual maximum absolute error = 4.9E-12.

of the subintervals by $1E-14$, $1E-15$, and $1E-16$), although for most z s the error is much less than that.

The above findings pertain to one-tailed p and, of course, the error for two-tailed p is twice as great. For two-tailed p , therefore, ZQUICK is accurate in all 10 decimal places that are displayed, ZSIMPIT is accurate to within 4 units in the 16th decimal place, ZQUICK.EXT to within 1 unit in the 16th place, and ZSTRECOK and ZKERCOOK to within 2 units in the 16th place. In an attempt to enhance these accuracies, we explored the impact of 17-place instead of 16-place constants in ZQUICK.EXT, ZSTRECOK, and ZKERCOOK and found that the 17-place constants did increase accuracy of the results by a few units in the 17th decimal place. We have not, however, used the 17-place constants in the programs because of the uncertainty in introducing them when the computer monitors and program listings display only 16 digits.

Figure 1 shows representative execution times for ZQUICK.EXT, ZSTRECOK, and ZKERCOOK. Because the execution times of ZQUICK.EXT vary greatly with the distance of z from the upper bound of the pertinent subinterval, the z values for which times are shown were selected from the middle of the subintervals. For most z values, ZKERCOOK is the fastest of these three highly accurate programs. Its time increases slightly with increasing z through 6.8 because of the greater number of terms evaluated, with a maximum time of 1.2 sec. ZSTRECOK is next fastest, maintaining essentially the same time (2.2–2.3 sec) for all z values through 6.9. In contrast to ZKERCOOK, ZQUICK.EXT tends to gain speed as z increases, reflecting the decreasing $f^{iv}(z)$ and the resulting decrease in the number of segments required to approximate the area. Although ZQUICK.EXT often is relatively

slow, sometimes taking up to 40.7 sec, it executes faster than ZSTRECOK or ZKERCOOK when $z = 6.75$. It loses the time advantage again with slightly larger z values, for which ZSTRECOK and ZKERCOOK employ the fast Peizer and Pratt (1968) approximation, which executes in 0.1 sec. ZSIMPIT is exceptionally slow when r is small, requiring up to 340 sec to evaluate as many as 8,192 segments when $r = 5E-16$. Even with $r = 5E-11$, a setting that produces results as accurate as the target accuracy of ZQUICK, ZSIMPIT takes up to 21 sec. ZQUICK, on the other hand, evaluates all z values in from 0.5 to 1.2 sec.

We used GW-BASICA instead of IBM Personal Computer BASIC to test the algorithms, because GW-BASICA is much faster. For example, the average times required to run ZQUICK.EXT, ZSTRECOK, and ZKERCOOK under GW-BASICA with the z values shown in Figure 1 were only 46%, 66%, and 37%, respectively, of the times required under IBM BASIC version 3.1.

Other Algorithms

Many other algorithms have been developed to approximate the normal distribution (Johnson & Kotz, 1970, chap. 13; Kennedy & Gentle, 1980, chap. 5), and comparative evaluations of a number of them have been published (e.g., Brophy, 1983; Hill & Joyce, 1967b; Kerridge & Cook, 1976). Only a few of the better known or especially accurate algorithms are mentioned here. Ibbetson's (1963) algorithm is a classic that is sometimes recommended to obtain fast 7-decimal-place results (Hill, 1969; Hill & Joyce, 1967b), although a simpler approximation of Zelen and Severo (1964, section 26.2.17), which was adapted from Hastings (1955), is almost as accurate. An algorithm of Milton and Hotchkiss (1969) is correct to

Figure 1. Execution time (in sec) of three normal curve programs as a function of z , run under GW-BASIC 3.2 on the IBM PC microcomputer.

9 decimal places, as are two compact approximations that Moran (1980) derived from Strecok (1968). Algorithms of Adams (1969a) and Hill (1973), which are nearly identical, are accurate to 10 places. A relatively slow algorithm of Cooper (1968) offers similar accuracy, and it can be modified for greater accuracy (Hill, 1969).

Hill and Joyce (1967a) developed a highly accurate algorithm, comprising a power series (Zelen & Severo, 1964, section 26.2.11) and a continued fraction (Zelen & Severo, 1964, section 2.26.14) for small and large z values, respectively. In fact, White (1970) used those methods in preparing his high-precision tables. The power series is rather frequently used by itself (e.g., Alf & Grossberg, 1987; Dunlap & Brown, 1983; Dunlap & Myers, 1984), but it is not extremely accurate when z is large. For example, on a computer with 16-digit arithmetic, Kerridge and Cook (1976) found that the power series could not attain 15-decimal-place accuracy when $z \geq 4$, or 10-decimal-place accuracy when $z = 6$. However, the complete Hill and Joyce (1967a) algorithm is essentially exact to the accuracy of the computer (Hill & Joyce, 1967b) and thus competitive with ZQUICK.EXT, ZSTRECOK, and ZKERCOOK; moreover, it is more accurate than those algorithms in evaluating z values larger than about 2. Adams (1969b) and Holmgren (1970) have reported that the Hill and Joyce algorithm sometimes runs into overflow, which is never a problem with the algorithms presented here. Holmgren recommended changes to correct that, and Adams (1969b) and Bergson (1968) recommended other changes to increase the speed of the algorithm.

Discussion

The algorithms explicated in this paper have been tested thoroughly against some of the best tables available, and these tests have led us to some conclusions that users of $f(z)$ programs should know.

First, it is clear that the usefulness of Simpson's rule in normal curve work depends on the level of precision desired. Wood's (1987) program NORMLCV.QIK uses Simpson's rule to find probabilities for $f(z)$ rapidly with 6- to 9-place accuracy, and our ZQUICK always delivers 10-place accuracy very quickly. Beyond 10 decimal places, Simpson's rule loses its efficiency, requiring that the program become long or slow or both. ZQUICK.EXT, for example, is longer than ZSTRECOK or ZKERCOOK and, with $z < 6.75$, it is slower.

Our second observation is that $f(z)$ programs using Simpson's rule have the advantage of increasing speed as a function of z because the fourth derivative rapidly diminishes as z increases, which in turn reduces the size of N returned by Inequality 2. Although $f^{iv}(z)$ has its maximum value of 1.197 at $z = 0$, $f^{iv}(z)$ drops rapidly, though not monotonically, to 0.484 at $z = 1$, to 0.270 at $z = 2$, to 0.133 at $z = 3$, to 0.022 at $z = 4$, and to 0.0007 at $z = 5$. This explains the rule's strength in finding $f(z)$ areas when z is large, whereas many other algorithms require help. Programs using Simpson's rule, which do not require add-on subroutines, are more straightforward and

more readily comprehended than hybrid algorithms like ZSTRECOK and ZKERCOOK. Indeed, Kerridge and Cook (1976, p. 402) expressed this thought articulately: "It is a great advantage to be able to use a single formula in computing, instead of the usual two: see for example Hill & Joyce (1967[a])."

Third, our tests of these programs have led us to recommend ZQUICK enthusiastically for these reasons: (1) Its accuracy is adequate for any task that behavioral researchers are likely to undertake. (2) It is quick, with running times in the 0.5 to 1.2 sec range, and most runs are completed in about 1 sec on the IBM PC. We use XT-compatible computers with 8- and 10-MHz clocks, and when run on these machines, ZQUICK gives almost instant results, always consuming less than 0.5 sec. (3) ZQUICK is not long, and because almost all microcomputers now have ample memory and disk drives, program length is far less of a consideration than it was formerly.

Fourth, ZKERCOOK is a useful routine when greater accuracy or program brevity is desired, and it is relatively fast with most values of z . Although its algorithm is not as intuitively clear as that of ZQUICK, ZKERCOOK has applications in situations calling for high precision, when tables such as those of the National Bureau of Standards (1953) or Zelen and Severo (1964, Table 26.1) were formerly required.

Fifth, we would warn that highly respected probability tables contain some errors. We were brought to this depressing conclusion by the fact that for some z s (e.g., 2.20) ZSTRECOK, ZKERCOOK, and ZQUICK.EXT agree to 16 decimal places, although they disagree with the 15-place probability value given by Zelen and Severo (1964, Table 26.1). These algorithms are fundamentally different, approaching $f(z)$ from disparate perspectives, and one assumes that an accurate result has been obtained when they agree.

Finally, we are aware that primitive $f(z)$ routines are still in use simply out of habit. Now, people who use normal curve areas should abandon slow programs or those having uncertain accuracy for programs that are both very fast and highly accurate.

Availability

Listings of the programs, as shown in the Appendixes, can be obtained without charge from D. Louis Wood, University of Arkansas at Little Rock, 2801 South University, Little Rock, AR 72204. For a disk copy, send a formatted (MS/PC-DOS 2.0 or later) 5.25-in. (13.335-cm) disk and a self-addressed, postage-paid mailer.

REFERENCES

- ADAMS, A. G. (1969a). Algorithm 39: Areas under the normal curve. *Computer Journal*, 12, 197-198.
- ADAMS, A. G. (1969b). Remark on Algorithm 304. *Communications of the ACM*, 12, 565-566.
- ALF, E. F., JR., & GROSSBERG, J. M. (1987). DORF2R.BAS: Analyzing signal-detection theory rating data in the BASIC programming language. *Behavior Research Methods, Instruments, & Computers*, 19, 475-482.

- ANTON, H. (1981). *Calculus*. New York: Wiley.
- BERGSON, A. (1968). Certification of and remark on Algorithm 304. *Communications of the ACM*, **11**, 271.
- BROPHY, A. L. (1983). Accuracy and speed of seven approximations of the normal distribution function. *Behavior Research Methods & Instrumentation*, **15**, 604-605.
- COOPER, B. E. (1968). Algorithm AS 2: The normal integral. *Applied Statistics*, **17**, 186-188.
- CYVIN, S. J. (1964). Algorithm 226: Normal distribution function. *Communications of the ACM*, **7**, 295.
- DAVIS, P. J., & RABINOWITZ, P. (1975). *Methods of numerical integration*. New York: Academic Press.
- DUNLAP, W. P., & BROWN, S. G. (1983). FORTRAN IV functions to compute expected normal scores. *Behavior Research Methods & Instrumentation*, **15**, 395-397.
- DUNLAP, W. P., & MYERS, L. (1984). FORTRAN IV functions for computing expected half-normal scores. *Behavior Research Methods, Instruments, & Computers*, **16**, 401-404.
- HASTINGS, C., JR. (1955). *Approximations for digital computers*. Princeton, NJ: Princeton University Press.
- HILL, I. D. (1969). Remark AS R2: A remark on Algorithm AS 2. *Applied Statistics*, **18**, 229-300.
- HILL, I. D. (1973). Algorithm AS 66: The normal integral. *Applied Statistics*, **22**, 424-427.
- HILL, I. D., & JOYCE, S. A. (1967a). Algorithm 304: Normal curve integral. *Communications of the ACM*, **10**, 374-375.
- HILL, I. D., & JOYCE, S. A. (1967b). Remarks on Algorithms 123, 180, 181, 209, 226, 272, 304. *Communications of the ACM*, **10**, 377-378.
- HOLMGREN, B. (1970). Remark on Algorithm 304. *Communications of the ACM*, **13**, 624.
- IBBETSON, D. (1963). Algorithm 209: Gauss. *Communications of the ACM*, **6**, 616.
- JOHNSON, N. L., & KOTZ, S. (1970). *Distributions in statistics: Continuous univariate distributions-1*. Boston: Houghton Mifflin.
- KENNEDY, W., & GENTLE, J. (1980). *Statistical computing*. New York: Marcel Dekker.
- KERRIDGE, D. F., & COOK, G. W. (1976). Yet another series for the normal integral. *Biometrika*, **63**, 401-403.
- MILTON, R. C., & HOTCHKISS, R. (1969). Computer evaluation of the normal and inverse normal distribution functions. *Technometrics*, **11**, 817-822.
- MORAN, P. A. P. (1980). Calculation of the normal distribution function. *Biometrika*, **67**, 675-676.
- NATIONAL BUREAU OF STANDARDS. (1953). *Tables of normal probability functions* (Applied Mathematics Series 23). Washington, DC: U.S. Government Printing Office.
- PEARSON, E. S., & HARTLEY, H. O. (Eds.) (1972). *Biometrika tables for statisticians* (Vol. 2). Cambridge, England: Cambridge University Press.
- PEIZER, D. B., & PRATT, J. W. (1968). A normal approximation for binomial, F, beta, and other common, related tail probabilities. *I. Journal of the American Statistical Association*, **63**, 1416-1456.
- SMITS, G. J. (1981). A FORTRAN IV function to compute the probability of a standard normal deviate. *Behavior Research Methods & Instrumentation*, **13**, 701.
- STRECOK, A. J. (1968). On the calculation of the inverse of the error function. *Mathematics of Computation*, **22**, 144-158.
- WHITE, J. S. (1970). Tables of normal percentile points. *Journal of the American Statistical Association*, **65**, 635-638.
- WOOD, D. L. (1985). High precision chi-square and normal curve integration. *Behavior Research Methods, Instruments, & Computers*, **17**, 429-431.
- WOOD, D. L. (1987). The last normal curve programs. *Behavior Research Methods, Instruments, & Computers*, **19**, 338-344.
- ZELEN, M., & SEVERO, N. C. (1964). Probability functions. In M. Abramowitz & I. A. Stegun (Eds.), *Handbook of mathematical functions* (pp. 925-995). Washington, DC: U.S. Government Printing Office.

APPENDIX A

Listing of Program ZSIMPIT

```

10 CLS: PRINT "ZSIMPIT": PRINT
20 DEFDBL C-H,P,S,X-Z: DEFINT I,N
30 X1=0: S4=0: X=0: H=0: C=-.5#
40 INPUT "Enter z ", Z
50 INPUT "Acceptable error " ; E
60 V1=TIMER
70 H=ABS(Z)/2: S4=EXP(H*H*C): S1=1+EXP(Z*Z*C): S2=0
80 N=2: Y=.3989422804014327#: S=Y*(S1+4*S4)*H/3
90 S2=S2+S4: S4=0: X=H/2
100 FOR I=0 TO N-1: X1=X+I*H: S4=S4+EXP(X1*X1*C): NEXT I
110 SS=Y*(S1+4*S4+2*S2)*X/3: N=N+N: IF ABS(S-S2)<E THEN I30
120 S=SS: H=X: GOTO 90
130 EE=ABS(S-SS): IF SS>.5# THEN SS=.5#
140 V2=TIMER
150 PRINT
160 PRINT "Area (0 - z) =" SS
170 P=.5#-SS: PRINT "p (one-tailed) =" P
180 PRINT
190 PRINT "Estimated error <=" EE
200 PRINT "Number of segments =" N
210 PRINT
220 PRINT "Elapsed Time =" USING "###.## sec"; V2-V1
230 END

```

APPENDIX B

Listing of Program ZQUICK

```

1 CLS
2 PRINT "This Program Computes Exact Normal Curve Probabilities "
3 FOR I=1 TO 2000:NEXT
4 CLEAR:DEFDBL A-Z:DEFINT J,M:DEFSNG V
5 PS="###.###":TS="###.###"
6 CLS:PRINT "ZQUICK":PRINT
7 INPUT "Enter z ",L
8 V1=TIMER
9 L=ABS(L):IF L<.0000000001# THEN AA=0#:A=.5#:GOTO 17
10 GOSUB 29
11 Q=(U-L)/(M*2)
12 X=L:GOSUB 28:S=S+Y:X=X:GOSUB 28:S=S+Y
13 FOR J=1 TO M:X=X+(2*J-1)*Q:GOSUB 28:T=T+Y:NEXT
14 S=S+4*T:M=M-1:T=0
15 FOR J=1 TO M:X=X+2*J*Q:GOSUB 28:T=T+Y:NEXT
16 S=S+2*T:A=S*Q/3+AREA:AA=.5#-A
17 V2=TIMER
18 PRINT "The One-tailed Probability Of z is "USING PS;A
19 PRINT
20 PRINT "The Area From 0 To z Is "USING PS;AA;
21 PRINT TAB(39)"The Area Between -z And z Is "USING PS;2*AA
22 PRINT
23 PRINT "The Two-tailed Probability Is "USING PS;2*AA
24 PRINT
25 PRINT "Elapsed Time="USING TS;V2-V1;PRINT " sec"
26 PRINT:PRINT "For Another Problem Press The Spacebar ":PRINT
27 Z$=INKEY$:IF Z$=""THEN 27 ELSE 4
28 Y=.3989422804014327#/EXP(X*X/2):RETURN
29 IF L>.6# THEN PRINT:PRINT "One-tailed P<.0000000001":PRINT:END
30 PRINT "C O M P U T I N G !"
31 IF L>.2# THEN 45
32 IF L<=.17# THEN M=11:U=.17#:AREA=.4325050683249616#:RETURN
33 IF L<=.34# THEN M=10:U=.34#:AREA=.3669282639639719#:RETURN
34 IF L<=.52# THEN M=10:U=.52#:AREA=.3015317875469662#:RETURN
35 IF L<=.72# THEN M=10:U=.72#:AREA=.2357624977792512#:RETURN
36 IF L<=.93# THEN M=10:U=.93#:AREA=.1761855422452579#:RETURN
37 IF L<=1.11# THEN M=10:U=1.11#:AREA=.1334995132427472#:RETURN
38 IF L<=1.3# THEN M=11:U=1.3#:AREA=.0968004845856103#:RETURN
39 IF L<=1.48# THEN M=11:U=1.48#:AREA=.0694366233333317#:RETURN
40 IF L<=1.67# THEN M=11:U=1.67#:AREA=.0474596818029473#:RETURN
41 IF L<=1.86# THEN M=11:U=1.86#:AREA=.0314427629807527#:RETURN
42 IF L<=2.05# THEN M=9:U=2.05#:AREA=.0201822154057044#:RETURN
43 IF L<=2.25# THEN M=9:U=2.25#:AREA=.012244726550447#:RETURN
44 IF L<=2.5# THEN M=9:U=2.5#:AREA=.0062096653257762#:RETURN
45 IF L<=2.7# THEN M=8:U=2.7#:AREA=.0034669738030406#:RETURN
46 IF L<=2.9# THEN M=8:U=2.9#:AREA=.001865813300384#:RETURN
47 IF L<=3.1# THEN M=8:U=3.1#:AREA=.0009676032132183#:RETURN
48 IF L<=3.3# THEN M=8:U=3.3#:AREA=.0004834241423838#:RETURN
49 IF L<=3.5# THEN M=7:U=3.5#:AREA=.0002326290790355#:RETURN
50 IF L<=3.7# THEN M=7:U=3.7#:AREA=.0001077997334774#:RETURN
51 IF L<=3.9# THEN M=6:U=3.9#:AREA=.0000480963440176#:RETURN
52 IF L<=4.1# THEN M=6:U=4.1#:AREA=.0000206575069125#:RETURN
53 IF L<=4.35# THEN M=6:U=4.35#:AREA=.0000068068765993#:RETURN
54 IF L<=4.6# THEN M=5:U=4.6#:AREA=.0000021124547025#:RETURN
55 IF L<=4.9# THEN M=5:U=4.9#:AREA=.0000004791832766#:RETURN
56 IF L<=5.3# THEN M=5:U=5.3#:AREA=.0000000579013404#:RETURN
57 IF L<=5.8# THEN M=4:U=5.8#:AREA=.000000003315746#:RETURN
58 M=4:U=6.5#:AREA=.0000000004016#:RETURN

```

APPENDIX C

Listing of Program ZQUICK.EXT

```

1 CLS
2 PRINT"The range of this program is z=-9 to z=9.":PRINT
3 DEFDBL A-Z:DEFINT J,V:DEFSNG D,E,M
4 INPUT"Enter the absolute value of the z score: ",L
5 PS="#.#####":TS="###.##"
6 IF L=0 THEN A=.5#:A2=0#:A3=1#:GOTO 20
7 ET1=TIMER
8 GOSUB 39
9 D=U-L
10 ER=1E-16
11 PRINT
12 GOSUB 33
13 H=(U-L)/M:X=L:GOSUB 32:S=S+Y:X=U:GOSUB 32:S=S+Y:V=M/2
14 FOR J=1 TO V:X=L+(2*J-1)*H:GOSUB 32:T=T+Y:NEXT
15 S=S+4*T:T=0:V=V-1
16 PRINT"H A L F D O N E !":PRINT
17 FOR J=1 TO V:X=L+2*J*H:GOSUB 32:T=T+Y:NEXT:S=S+2*T
18 A=AREA+ABS(H*S/3):A2=.5#-A:A3=1#-2*A
19 ET2=TIMER
20 PRINT"The 1 tailed probability ="USING PS;A
21 PRINT
22 PRINT"The 2 tailed probability ="USING PS;2#*A
23 PRINT
24 PRINT"The area from 0 to z ="USING PS;A2
25 PRINT
26 PRINT"The area from -z to z ="USING PS;A3
27 PRINT
28 PRINT"The no. segments ="M TAB(26)"Elapsed time ="USING TS;ET2-ET1;
29 PRINT" secs"
30 PRINT
31 CLEAR:GOTO 3
32 Y=.3989422804014327#/EXP(X*X/2):RETURN
33 PRINT"C O M P U T I N G !" SPC(4)
34 IF D4=0 THEN D4=.3989423/EXP(L*L/2)*(L^4-6*L*L+3)
35 M=(ABS(D4)*D^5/(180*ER))^-.25
36 IF M<10 THEN M=10:RETURN
37 M=INT(M+1):IF M/2>INT(M/2) THEN M=M+1
38 RETURN
39 IF L>9# THEN PRINT:PRINT"One-tailed P < 1E-16":PRINT:END
40 IF L>2.75# THEN 52
41 IF L<=.25 THEN U=.25#:AREA=.4012936743170763#:RETURN
42 IF L<=.5# THEN U=.5#:AREA=.3085375387259869#:RETURN
43 IF L<=.75# THEN U=.75#:AREA=.2266273523768682#:RETURN
44 IF L<=1# THEN U=1#:AREA=.1586552539314571#:D4=.484:RETURN
45 IF L<=1.25# THEN U=1.25#:AREA=.1056497736668553#:D4=.719:RETURN
46 IF L<=1.5# THEN U=1.5#:AREA=.680720126885807D-02:D4=.74:RETURN
47 IF L<=1.75# THEN U=1.75#:AREA=.4005915686381708D-02:RETURN
48 IF L<=2# THEN U=2#:AREA=.0227501319481792#:RETURN
49 IF L<=2.25# THEN U=2.25#:AREA=.1222447265504471D-02:RETURN
50 IF L<=2.5# THEN U=2.5#:AREA=.6209665325776152D-03:D4=.08:RETURN
51 IF L<=2.75# THEN U=2.75#:AREA=.2979763235054556D-03:D4=.135:RETURN
52 IF L<=3# THEN U=3#:AREA=.134989803163008D-03:D4=.1392:RETURN
53 IF L<=3.25# THEN U=3.25#:AREA=.57702504239078D-04:RETURN
54 IF L<=3.5# THEN U=3.5#:AREA=.232629079035547D-04:RETURN
55 IF L<=3.75# THEN U=3.75#:AREA=.841728520081D-05:RETURN
56 IF L<=4# THEN U=4#:AREA=.1671241833134D-05:RETURN
57 IF L<=4.25# THEN U=4.25#:AREA=.10688525774939D-05:RETURN
58 IF L<=4.5# THEN U=4.5#:AREA=.397673124718D-06:RETURN
59 IF L<=4.75# THEN U=4.75#:AREA=.101708324257D-06:RETURN
60 IF L<=5# THEN U=5#:AREA=.286651571882D-07:RETURN
61 IF L<=5.5# THEN U=5.5#:AREA=.18989562489D-08:RETURN
62 IF L<=6# THEN U=6#:AREA=.9865876865D-10:RETURN
63 IF L<=6.5# THEN U=6.5#:AREA=.0000000004016#:RETURN
64 IF L<=7# THEN U=7#:AREA=.12798226D-12:RETURN
65 IF L<=7.5# THEN U=7.5#:AREA=.3191D-14:RETURN
66 IF L<=8 THEN U=8#:AREA=.621D-16:RETURN
67 U=9#:RETURN

```

APPENDIX D

Listing of Program ZSTRECOK

10 CLS: PRINT"ZSTRECOK": PRINT	100 V2=TIMER
20 DEFDBL P,W-Z: DEFINT I	110 PRINT
30 INPUT"Enter z ", Z	120 PRINT"Area (0 - z) =" P
40 V1=TIMER	130 PRINT"p (one-tailed) =" P1
50 Z=ABS(Z)	140 PRINT
60 IF Z>6.9# THEN ZZ=Z*Z: W=1-1/(ZZ+3-1/(.22#*ZZ+.704#)):	150 PRINT"Elapsed Time =" ; USING" ###.## sec"; V2-V1
PI=.3989422804014327#*EXP(-ZZ/2)*W/Z: P=.5#-PI: GOTO 100	160 END
70 P=.1414213562373095#*Z: X=P+P: Y=-.04#	170 REM: .3989422804014327 = 1/SQR(2*PI) (Line 60)
80 FOR I=1 TO 29: P=P+EXP(I*I*Y)/I*SIN(I*X): NEXT I	180 " .1414213562373095 = 1/(SQR(2)*5) (Line 70)
90 P=.3183098861837907#*P: PI=.5#-P	190 " .3183098861837907 = 1/PI (Line 90)

APPENDIX E Listing of Program ZKERCOOK

```

10 CLS: PRINT"ZKERCOOK": PRINT
20 DEFDBL P,T,W,Z: DEFINT I
30 INPUT"Enter z ", Z
40 V1=TIMER
50 Z=ABS(Z): PS=.3989422804014327#
60 IF Z>6.8# THEN ZZ=Z*Z: W=1-1/(ZZ+3-1/ (.22#*ZZ+.704#)):
    P1=PS*EXP(-ZZ/2)*W/Z: P=.5#-P1: GOTO 110
70 I=2: T=1: P=1: T1=Z*Z/4: Z1=T1: P1=0
80 T=Z1*(T1-T)/I: I=I+1: P=P+T/I: IF P=P1 THEN 100
90 T1=Z1*(T-T1)/I: I=I+1: P1=P: GOTO 80

100 P=PS*Z*EXP(-Z1/2)*P: P1=.5#-P
110 V2=TIMER
120 PRINT
130 PRINT"Area (0 - z) =" P
140 PRINT"p (one-tailed) =" P1
150 PRINT
160 PRINT"Elapsed Time =" ; USING" ### sec"; V2-V1
170 END
180 REM: .3989422804014327 = 1/SQR(2*PI) (Line 50)

```

(Manuscript received February 7, 1989;
revision accepted for publication May 24, 1989.)