Notes and Comment

Constructability and the power function for line judgments

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Engen (1971) presented subjects with the task of making ratio estimation judgments of line length and concluded that subjective line length conformed to a power function with exponent greater than one. But Bogartz (1979) correctly pointed out an error in Engen's application of the Torgerson (1958) method of scale construction. Bogartz reanalyzed the Engen data and concluded that subjective line length was, in fact, proportional to physical line length. However, the Bogartz conclusion of proportionality may not be routinely accepted, because the Torgerson procedure is based on a simple ratio scaling model that has been falsified for numerous data sets (Fagot, 1978, 1979, 1981). We take the view that an unambiguous interpretation of the psychophysical function requires an explicit statement of the scaling model, including a test of the model prior to scale construction. It is with these considerations in mind that we shall reexamine the data of Engen (1971) and the reply of Bogartz (1979) to the conclusion reached by Engen.

The scaling model implicit in the Torgerson procedure for scale construction from ratio estimation data is the classical (C) model (Fagot, 1978):

$$R_{ab} = \psi_a / \psi_b, \quad (1)$$

where ψ is the psychological scaling function and R_{ab} is the observed ratio estimate of stimulus a to b . Equation 1 entails the following structural condition (testable consequence):

$$R_{ac} = R_{ab}R_{bc} \quad (C1), \quad (2)$$

which requires that subjective ratios behave like numerical ratios (Eisler, 1960; Fagot, 1978; Sjöberg, 1971).

Since C1 has been shown to be inconsistent with most published data, Equation 1 has been modified by the introduction of bias parameters. One such

bias model, which shall be used to analyze the Engen (1971) data is the following:

$$R_{ab} = \beta \psi_a / \psi_b, \quad (3)$$

where β is a bias parameter (Fagot, 1978; Sjöberg, 1971). Model Equation 3, the constant bias (CB) model, entails that

$$R_{ab}R_{bd} = R_{ac}R_{cd} \quad (C2). \quad (4)$$

For further discussion and interpretation of these models, along with empirical tests, see Fagot (1978, 1979, 1981).

Structural conditions C1 and C2 are the testable consequences of the C and CB models, respectively. If C1 is not satisfied, then a scale ψ satisfying Equation 1 does not exist, and a psychophysical function based on Equation 1 is *not constructable*. In this case, one considers models, such as Equation 3, that place weaker constraints on the structure of the data. [A weaker bias model (Fagot, 1978, Equations 3 and 10) might also be applied, but is not considered here due to the satisfactory fit of the stronger CB model.] The relationship of the Torgerson procedure to these models will now be established.

The Torgerson (1958, pp. 108-112) procedure for constructing a ratio scale based on ratio estimation data requires a complete square matrix of observations R_{ab} , R_{ba} , and R_{aa} . Unfortunately, as is commonly the case, the Engen (1971) data consist only of observations R_{ab} ($a < b$). In order to obtain the R_{ba} and the R_{aa} , the Torgerson procedure uses the relations $R_{aa} = 1$ and $R_{ba} = 1/R_{ab}$, derived from Equation 1. These assumptions allow one to write the complete matrix required for a solution, but note that only $(1/2)n(n-1)$ of the n^2 entries constitute independent observations. Given this matrix, the logs of the ratios (some observed, some inferred) are averaged for each stimulus and the antilogs of these means are the scale values of the stimuli.

Since the Torgerson scaling procedure applied by Bogartz is based on Equation 1, we need to inquire if structural condition C1 is satisfied. Figure 1 shows a plot of R_{ac} as a function of $R_{ab}R_{bc}$ for the Engen (1971) data. It is evident that there are marked systematic deviations of the form $R_{ab}R_{bc} > R_{ac}$. This finding is typical of other modalities and was reported by Fagot (1978) for judgments of angles (Goude, 1962), brightness (Fagot, Stewart, & Kleinkecht, 1975), velocity (Mashhour, Note 1), heaviness (Eisler, 1960), darkness and area (Ekman, Goude, & Waern, 1961), and weight (Goude, 1962).

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Figure 1. C1: R_{ac} as a function of $R_{ab}R_{bc}$ ($a < b < c$). The diagonal line is the identity line $R_{ac} = R_{ab}R_{bc}$.

Figure 2. C2: $R_{ac}R_{cd}$ as a function of $R_{ab}R_{bd}$ ($a < b < c < d$). The diagonal line is the identity line $R_{ac}R_{cd} = R_{ab}R_{bd}$.

Figure 2 shows a plot for C2 of $R_{ac}R_{cd}$ as a function of $R_{ab}R_{bd}$. Although the number of points is small, the data of Figure 2 are consistent with C2, showing no signs of systematic error.

The failure of C1 indicates conclusively that a ratio scale ψ satisfying Equation 1 does not exist, that a psychophysical function based on Equation 1 is not constructable, and, therefore, that the application of the Torgerson procedure by Bogartz (1979)

and Engen (1971) was inappropriate and their conclusions unwarranted.

Since the Engen data do not reject C2, a power function for the CB model is constructable. The regression approach is outlined below.

Following Bogartz (1979) and Engen (1971), we consider the simple power function hypothesis $\psi = c S^k$, where c is an arbitrary constant and S is the physical scale. Since ψ and S are assumed to be ratio scales and the data are not near threshold, an additive constant may be omitted. This hypothesis, together with Equation 1, entails that

$$R_{ab} = S_{ab}^k, \quad (5)$$

where $S_{ab} = S_a/S_b$. Equation 5 was stated previously by Baird and Noma (1978). The CB power function is then

$$R_{ab} = \beta S_{ab}^k. \quad (6)$$

The chief advantage of the power functions given by Equations 5 and 6 is their expression entirely in terms of observables. These equations avoid the use of inferred quantities R_{ba} and R_{aa} , as well as prior construction of a scale as required by the Torgerson procedure. R_{ba} and R_{aa} would be included as part of the construction set only if observed. If there are systematic deviations in the observed R_{ab} (such as a violation of C1), then these effects will surface in using Equations 5 and 6 to construct the power function. But if a ψ scale is constructed and used to construct the power function, these systematic deviations may be obscured by the averaging procedure used in estimating scale values. Note also that the use of Equations 5 and 6 eliminates the arbitrary constant c from the power functions, resulting in only one parameter, the exponent k , to be estimated.

Proceeding with a regression analysis of the power function, we express the CB power function in log linear form.

$$\log R_{ab} = k \log S_{ab} + \log \beta \quad (7)$$

and apply the linear regression model $Y = b_1 X + b_0$.

Figure 3 shows a plot of $\log R_{ab}$ as a function of $\log S_{ab}$. The solid line is the CB line (Equation 7), and the dashed line is the theoretical line indicating proportionality of R_{ab} and S_{ab} . The data points fit reasonably well about line 7, but show systematic deviations from the proportionality line, indicating the need for the additive constant bias "correction," as well as a slope (exponent) "correction," and demonstrating the unacceptability of the power function based on the C model.

Although Figure 3 shows a reasonably good fit to the CB power function and a corresponding rejection of linearity, the estimated exponent was

Figure 3. $\log R_{ab}$ as a function of $\log S_{ab}$ ($a < b$). The solid line is the line estimated by the CB power function (Equation 7). The dashed line is the proportionality line.

$\hat{k}=1.065$, not markedly different from unity. In terms of meaningful interpretation of parameters, we should ask if it is safe to conclude that the additive constant is $b_0 = \log \beta$ as entailed by Equation 7, and not some uninterpretable constant that fortuitously improves the fit. Fortunately, we do have an independent estimate of β that may be obtained entirely from the observed R_{ab} , namely, $\beta = R_{ab}R_{bc}/R_{ac}$ (Fagot, 1978). Taking the geometric mean of β obtained from each of the 10 triples of stimuli, we get $\hat{\beta}=1.0646$, and taking the regression estimate, we get $\hat{\beta}=1.0640$. Thus, the estimate of β obtained via the additive constant in the regression of $\log R_{ab}$ on $\log S_{ab}$ (Equation 7) agrees remarkably well with the independent estimate expressed entirely in terms of observed R_{ab} , and strengthens our confidence in the CB power function.

Bogartz (1979), in his reanalysis of the Engen (1971) data using the inappropriate Torgerson procedure based on the C model, derived an estimate of the exponent of .998, and of the intercept of 1.004, on the basis of which he concluded that subjective line length was proportional to physical line length. It can be shown that, if Equation 7 is correct and $\beta > 1$, the least squares estimate of k assuming the C model ($\beta = 1$) will underestimate k , and hence be biased in the direction of linearity, explaining the result obtained by Bogartz. Several other studies question the veridicality of line judgments (e.g., Baird, Kreindler, & Jones, 1971; Ekman & Junge, 1961).

Comrey (1950) presented ratio estimation data similar in form to that of Engen (1971), but derived from Comrey's constant-sum method (divide 100 points between the two stimuli). Bogartz indicated that his reanalysis of Comrey's data also resulted in a finding of proportionality. However, not only are the Comrey data remarkably variable, especially for line judgments, but neither C1 nor C2 is satisfied for these data. Hence a ratio scale satisfying Equation 1 or Equation 3 does not exist for the Comrey data, indicating that the psychophysical function is not constructable, using either of these models.

REFERENCE NOTE

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