METHODS & DESIGNS

Computing the von Neumann-Morgenstern characteristic function $v(S)$ for cooperative n -person transferable utility normal form games: LP and saddlepoint solutions

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An APL computer program is described for computing the von Neumann-Morgenstern *characteristic function* of cooperative n -person transferable utility games in normal form. The program is designed to stand alone, or it can be used in conjunction with other programs for game theoretic computation that will be introduced in future articles.

Cooperative n -person game theory is broadly concerned with the normative and prescriptive analysis of multilateral conflicts of interest (von Neumann & Morgenstern, 1944). The characteristic function (symbolized by v) is a prominent descriptive model within the general theory. It numerically indexes the expected value or worth of every possible faction in a conflict. These values provide the basis for subsequent analyses aimed at prescribing how increments and decrements can be rationally apportioned among opposing parties to resolve the conflict. Multilateral conflicts of interest are labeled *games*, and the contesting parties are called *players*. A game is said to be in *normal form* when the respective outcomes associated with each player's set of strategy alternatives are represented tabularly in a matrix.

Although the general theory contains many different competing descriptive models, v dominates the representation of games where *transferable utility* is assumed.¹ Despite this wide application, v is not easily derived for normal-form games. In most cases, a long error-prone iterative process is involved. This is especially vexing in application settings where a matrix is generally the best means available for capturing strategic properties of real-

life conflicts. Experimental work related to conflict decision making can also be hindered since game matrices are often used to avoid the undesirable confounding of strategic and coalition formation factors (Sakurai, 1985). Given these difficulties and the fundamental role that v serves in most game theory application and research, we created a general computer program for the convenient and accurate computation of v .

The Characteristic Function

As already noted, v presumes the existence of freely transferable utility. This means that players have a common medium of exchange that permits them to negotiate transfers so that increments (utility) are proportional to the amounts transferred and are conserved under reapportionment. A different generalized characteristic function, V , should be used in case utility is *nontransferable* (e.g., Aumann, 1961, 1967; Aumann & Peleg, 1960; Peleg, 1963a, 1963b).

The formal theory is developed as follows: Let the finite set of players in a game be denoted by $N = 1, 2, \dots, n$. The characteristic function is a real-valued set function v defined on all possible subsets of players, or *coalitions*, in the game. It represents the expected worth of every coalition $S \in N$ by a single real number $v(S)$, where the empty set $v(\emptyset) = 0$. The expected worth of a coalition S coincides with the *minimax* value of a reduced game between S and its complement $-S = (N - S)$.

The minimax theorem postulates that every coalition, S , is opposed by a countercoalition of all remaining players who will single-mindedly attempt to minimize any

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collective gain by S . Accordingly, the best course for members of S is to adopt a coordinated joint strategy that maximizes their expected return under the most damaging opposition strategy. By acting in such a conservative manner, S can assure itself of some minimum guaranteed outcome regardless of how the other players conduct themselves. Sometimes S can secure its minimax value when members make single pure selections from their respective strategy choice sets. Other circumstances may require a probability mix of choice selections.

To see the difference between pure and mixed probability strategies, let each player $i \in N$ have a set of individual strategy choice alternatives $C_i = (c_1, c_2, \dots, c_n)$. Each strategy choice is associated with distinct outcomes that are also influenced by the choices of all other players.² Similarly, let every coalition S have a set of joint strategy choice alternatives $C_S = \sum_{i \in S} C_i$ that is the topological product of individual member strategy sets. Then, represent the reduced game between S and its complement $-S$ as a matrix A_S , with row dimensions defined by C_S , columns defined by C_{-S} , and corresponding outcome cell entries computed by $a_{\sigma\tau} = \sum_{i \in S} u_i(\sigma, \tau)$, where σ and τ are joint strategies selected from C_S and C_{-S} , respectively.

If S can maximize its minimax interests in A_S by using a pure unmixed joint strategy, the reduced game is said to have a *pure strategy saddlepoint*. This means there is an element in A_S that is *simultaneously* minimal in its row and maximal in its column. To illustrate, suppose A_S is an $m \times n$ matrix; a_{lh} is a pure saddlepoint if

$$a_{lh} = \min_{j=1, \dots, n} (a_{lj}) = \max_{i=1, \dots, m} (a_{ih}),$$

where

$$\min_{j=1, \dots, n} (a_{lj})$$

is a minimal element of the collection $(a_{l1}, \dots, a_{lj}, \dots, a_{ln})$, which corresponds with the l th row of A_S , and

$$\max_{i=1, \dots, m} (a_{ih})$$

is the maximal element of column h . Thus, a_{lh} is the element that is both minimal in row l and maximal in column h . When pure strategy saddlepoints exist, they represent coalition S 's expected worth or characteristic function value, $v(S)$.

If there are no pure strategy saddlepoints, coalition S can maximize its interests by adopting a probability mix of strategy choice selections (a *mixed strategy*). Let P be a set of probabilities (summing to one) that weight S 's random selection of specific joint strategy alternatives such that p_σ is the probability of selecting joint strategy σ . Define Q as a similar set of probability weights for $-S$. According to the minimax theorem, every reduced game between complementary coalitions has a (minimax) *mixed strategy saddlepoint* given by

$$\max_{p \in P} \min_{q \in Q} \sum_{\sigma} \sum_{\tau} p_{\sigma} a_{\sigma\tau} q_{\tau}.$$

This saddlepoint is defined as the characteristic function value of S and can be found by calculating the primal or dual of a linear program (Charnes, 1953; Charnes & Sorensen, 1974):

$$\begin{aligned} v(S) = \max \gamma & \quad v(S) = \min \delta \\ \text{such that } \gamma - \sum_{\sigma} p_{\sigma} a_{\sigma\tau} & \leq 0 & \quad \text{such that } \delta - \sum_{\tau} a_{\sigma\tau} q_{\tau} & \geq 0 \\ \sum_{\sigma} p_{\sigma} & = 1 & \quad \sum_{\tau} q_{\tau} & = 1 \\ p_{\sigma} & \geq 0 & \quad q_{\tau} & \geq 0. \end{aligned}$$

The APL Program

This program automatically computes the characteristic function for all normal form games having up to nine players. Each player is allowed a maximum of nine pure strategy choice alternatives. Every player can have a different number of strategy choices. Negative outcomes are also permitted.

The program begins by designating the largest row sum of the overall game matrix as $v(N)$. A reduced game matrix is then constructed for every individual player and coalition. Each reduced matrix is first evaluated to identify pure strategy saddlepoints that can be assigned as the characteristic function value of corresponding players/coalitions. Multiple pure saddlepoints always have the same value (Charnes & Cooper, 1961), and any one can be selected as $v(S)$.

If no pure strategy saddlepoints exist, the reduced game is automatically transformed into a linear program and a maximum value (primal) solution computed using Smillic's (1983) simplex program. This solution is subsequently assigned as the characteristic function for corresponding players/coalitions.

Input. To execute the program, users must (1) indicate the number of strategy alternatives available to each player, (2) type MAIN, and (3) specify the overall normal-form game outcome matrix. More than one line may be used for input.

Use the following notational convention: (1) label individual players using uppercase letters (e.g., A, B, ..., N), (2) indicate coalitions by grouping the constituent members with no space between player labels (e.g., AC is the coalition between player A and player C), (3) sequentially number each player's set of pure strategy alternatives (i.e., 1, 2, ..., n), (4) represent the various pure strategy combinations resulting from individual player strategy choice selections by vectors (e.g., the vector 1 1 2 signifies that player A and player B select their respective first strategy alternatives and player C selects his/her second alternative), and (5) use lowercase letters corresponding to player labels in combination with appropriate strategy vector subscripts to indicate individual player outcomes associated with any given pure strategy

selection combination (e.g., a_{121} represents player A's outcome for the strategy combination 1 2 1).

The number of strategy alternatives available to players is collectively specified by a vector of appropriate dimension (leave a blank space between each vector element). For example, the vector 3 2 4 describes a three-person game in which player A has three strategy alternatives, player B two alternatives, and player C four alternatives.

A vector is also used to specify the normal form game outcome matrix. The vector is constructed by sequentially stringing together successive rows of the outcome matrix. This assumes that rows of the outcome matrix are defined by a complete logically ordered listing of all possible pure strategy selection combinations and that columns are alphabetically ordered by player labels. Cells within the matrix indicate corresponding outcomes for each player from the respective strategy selection combinations (row labels). For example, Table 1 represents the normal form outcome matrix of a three-person game in which each player has two strategy alternatives. To input this matrix, specify the vector $a_{111} b_{111} c_{111} a_{112}, \dots, b_{122}, \dots, c_{222}$. The characteristic function for this example game is obtained by simply specifying the following: 2 2 2 MAIN $a_{111} b_{111}, \dots, c_{222}$.

Example/Output. A sample program output is shown in the Appendix. Notice that it adopts the program syntax described above. In the sample game player A has three strategy choices, whereas players B and C each have two choices. Both saddlepoint and linear program (LP) solutions are illustrated. This particular game will serve as a recurring example in future articles to help demonstrate computational and theoretical differences between competing descriptive models. For example, see Sakurai and Brennan (1988a, 1988b) for computational examples of the constrained game function CGF(S) (Charnes, 1953; Charnes & Sorensen, 1974) and the Harsanyi function H(S) (Harsanyi, 1959, 1963) using this same game.

Program outputs begin by listing the overall normal-form game outcome matrix for user confirmation. The reduced game matrices for every player and coalition are also printed to aid in designing experimental test games. Analytic results include (1) the determination of $v(N)$ and

associated strategy combination(s), (2) pure strategy saddlepoints, and (3) linear program maximum value (primal) solutions with strategy selection probability weights (negative correction coefficients are also printed as indicated).

Smillie's (1983) simplex program identifies individual and joint coalitional strategy alternatives as consecutively numbered (equation) variables, beginning with variable number 2. Variable number 1 is reserved for the maximum value solution, and only variables having probability weights greater than zero are listed (recall that negative probabilities are disallowed). For example, suppose coalition AB has four joint strategy alternatives, but only the first joint strategy (i.e., variable 2) and third joint strategy (variable 4) are required for an optimal mixed probability strategy. In that case, the program would only list results for variables 1 (the primal solution), 2, and 4. Whenever a reduced game matrix contains negative outcomes, the printed negative correction coefficient must be subtracted from each listed "LEVEL" value to obtain correct strategy selection probabilities.

Program outputs end with a consolidated factorial listing of the characteristic function (Searle, 1971). For example, the characteristic function for a three-person game will be listed as follows: $v(A)$, $v(B)$, $v(AB)$, $v(C)$, $v(AC)$, $v(BC)$, $v(ABC)$.

Language and System Requirements. The program requires STSC/APL*PLUS for the IBM PC and compatibles with 640K and one floppy disk drive.

Availability. A program listing is available from the first author. In addition, a transfer copy will be made for users who send a blank double-sided, double-density floppy disk formatted for the IBM PC and compatibles.

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Table 1
A Three-Person Two-Choice Normal Form Game Outcome Matrix

Strategy Selection Combinations			Corresponding Outcomes		
A	B	C	A	B	C
1	1	1	a_{111}	b_{111}	c_{111}
1	1	2	a_{112}	b_{112}	c_{112}
1	2	1	a_{121}	b_{121}	c_{121}
1	2	2	a_{122}	b_{122}	c_{122}
2	1	1	a_{211}	b_{211}	c_{211}
2	1	2	a_{212}	b_{212}	c_{212}
2	2	1	a_{221}	b_{221}	c_{221}
2	2	2	a_{222}	b_{222}	c_{222}

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NOTES

1. This domination appears to be more a matter of convention than of any proven empirical superiority. In fact, there are important theoretical objections to v (e.g., Shapley & Shubik, 1973; Shubik, 1982). Empirical findings also suggest that significant error can be introduced in game theory analyses by invalid applications of v (Charnes & Golany, 1983; Michener, Macheel, Depies, & Bowen, 1983; Sakurai, 1980, 1986a, 1986b).

2. These payoffs are not self-determined and depend on the choices made by all other players. Thus, for every player i , there is a payoff function $U_i(c_i, c_1, \dots, c_n)$, indicating a utility payoff outcome for the combination of i 's k th choice selection, player j 's l th choice, $\dots$, and player n 's m th selection.

APPENDIX

Sample Output from Computing the Characteristic Function of a Three-Person Normal-Form Game

OPTSADLP PLPTAB

Input is a flat matrix or vector.

3 Person game with choices per person: 3 2 2

Game Matrix

ABC	A	B	C
111	-1	100	20
112	13	8	56
121	37	0	37
122	1	-11	8
211	39	9	39
212	21	12	38
221	67	15	11
222	5	3	-6
311	65	17	82
312	9	8	7
321	37	37	1
322	5	9	16

V(ABC) Largest row sum = 164 for 311

Reduced Matrices

P(A)

A	B	1	1	2	2
C	1	2	1	2	

1	-1	13	37	1
2	39	21	67	5
3	65	9	37	5

SADDLE V(A) = 5 5

P(B)

B	A	1	1	2	2	3	3
C	1	2	1	2	1	2	

1	100	8	9	12	17	8
2	0	-11	15	3	37	9

SADDLE V(B) = 8

P(C)

C	A	1	1	2	2	3	3
B	1	2	1	2	1	2	

1	20	37	39	11	82	1
2	56	8	38	-6	7	16

Input to simplex is:

1	0	0	0	0	0	0	0	0	0	0	0
1	-26	-62	1	0	0	0	0	0	0	0	0
1	-43	-14	0	1	0	0	0	0	0	0	0
1	-45	-44	0	0	1	0	0	0	0	0	0
1	-17	0	0	0	0	1	0	0	0	0	0
1	-88	-13	0	0	0	0	1	0	0	0	0
1	-7	-22	0	0	0	0	0	1	0	0	0
0	1	1	0	0	0	0	0	0	1	1	1

Smillie simplex program results with 6 subtracted.

MAXIMUM VALUE 5.68750

VAR.

LEVEL	5.68750
1	-5.31250
2	-5.68750
3	19.56250
4	16.25000
5	27.00000
6	46.87500

P(AB)

AB	C	1	2
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11	99	21
12	37	-10
21	48	33
22	82	8
31	82	17
32	74	14

SADDLE V(AB) = 33

APPENDIX (Continued)

P(AC)

AC B 1 2

11 19 74

12 69 9

21 78 78

22 59 -1

31 147 38

32 16 21

SADDLE V(AC) = 78

P(BC)

BC A 1 2 3

11 120 48 99

12 64 50 15

21 37 26 38

22 -3 -3 25

Input to simplex is:

1 0 0 0 0 0 0 0 0 0

1 -123 -67 -40 0 1 0 0 0 0

1 -51 -53 -29 0 0 1 0 0 0

1 -102 -18 -41 -28 0 0 1 0 0

0 1 1 1 1 0 0 0 1 1

Smillie simplex program results with 3 subtracted.

MAXIMUM VALUE 49.18605

VAR. LEVEL

1 49.18605

2 -2.59302

3 -2.40698

6 34.60465

5 8 33 5.6875 78 49.18605 164

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