FORTRAN IV functions for computing expected half-normal scores

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A half-normal distribution results if all those values that are negative in a standard normal distribution are made positive; that is, it is the distribution of the absolute values of normally distributed data. The mathematical formula for this distribution is

$$(2/\pi)^{1/2} \text{EXP}(-Z^2/2),$$

for Z greater than or equal to zero. If one were to sample N values from the half-normal distribution and put those scores in order, the smallest value would be near zero and the largest value in the neighborhood of 2 to 3. The program described in this paper computes the expected value of the k th of N ordered half-normal scores.

Although there are a number of potential applications of expected half-normal scores, only three will be described in detail. Hills (1969) and Stavig and Acock (1976) proposed a test for the significance of correlation coefficients in a matrix, in which there may be a great number of coefficients to be tested. In such a case, there should be several rather large coefficients by chance alone. Their procedure is to rank the correlations in terms of absolute value, convert the ordered correlations to normal scores by Fisher's z transformation, and then compare these with the expected values of half-normal scores. Only those correlations whose z -score equivalents surpass the corresponding expected ordered half-normal score by a sufficient amount are considered significant.

Daniel (1959) and Zahn (1975a, 1975b) described the use of expected half-normal scores for interpreting the results of factorial experiments in which all the independent variables have only two levels. The idea here is that if one runs a five-way factorial, 31 tests of significance would be made, some of which would appear to be quite large by chance alone. Since each effect is on a single degree of freedom, each can be considered as a linear contrast of the cell means in the experiment. These contrasts should generally have normal distributions, so that if ranked, they may be compared with corresponding expected half-normal scores, as were the transformed correlations above.

Third, an easy way to compare the absolute values of quantities that should be normal with the expected values of ordered half-normal scores is to construct a

half-normal probability plot. Such plots are produced by programs such as BMDP5D, from the BMDP series of programs (Dixon, 1981). Essential to such plots is the computer generation of expected half-normal scores. Other applications for these scores exist: Best, Cameron, and Eagleson (1983) described the use of half-normal probability plots for testing large sets of values of Kendall's Tau; Cox and Lauh (1967) described the use of half-normal probability plots in analyzing multi-dimensional contingency tables; and Stavig (1978) provided transformational equations that permit the application of half-normal score techniques to t , F , and χ^2 statistics.

The critical problems for all of these potential applications are the lack of available tables of expected half-normal scores and the lack of accurate algorithms to approximate these scores. The only table we were able to find is in Zahn (1975b), which goes up to $N = 20$, clearly insufficient for many of the applications previously described. Although several approximating algorithms have been suggested, the accuracy of most currently available algorithms is poor. Therefore, the purposes of the present paper are to provide a computer program that accomplishes an exact solution via numerical integration, to assess the relative merits of approximating algorithms, and to provide a quickly executing function for estimating expected half-normal scores with reasonable accuracy.

Table 1 compares the accuracy of each approximation with the exact solution; the approximations are ordered in terms of poorest to best accuracy. The first two use the inverse of the normal probability integral, for which we used the approximating function of Emerson (1979). The function labeled S&A-1 was proposed by Stavig and Acock (1976) to be used when tables of expected normal scores were not available. The function is

$$z^{-1}(1 - (J-1/2)/2N),$$

where N is the number of half-normal scores and J is the position of the score in order. The second function is that used by the program BMDP5D in making half-normal probability plots (Dixon, 1981); the function is

$$z^{-1} [(3N+3J-1)/(6N+1)].$$

It appears that both of these functions are inaccurate to an extent that one should question using them in half-normal plots, much less in hypothesis testing. The remaining three functions tested are based upon the use of expected ordered normal scores to approximate expected ordered half-normal scores; the former are computed quickly via an algorithm described by Dunlap and Brown (1983). The function labeled S&A-2, also described by Stavig and Acock (1976), is

Table 1
Absolute Error of Five Algorithms Approximating Ordered Half-Normal Scores as Functions of Number of Scores

Algorithm		Number of Half-Normal Scores						
		2	4	8	16	32	64	128
S&A-1	1st	.022	.070	.080	.077	.071	.064	.0584
	Max	.149	.105	.080	.077	.071	.064	.0584
BMDP81	1st	.108	.059	.042	.036	.034	.033	.0330
	Max	.174	.111	.065	.036	.034	.033	.0330
S&A-2	1st	.099	.041	.017	.007	.002	.001	.0002
	Max	.171	.110	.064	.019	.019	.009	.0048
Alt-1	1st	.034	.020	.011	.006	.004	.002	.0010
	Max	.034	.027	.011	.006	.004	.002	.0010
Alt-2	1st	.000	.000	.000	.000	.000	.000	.0002
	Max	.007	.008	.007	.004	.003	.002	.0012

Note—Rows labeled 1st contain errors for the largest score, and Max, the largest difference between estimated and actual scores.

$$\text{ENS}(K,M); M=2N, K=1 \text{ to } N,$$

where ENS represents expected ordered normal scores. Somewhat better overall accuracy can be obtained by using a slightly altered function.

$$\text{ENS}(K,M); M=2N+1, K=1 \text{ to } N.$$

This function is labeled Alt-1 in Table 1; however, it consistently produces slight overestimates of the correct values. The final function, labeled Alt-2, is the same as Alt-1 except that an error correction based on N is subtracted to improve the estimate. The error correction (EC) is

$$\text{EC} = .092263N^{-1} - .057445N^{-2} + .012751N^{-3} + .000443.$$

The FORTRAN IV code for Alt-2 appears in Appendix A; it is quite accurate for the large half-normal scores, which are most crucial to hypothesis testing, and is accurate to at least two decimal places for all other scores.

The exact solution for expected ordered half-normal scores via numerical integration is quite parallel to a similar exact solution for expected normal scores described by Dunlap and Brown (1983). The half-normal probability function is calculated from the normal probability function using an approximation (number 26.2.11) from Zelen and Severo (1965). Simpson's rule is used to numerically integrate the function.

$$\frac{N!}{(N-J)!(J-1)!} P^{(N-J)} Q^{(J-1)} z \, dx,$$

from zero to positive infinity, where P is the proportion below x and Q the proportion above x on the half-normal distribution, and z is the ordinate of the half-normal curve at x . First, the program searches for a usable range for the integration; then Simpson's rule is

applied over 101 steps. The values computed agree exactly with the three-decimal place values tabled by Zahn (1975b); however, the present program is probably accurate to five decimal places because it uses the same computational scheme that was used for expected normal scores by Dunlap and Brown (1983), which was accurate to that level. The FORTRAN code for the exact solution is available in Appendix B.

Time Requirements. The shortcut approximate function computes a single expected half-normal score in approximately 0.28 msec on the DEC-20 computer. Computation times for the exact solution are considerably longer and depend on both N and J . The shortest computing times, of approximately 175 msec, were found when the smallest score was computed and the N was approximately 90. The longest computing times were found when computing the largest half-normal scores; these increased as a function of N , taking as long as 590 msec at $N = 2,000$.

Program Availability. These functions are written in standard FORTRAN IV and run on a DEC-20 computer. Main routines are included before each function for instructive purposes. These main routines contain some DEC-specific FORTRAN statements; users will probably wish to write their own main programs, depending upon the use they have in mind. Neither function uses dimensioned storage; therefore, memory size need only be sufficient to load and run the programs. Listings of these functions can be obtained at no cost from William P. Dunlap, Department of Psychology, Tulane University, New Orleans, Louisiana 70118.

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Appendix A

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C — MAIN ROUTINE FOR APPROXIMATE HALF NORMAL SCORES
1  TYPE 2
2  FORMAT(' ENTER N ')
  ACCEPT *,N
  DO 4 I=1,N
    SHN=HNS(I,N)
    TYPE 3,I,SHN
3  FORMAT(1X,I4,F10.5)
4  CONTINUE
  GOTO 1
  END

C
  FUNCTION HNS(J,N)
C — RAPID APPROXIMATION OF THE JTH OF N HALF NORMAL SCORES
C — ENS(K,M) WHERE M=2N+1 & K=M-J+1
C — DUNLAP & BROWN, BRMI, 1983, 15, 395-397.
  M=2*N+1
  K=M-J+1
  HNS=SCR(K,M)
C — ERROR CORRECTION
  XN=N
  A=1./XN
  EC=.092263*A-.057445*A*A+.012751*A*A*A+.000443
  HNS=HNS-EC
  RETURN
  END

C
  FUNCTION SCR(J,N)
C — RAPID APPROXIMATION TO THE JTH OF N EXPECTED NORMAL SCORES
C — USES BLOMS ALGORITHM WITH CORRECTIONS PROPOSED BY
C — HARTLEY, BIOMETRIKA, 1961, 48, 151-165.
  XJ=J
  XN=N
  X=ALOG10(XN)
  A1=.315065+.057974*X-.009776*X*X
  A2=.327511+.058212*X-.007909*X*X
  A=A2
  IF(J.EQ.1.OR.J.EQ.N)A=A1
  SCR=ZINV((XJ-A)/(XN-2.*A+1.))
  RETURN
  END

C
  FUNCTION ZINV(P)
C — QUICK APPROXIMATION TO THE INVERSE OF THE NORMAL DIST FUNCT.
C — EMERSON, BRMI, 1979, 11, 397-398.
  SGN=-1.
  Q=P
  IF(Q-.5)2,2,1
1  SGN=1.
  Q=1.-Q
2  Z=SQRT(-2.*ALOG(Q))
  W=Z*(.186264+Z*.001308)
  W=1.+Z*(1.432788+W)
  Q=.802853+Z*.010322
  W=(2.515517+Z*Q)/W
  ZINV=(Z-W)*SGN
  RETURN
  END

```

Appendix B

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C — MAIN ROUTINE FOR NUMERICAL CALCULATION OF EXPECTED
C — ORDERED HALF NORMAL SCORES
1  TYPE 2
2  FORMAT(' ENTER J, N ')
  ACCEPT *,J,N
  SN=HNSCR(J,N)
3  TYPE 4,N,J,SN
4  FORMAT(1X,2I5,F15.5)
  GOTO 1
  END

C
  FUNCTION HNSCR(J,N)
C — COMPUTES THE EXPECTED VALUE OF THE JTH OF N
C — ORDERED HALF NORMAL SCORES
C — COMPUTE THE CONSTANT OF INTEGRAL IN LOG FORM
  C=FACL(N)-FACL(J-1)-FACL(N-J)
  HNSCR=0.
  STR=0.
  FIN=0.
C — FIND LOWER (STR) AND UPPER (FIN) LIMITS OF INTEGRATION
  DO 1 I=1,50
    XI=1
    XI=XI*.14
    FUNC=FUN(XI,C,J,N)
    IF(STR.EQ.0..AND.FUNC.GT.1.E-7)STR=XI
    IF(STR.GT.0..AND.FUNC.LT.1.E-7)GOTO 2
1  CONTINUE
2  FIN=XI
  STR=STR+.14
  IF(STR.GE.0.)GOTO 3
  STR=0.
  FIN=.28
3  CONTINUE
  W=(FIN-STR)/100.
  XL=STR
  FL=FUN(XL,C,J,N)
C — BEGIN SIMPSONS RULE INTEGRATION
  DO 4 I=1,101
    XU=XL+W
    XM=(XU+XL)/2.
    FM=XM*FUN(XM,C,J,N)
    FU=XU*FUN(XU,C,J,N)
    HNSCR=HNSCR+(FU+4.*FM+FL)*W/6.
    XL=XU
    FL=FU
4  CONTINUE
  RETURN
  END

C
  FUNCTION FUN(X,C,J,N)
C — COMPUTES VALUE OF FUNCTION TO BE INTEGRATED - FIRST IN LOG FORM
C — THEN IN DECIMAL FORM IF LARGE ENOUGH
C — C=N!/((J-1)!/(N-J)!) THE CONSTANT OF INTEGRATION
C — FUN=C*P**(N-J)*Q**(J-1)*2
C — WHERE P IS PROPORTION BELOW & Q THE PROPORTION ABOVE X
C — ON THE HALF NORMAL CURVE; AND Z IS THE ORDINATE OF THE
C — HALF NORMAL CURVE AT X.
  DOUBLE PRECISION X,POFZ,P,C,XJ,XN,FUN,FL
  XJ=J
  XN=N
  P=POFZ(X)*2.D0-1.D0D0
  Q=1.D0-P
  FUN=0.D0
  IF(P.LE.G.D0.OR.C.LE.0.D0)RETURN
  FL=C*DLOG(P)*(XN-XJ)+DLOG(Q)*(XJ-1.D0)-X*X/2.D0-.2257914D0
  IF(FL.LT.-40.)RETURN
  FUN=DEXP(FL)
  RETURN
  END

C
  FUNCTION FACL(K)
C — COMPUTE THE NATURAL LOG OF K!
C — ABOVE 36 USES STIRLINGS APPROXIMATION
  IF(K.GT.36)GOTO 2
  FACL=0.
  IF(K.LE.1)RETURN
  DO 1 I=2,K
    X=I
1  FACL=FACL+ALOG(X)
  RETURN
2  X=K
  FACL=.918938533+ALOG(X)*(X+.5)-X+1./(12.*X)
  RETURN
  END

C
  DOUBLE PRECISION FUNCTION POFZ(X)
C — NORMAL PROBABILITY INTEGRAL, -INFINITY TO X; X>0
C — ALGORITHM 26.2.11 P.932 ZELEN & SEVERO
C — FOR HALF NORMAL PROBABILITIES USE 2*(POFZ-.5)
  DOUBLE PRECISION X,XX,S,T,C,XN,SN
  POFZ=.5D0
  IF(XX.EQ.0)RETURN

```

```
X=DABS(XX)
POFZ=1,D0
IF(X,GT,8.35)GOTO 3
S=X
T=X
C=X*X
XN=0,D0
1 XN=XN+1,D0
T=T*C/(2,D0*XN+1,D0)
SN=S+T
IF(SN,EQ,S)GOTO 2
```

```
S=SN
GOTO 1
2 POFZ=,5D0+,3989422804014327D0*DEXP(-C/2,D0)*S
3 IF(XX,LT,0,D0)POFZ=1,D0-POFZ
RETURN
END
```

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