The perceptual organization of dot lattices

MICHAEL KUBOVY

University of Virginia, Charlottesville, Virginia

Bravais (1850/1949) demonstrated that there are five types of periodic dot patterns (or lattices): oblique, rectangular, centered rectangular, square, and hexagonal. Gestalt psychologists studied grouping by proximity in rectangular and square dot patterns. In the first part of the present paper, I (1) describe the geometry of the five types of lattices, and (2) explain why, for the study of perception, centered rectangular lattices must be divided into two classes (*centered rectangular* and *rhombic*). I also show how all lattices can be located in a two-dimensional space. In the second part of the paper, I show how the geometry of these lattices determines their grouping and their multistability. I introduce the notion of *degree of instability* and explain how to order lattices from most stable to least stable (hexagonal). In the third part of the paper, I explore the effect of replacing the dots in a lattice with less symmetric motifs, thus creating wallpaper patterns. When a dot pattern is turned into a wallpaper pattern, its perceptual organization can be altered radically, overcoming grouping by proximity. I conclude the paper with an introduction to the implications of motif selection and placement for the perception of the ensuing patterns.

The Gestalt psychologists used periodic dot patterns—also known as lattices or nets—such as those shown in Figures 1 and 2 to demonstrate grouping by proximity. The study of perceptual organization in periodic dot patterns has not gone beyond these simple patterns. In this paper I will draw on the work of Bravais (1850/1949), who partitioned the dot patterns into five classes. I will show that, from a perceptual point of view, one of these classes should be split. I will then examine the implications of grouping by proximity for the perceptual organization and the multistability of the various types of dot patterns.

THE CLASSIFICATION OF LATTICES

Consider a point O in the plane and two vectors, $\mathbf{a}$ and $\mathbf{b}$, whose directions are neither parallel nor antiparallel (Figure 3). Let us impose certain conditions on the parallelogram they define: $\|\mathbf{a}\| \leq \|\mathbf{b}\| \leq \|\mathbf{a}-\mathbf{b}\| \leq \|\mathbf{a}+\mathbf{b}\|$ ¹ (i.e., $\|\vec{OA}\| \leq \|\vec{OB}\| \leq \|\vec{BA}\| \leq \|\vec{OC}\|$). The set of points L , which consists of all linear combinations $m\mathbf{a} + n\mathbf{b}$, where m and n are integers, is the *lattice* spanned by $\mathbf{a}$ and $\mathbf{b}$ (Figure 4). The first inequality is adopted for convenience, without loss of generality. The second and third inequality ensure that $\mathbf{a}$ and $\mathbf{b}$ are the shortest translations that map L onto itself. (The preceding presenta-

tion follows Armstrong, 1988, pp. 148–150.) The pair $\mathbf{a}, \mathbf{b}$ is sometimes called the *basis* of L (Hoggar, 1992, p. 53).

From the point of view of the crystallographer, there are five types of lattices (Bravais, 1850/1949, pp. 27–28).²

Figure 1. This dot pattern (a square lattice of points) serves as reference for the pattern in Figure 2.

Figure 2. A dot pattern (a rectangular lattice of points) demonstrating grouping by proximity.

This work is dedicated to the memory of Gtano Kanizsa, master of phenomenological analysis. This research was supported by PHS Grant 5 R01 MH473717 to the University of Virginia (M.K., PI). I thank S. Boker, J. Hollier, G. R. Lockhead, M. Peterson, J. R. Pomerantz, A. Schulman, R. N. Shepard, and J. Wagemans for their helpful suggestions. Request reprints from M. Kubovy, Department of Psychology, Gilmer Hall, The University of Virginia, Charlottesville, VA 22903-2477 (e-mail: kubovy@virginia.edu).

Figure 3. Vectors $\mathbf{a}$ and $\mathbf{b}$ are the basis of (i.e., they are the translation vectors that generate) the lattice in Figure 4. The triangle OAB is known as the lattice's *basic* (or *principal*) *triangle*. For all lattices, $60^\circ \leq \angle AOB \leq 90^\circ$; $45^\circ \leq \angle BAO \leq 90^\circ$; $0^\circ \leq \angle OBA \leq 60^\circ$ (Bravais, 1850/1949, p. 20).

Figure 4. Part of the point lattice spanned by vectors $\mathbf{a}$ and $\mathbf{b}$. The point P is given by $2\mathbf{a} - \mathbf{b}$.

The classification is based on the shape of the basic parallelogram (the shaded region in Figure 4) defined by the vectors $\mathbf{a}$ and $\mathbf{b}$ (Armstrong, 1988, pp. 148–149). Table 1 lists the five Bravais lattices and the relations among $\|\mathbf{a}\|$, $\|\mathbf{b}\|$, $\|\mathbf{a} - \mathbf{b}\|$, and $\|\mathbf{a} + \mathbf{b}\|$.

Bravais's classification does not distinguish between the case $\|\mathbf{a}\| < \|\mathbf{b}\| = \|\mathbf{a} - \mathbf{b}\| < \|\mathbf{a} + \mathbf{b}\|$ and the case $\|\mathbf{a}\| = \|\mathbf{b}\| < \|\mathbf{a} - \mathbf{b}\| < \|\mathbf{a} + \mathbf{b}\|$, because both generate a centered rectangular pattern. (This claim will become clear by studying Figure 5.) However, since we are investigating perceptual grouping by proximity, the difference is crucial. A nomenclature that recognizes this difference is given in Table 1.

In Figure 5, I compare the six lattice types (all share the same length of $\mathbf{a}$) and their properties. In Figure 6, I show how these six lattice types can be located in a two-dimensional space (in which $\mathbf{a}$ is held constant), whose coordinates are $\|\mathbf{b}\|$ and $\gamma = \angle(\mathbf{a}, \mathbf{b})$. It may be useful to read the following description of Figure 6 while consulting Figure 5 from time to time.

Recall the assumption that none of the vectors is shorter than $\mathbf{a}$. Therefore, all the lattices are limited on the left by $\|\mathbf{b}\| = \|\mathbf{a}\|$. Second, to ensure that $\|\mathbf{a} - \mathbf{b}\| \leq \|\mathbf{a} + \mathbf{b}\|$, we require that $\gamma \leq 90^\circ$ (to see why, look at Figure 3). Therefore, all the lattices lie on or below the limit $\gamma = 90^\circ$. Let us first consider the three types of lattices for which $\|\mathbf{b}\| = \|\mathbf{a}\|$. We note that $\gamma \geq 60^\circ$; it is easy to see (Figure 3) that if $\|\vec{OA}\| = \|\vec{OB}\|$ and $\gamma < 60^\circ$, then $\|\mathbf{b}\| \geq \|\mathbf{a} - \mathbf{b}\|$, which is contrary to our assumption. If we let $\gamma = 60^\circ$, the lattice is hexagonal. If we increase γ , the lattice becomes rhombic. When $\gamma = 90^\circ$, the lattice is square. Now let us hold γ constant and increase $\|\mathbf{b}\|$; the lattice becomes rectangular.

Let us now return to the hexagonal lattice. If we increase $\|\mathbf{b}\|$ —which transforms the hexagonal lattice into a centered rectangular lattice— γ must increase too. To see why, look at the triangle ABC in the fourth panel of the “centered rectangular” column in Figure 5. $\gamma = \angle ACB = \angle BAC$, and $\|\mathbf{b}\| = \|\vec{CB}\| = \|\vec{BA}\|$. These two equalities force the following relation between γ and $\|\mathbf{b}\|$:

$$\gamma = \cos^{-1} \left(\frac{1}{2\|\mathbf{b}\|} \right).$$

This limit function is shown in Figure 6, and its equation appears in the geometric description of the centered rectangular lattice in Figure 5. The open set of lattices limited on the left by the rhombic lattices (of which the hexagonal and the square are special cases), at the top by rectangular lattices (of which the square is a special case), and at the bottom by centered rectangular lattices (of which the hexagonal is a special case), and not limited on the right, contains the oblique lattices.

THE MULTISTABILITY OF LATTICES

A figure is perceptually ambiguous (or reversible, or bistable) when we spontaneously see it shift from one organization to another. The nature of the shift varies among

Table 1
The Five Bravais Lattices as Specifications of
 $\|\mathbf{a}\| \leq \|\mathbf{b}\| \leq \|\mathbf{a} - \mathbf{b}\| \leq \|\mathbf{a} + \mathbf{b}\|$

Bravais Lattice		New Nomenclature
Hexagonal	$\ \mathbf{a}\ = \ \mathbf{b}\ = \ \mathbf{a} - \mathbf{b}\ \leq \ \mathbf{a} + \mathbf{b}\ $	Hexagonal
Square	$\ \mathbf{a}\ = \ \mathbf{b}\ \leq \ \mathbf{a} - \mathbf{b}\ = \ \mathbf{a} + \mathbf{b}\ $	Square
Centered rectangular	$\ \mathbf{a}\ = \ \mathbf{b}\ \leq \ \mathbf{a} - \mathbf{b}\ \leq \ \mathbf{a} + \mathbf{b}\ $	Rhombic
Centered rectangular	$\ \mathbf{a}\ \leq \ \mathbf{b}\ = \ \mathbf{a} - \mathbf{b}\ \leq \ \mathbf{a} + \mathbf{b}\ $	Centered rectangular
Rectangular	$\ \mathbf{a}\ \leq \ \mathbf{b}\ \leq \ \mathbf{a} - \mathbf{b}\ = \ \mathbf{a} + \mathbf{b}\ $	Rectangular
Oblique	$\ \mathbf{a}\ \leq \ \mathbf{b}\ \leq \ \mathbf{a} - \mathbf{b}\ \leq \ \mathbf{a} + \mathbf{b}\ $	Oblique

Figure 6. The six types of lattices organized in a two-dimensional space, in which $\|a\|$ is held constant and $\|b\|$ and γ vary.

ambiguous figures—in Rubin's (1915/1921) demonstrations the figure-ground organization shifts (Figure 7); in crystallographer Necker's (1832) rhomboid (Figure 8)³ we see reversals in depth; and Schumann (1904) observed that a square lattice (similar to Figure 1, but instead of circles it had black squares, as in Figure 15, Pattern S) alternates between a collection of columns and a collection of rows.

A figure is multistable when we can see it in more than two ways. For example, an equilateral triangle is mul-

tistable (Attneave, 1968, 1971): at different times it points in the direction of each of its vertices. Similarly, a field of equilateral triangles points ambiguously (Figure 9). If we refer the pointing directions to the direction of the hour hand of a clock, the triangles in Figure 9 point toward half past noon ($12\frac{1}{2}$), half past four ($4\frac{1}{2}$), or half past eight ($8\frac{1}{2}$).

Lattices are yet another example of multistability. We spontaneously see each lattice as a collection of parallel strips of dots. We are most likely to see the strips oriented parallel to the direction of the shortest interdot distance in the lattice—that is, parallel to $\mathbf{a}$. As I mentioned before, this is the phenomenon of *grouping by proximity*.

Although we tend to organize the lattice by proximity, we sometimes see each lattice organized in strips parallel to the directions of vectors $\mathbf{b}$, $\mathbf{a} - \mathbf{b}$, or $\mathbf{a} + \mathbf{b}$. Because this occurs spontaneously, and without much scrutiny, we say that lattices are multistable.

The Instability of the Six Types of Lattices

Hitherto, figures have been classified as ambiguous or unambiguous. Ambiguity has not been considered to be a matter of degree. Nevertheless, the idea of degrees of ambiguity has been implicit in manipulations of ambiguous figures that increased the likelihood of one or the other of its interpretations. A biased ambiguous figure is more stable—less ambiguous—than a balanced one. For instance, Graham (1929) showed how to bias the perception of a Maltese cross inscribed in a circle by varying the area, color, and brightness of the four sectors that form the cross (Figure 10), thus decreasing its ambiguity.

Figure 7. Rubin's figure-ground reversible figure (after Rubin, 1921, Figure 2).

Figure 8. The Necker rhomboid is reversible in depth (after Necker, 1832).

Figure 9. An array of equilateral triangles is multistable (inspired by Attneave, 1968, 1971).

The multistability of lattices is variable in a similar fashion. Let us compare the distribution of the lengths of the four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a}-\mathbf{b}$, and $\mathbf{a}+\mathbf{b}$ for the six types of lattices (Figure 5, third row). These distributions tell us the degree to which each lattice is unstable or ambiguous. The most ambiguous lattices are the hexagonal ones, because they are the only lattices for which three of the four vectors are equal in magnitude. Therefore, *ceteris paribus*, the probability of seeing a hexagonal lattice partitioned into strips parallel to the directions of $\mathbf{a}$, $\mathbf{b}$, or $\mathbf{a}-\mathbf{b}$ is equal.

For our purposes, the classification of lattices into disjoint classes is less useful than the continuous variation in the distribution of vector lengths in the space depicted in Figure 6. In Figure 11, I summarize the qualitative phenomenological observations of this and the preceding paragraph. (A paper describing a mathematical model and laboratory data supporting it is in preparation—Kubovy & Wagemans, 1994.) It is a contour plot drawn on the space defined by Figure 6. Along the direction representing γ , the two plots cover the same domain ($60^\circ \leq \gamma \leq$

90°). In Figure 6, $\|\mathbf{b}\|$ runs from $\|\mathbf{b}\| = \|\mathbf{a}\|$ to $\|\mathbf{b}\| = 10\|\mathbf{a}\|$. In Figure 11, $\|\mathbf{b}\|$ runs from $\|\mathbf{b}\| = \|\mathbf{a}\|$ to $\|\mathbf{b}\| = 2\|\mathbf{a}\|$. The plot represents the surface by contours—equal degrees of ambiguity are connected by equal-ambiguity contours—and by gray level—the lighter the gray, the higher the surface. The stippled region represents $(\|\mathbf{b}\|, \gamma)$ pairs for which no lattices exist. The region around the hexagonal lattice has the lightest shading, indicating that these lattices are more ambiguous than the rest.

As we move upward along the line $\|\mathbf{a}\| = \|\mathbf{b}\|$ in Figure 6, or—equivalently—the line $\|\mathbf{b}\|/\|\mathbf{a}\|$ in Figure 11, the difference between $\|\mathbf{b}\|$ and $\|\mathbf{a}-\mathbf{b}\|$ grows, while the difference between $\|\mathbf{a}-\mathbf{b}\|$ and $\|\mathbf{a}+\mathbf{b}\|$ shrinks. As a result, the ambiguity of the lattices decreases (Figure 11). Let us dwell briefly on the square lattices. Because $\|\mathbf{a}\| = \|\mathbf{b}\|$, we are most likely to see them partitioned into strips parallel to $\mathbf{a}$ or $\mathbf{b}$. In addition, an observer is just as likely to notice an organization parallel to $\mathbf{a}-\mathbf{b}$, as to notice the perpendicular organization parallel to $\mathbf{a}+\mathbf{b}$. So the square

Figure 10. Top: An ambiguous figure. A Maltese cross inscribed in a circle. The gray sectors are congruent with the white ones. Bottom: A less ambiguous figure. The gray areas are smaller than the white areas; the gray areas are more salient (inspired by Graham, 1929).

Figure 11. A contour representation of the degree of instability of different lattices, superimposed on the space depicted in Figure 6. The lighter the shading, the more unstable the lattice.

lattices have two layers of ambiguity: the $\mathbf{a}, \mathbf{b}$ pair of directions, and the $\mathbf{a} - \mathbf{b}, \mathbf{a} + \mathbf{b}$ pair of directions.

Now let us think about the ambiguity of lattice as $\|\mathbf{b}\|$ varies. All other things being equal, the ambiguity of lattices should decrease as $\|\mathbf{b}\|$ grows. For each value of $\|\mathbf{b}\|$, the centered rectangular lattice should be the most ambiguous, since $\|\mathbf{b}\| = \|\mathbf{a} - \mathbf{b}\|$.

Manipulating Lattice Instability

Reference frames and texture can reduce the ambiguity of an array of equilateral triangles (Palmer, 1989; Palmer & Bucher, 1982). In Figure 12, I have placed an array of equilateral triangles (the ones shown in Figure 9) on a ruled background in six different orientations and in a rectangular frame in three different orientations. As mentioned earlier—using the face of the clock as reference—the vertices of the triangles point toward $12\frac{1}{2}$, $4\frac{1}{2}$, and $8\frac{1}{2}$. The reference frame and texture effectively bias the pointing. We tend to see the two arrays on the top pointing to $12\frac{1}{2}$, the two arrays in the middle pointing to $4\frac{1}{2}$, and the two arrays on the bottom pointing to $8\frac{1}{2}$.

Reference frames can also affect the way we perceive lattices. Consider, for example, a rectangular dot lattice ($\|\mathbf{b}\|/\|\mathbf{a}\| \approx 1.5$) on a striped background (Figure 13). The direction of the background stripes is parallel to $\mathbf{c}$ (or $\mathbf{d}$). Without the background stripes, we are unlikely to see the lattice organized in the $\mathbf{c}$ or $\mathbf{d}$ directions. The background, which serves as a reference frame, increases the likelihood that we will see the lattice as a collection of strips parallel to the background stripes.

Gestalt laws, such as good continuation, can also affect the organization of lattices. Consider, for example,

what happens when I construct a discrete wallpaper pattern⁴ (Figure 14) on an oblique lattice by replacing each dot (which has no orientation) with an elongated asymmetric motif (which has an intrinsic orientation).⁵ On the top of Figure 14, I replaced the dots in an oblique lattice with a “P” motif whose straight stem is parallel to $\mathbf{a}$.⁶ On the bottom of Figure 14, I rotated the motif so that its straight stem would be parallel to $\mathbf{a} - \mathbf{b}$. The lattices underlying these two patterns are the same; the basis of both lattices is $\mathbf{a}, \mathbf{b}$. The basic parallelogram of both patterns remains the same; it is the hatched parallelogram in the upper pattern. In the pattern on the bottom, however, the orientation of the motif has increased the salience of the $\mathbf{a} - \mathbf{b}$ direction, so that the most salient vectors are now $\mathbf{a}$ and $\mathbf{a} - \mathbf{b}$. As a result, the *perceived* basic parallelogram, which is the hatched parallelogram in the right-hand pattern, does not coincide with the lattice’s true, basic parallelogram.

As my last example, consider a discrete wallpaper pattern based on a square lattice, in which I replaced each dot with a square motif (Figure 15).⁷ At the top of Fig-

Figure 12. The orientation of a textured background and reference frames bias the pointing of an array of equilateral triangles. The same array is shown on six different backgrounds and three different frame orientations (after Palmer, 1989, and Palmer & Bucher, 1982).

Figure 13. A rectangular dot lattice with a textured background. The background increases the salience of the direction $a-b$ (or $a+b$).

ure 15, I replaced the dots in a square lattice with squares whose sides are parallel to a and b . I will refer to it as Pattern S. At the bottom, I rotated each motif by 45° so that its sides would be parallel to $a-b$ and $a+b$. I will refer to it as Pattern D. The lattices that underlie these patterns are still the same—the basic parallelogram of both is still a square.

In Pattern S, in which the a, b directions are dominant, we see the motif as a square, and the predominant overall organization as rows and columns. We rarely notice the diagonal organization (parallel to $a-b$ or $a+b$). In Pattern D, grouping by proximity still determines how we perceive the pattern—the organization parallel to a, b dominates. However, if grouping by proximity were the only factor determining the perceptual organization of this lattice, we would expect Pattern D to be more stable than Pattern S in its organization parallel to a, b ; the smallest distance between the corners of the motifs is smaller in D than in S. Nevertheless, the salience of the directions parallel to $a-b$ and $a+b$ is greater in Pattern D than in Pattern S. As a result, our interpretation of Pattern D readily alternates between an organization parallel to a, b and an organization parallel to $a-b, a+b$. Thus, Pattern D is more ambiguous than Pattern S.

One can see a phenomenological demonstration of this point in Figure 16. At the top of this figure is Pattern S, and at the bottom is Pattern D. To each, I have added a background intended to bias an observer away from the primary organization (a, b) to the secondary organization ($a-b, a+b$). The background does not distract the perception of Pattern S from a, b , whereas it does so easily for Pattern D. This is evidence that Pattern S is more stable than Pattern D.

There are, moreover, other phenomenological consequences of this change of motif. First, the change affects the appearance of the motif itself. When the a, b directions are dominant in Pattern D, we see the motif as a diamond (an observation due to Mach, 1906/1959). When we see

Pattern D organized parallel to $a-b, a+b$, we tend to see the motif as a square. In contrast, when we see Pattern S organized parallel to $a-b, a+b$, we do not necessarily see the motif as a diamond (although we sometimes do).

Second, the change from Pattern S to Pattern D affects the global organization of the pattern; when the a, b directions are dominant in Pattern D, we see it as an (alternating) collection of rows and columns of diamonds. However, when the $a-b, a+b$ directions are dominant in Pattern D, the rows-and-columns organization is weaker. Several other organizations can be seen; two of the most common are the checkerboard and interlocking crosses.

Third, the change from Pattern S to Pattern D affects the organization of the background. In Pattern S, the dominant organization (squares in rows and columns) produces a background that, with little effort, we can see as a rectangular mesh of white lines on a black background. The dominant organization of Pattern D (rows and columns of diamonds) creates a complex background that does not

Figure 14. Two isomorphic wallpaper patterns (based on the same oblique lattice) with congruent motifs may appear to be based on different lattices.

Figure 15. Two isomorphic wallpaper patterns (based on the same square lattice) with congruent motifs sometimes appear to be based on different lattices. The top one (Pattern S) is more stable than the bottom one (Pattern D).

readily reverse into figure. However, the alternative organization of Pattern D (checkerboard of squares) creates a background that can be seen in three ways. (1) We may see white squares between the black ones. The size of these white squares is difficult to ascertain, but they are roughly the size of the black squares. The remaining white regions form anomalous (also known as illusory, or subjective, or cognitive) strips that frame the white and black squares of the checkerboard (Kanizsa, 1955, 1971/1979; Schumann, 1904). (2) We may see large white squares between the black ones. (3) We may see white octagons between the squares.

CONCLUSIONS

The evidence in this paper is phenomenological. The analysis that motivates these phenomenological observations is based on an effort to acquire a thorough understanding of the stimulus domain. The phenomenological bent of the present research is not popular. For instance, Watt (1991) writes:

I have warned against being confused between the subjective phenomenology of vision . . . and the objective, action-controlling side of vision. . . . [O]ur experience of vision might be a representation of the potential actions that we could make in the scene. . . . This representation must in-

volve memory. . . . [T]he visual system delivers directions to . . . effector organs . . . independently of phenomenology. . . . [T]his is real vision. (p. 296)

I do not recommend confusing phenomenology with the "objective action-controlling side of vision." Yet phenomena such as those described in this article—which are most likely to be discovered through phenomenological exploration—are relevant to theories of vision, even if their place in such theories is still unknown. Nor do I recommend that we stop at phenomenology (neither did the Gestalt psychologists). Indeed, Wagemans and I (Kubovy & Wagemans, 1994) are preparing a report on relevant experimental work.

Many related phenomena await mathematical and phenomenological analysis. Two directions of exploration are worth noting. (1) There are 17 types of wallpaper patterns, each based on one of the five types of Bravais lattices. Each of these wallpaper patterns allows for a variety of motifs (for the diversity of wallpaper patterns, see Stevens, 1980). We do not yet know all the ways in which the choice of motif can affect the perceptual organization of otherwise isomorphic wallpaper patterns. The examples discussed in this paper suggest that this choice can have a profound effect. (2) Each type of wallpaper pattern is defined by its lattice type and a list of isometric transformations (i.e., transformations that preserve dis-

Figure 16. The wallpaper patterns shown in Figure 15, with textured backgrounds designed to bias the viewer's perception away from the preferred organization. The top one (Pattern S) lends itself to this influence less than the bottom one (Pattern D).

tance: translation, rotation, reflection, and glide reflection⁸) that map the pattern onto itself. When different perceptual organizations are induced in wallpaper patterns by different motifs, different isometries become salient to the observer. As a result, formally isomorphic wallpaper patterns can appear to be of different types, and formally different wallpaper patterns may appear to be similar. It may be possible to create a perceptual taxonomy of regular patterns.

REFERENCES

- ARMSTRONG, M. A. (1988). *Groups and symmetry*. New York: Springer-Verlag.
- ATTNEAVE, F. (1968). Triangles as ambiguous figures. *American Journal of Psychology*, **81**, 447-453.
- ATTNEAVE, F. (1971, December). Multistability in perception. *Scientific American*, pp. 62-71.
- BRAVAIS, A. (1949). *On the systems formed by points regularly distributed on a plane or in space* (A. L. Shaler, Trans.). Crystallographic Society of America. (Original work published 1850)
- GRAHAM, C. H. (1929). Area, color, and brightness difference in a reversible configuration. *Journal of General Psychology*, **2**, 470-481.
- HOGGAR, S. G. (1992). *Mathematics for computer graphics*. Cambridge: Cambridge University Press.
- KANIZSA, G. (1955). Marginal quasi-perceptivi in campi con stimolazione omogenea [Quasi-perceptual edges in fields with homogeneous stimulation]. *Rivista di Psicologia*, **49**, 7-30.
- KANIZSA, G. (1979). Anomalous contours and surfaces. In *Organization in vision: Essays on Gestalt perception* (pp. 192-221). New York: Praeger.
- KUBOVY, M., & WAGEMANS, J. (1994). *On the entropy of lattices*. Manuscript in preparation.
- MACH, E. (1959). *The analysis of sensations and the relation of the physical to the psychological* (S. Waterlow, Trans.). New York: Dover. (5th German ed. 1906)
- NECKER, L. A. (1832). Observations on some remarkable phenomena seen in Switzerland; and an optical phenomenon which occurs on viewing of a crystal or geometrical solid. *Philosophical Magazine* (3rd series), **1**, 329-337.
- PALMER, S. E. (1989). Reference frames in the perception of shape and orientation. In B. E. Shepp & S. Ballesteros (Eds.), *Object perception: Structure and process* (pp. 121-164). Hillsdale, NJ: Erlbaum.
- PALMER, S. E., & BUCHER, N. M. (1982). Textural effects in perceived pointing of ambiguous triangles. *Journal of Experimental Psychology: Human Perception & Performance*, **7**, 88-114.
- RUBIN, E. (1921). *Visuell wahrgenommene Figuren: Studien in psychologischer Analyse* [Visually perceived figures: Studies in psychological analysis]. København: Gyldendalske Boghandel. (Original work published 1915. Photocopy available from University Microfilms, Ann Arbor, MI.)
- SCHUMANN, F. (1904). Beiträge zur Analyse der Gesichtswahrnehmungen: 1°. Einige Beobachtungen über die Zusammenfassung von Gesichtseindrücken zu Einheiten [Contributions to the analysis of visual perceptions: 1. Some observations on the grouping of visual impressions into wholes]. *Zeitschrift für Psychologie*, **23**, 1-32.
- SENECHAL, M. (1990). *Crystalline symmetries: An informal mathematical introduction*. Bristol, U.K.: Adam Hilger.
- STEVENS, P. S. (1980). *Handbook of regular patterns: An introduction to symmetry in two dimensions*. Cambridge, MA: MIT Press.
- WATT, R. J. (1991). *Understanding vision*. London: Academic Press.

NOTES

1. The magnitude (or length, or norm) of vector $\mathbf{x}$ is $\|\mathbf{x}\|$. The symbols "+" and "-" represent vector addition and subtraction.
2. The French physicist and naval officer Auguste Bravais (1811-1863) was the first to enumerate the 5 plane lattices (which, in Bravais, 1850/1949, are called *nets*) and the 14 lattices in three-dimensional space. In his honor, crystallographers call plane and three-dimensional lattices *Bravais lattices* (Senechal, 1990). (Bravais also studied terrestrial magnetism, the northern lights, botanical geography, astronomy, and hydrography.)
3. To the best of my knowledge, Necker never discussed the reversals of a cube.
4. Each of the five Bravais lattices corresponds to several kinds of wallpaper patterns: two for the oblique lattice, five for the rectangular lattice, three for the square lattice, two for the centered rectangular lattice, and five for the hexagonal lattice. Thus, there are 17 types of wallpaper patterns (Armstrong, 1988, chap. 26). It is beyond the scope of this paper to focus on each of these.
5. The international crystallographic symbol of this wallpaper pattern is $p1$.
6. As long as the same part of each copy of the motif coincides with the dot it replaces, it is of no consequence which part of the motif coincides with the dot.
7. The international crystallographic symbol of this wallpaper pattern is $p4mm$.
8. Glide reflection is the transformation that relates right and left footprints.

(Manuscript received September 29, 1993;
revision accepted for publication March 1, 1994.)