METHODS & DESIGNS

Generation of internally restricted random-dot patterns*

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The probability of a successful draw of a dot position within a random-dot pattern is considered for the special case where a minimal city block distance is imposed separating any two dots within the random pattern. An analytical solution is derived and is tested against Monte Carlo simulations. The agreement is considered to be good, except at extremely low probabilities of a successful draw. An alternative method, based upon the utilization of successively coarser grids, is also given. Formulas are also provided for circular and for diamond-shaped separation regions.

Random-dot patterns have been used in a variety of psychophysical experiments. For example, in our own examination of the memory for spatial position for random-dot visual patterns (e.g., Pollack, 1972), we wanted to guarantee that any two dots would be separated by a given minimal distance from each other. This "protection" feature was introduced in order to separate dots within defined "signal" and "noise" dot fields. The simplest method to ensure the required conditions was to construct an imaginary square of side $2D$ around each randomly drawn point and to test later points for encroachment upon the protected area. The protection is illustrated in Fig. 1a. Each later random draw that encroached upon earlier draws was rejected and a new random draw was made.

For a fixed protection, the probability of a successful draw for the n^{th} point, $P(n)$, will decrease as n is increased. For a given number of points, $P(n)$ will decrease as the protection, D , is increased. Despite the microsecond speed of a small computer, at large values of D or n , long delays were encountered in generating the random-dot fields. Since we wished to employ dot fields with nearly maximum density, it was decided to seek an analytical solution and to verify the solution.

As each new point is successfully added, the probability of locating the next point on a given random draw is decreased by an amount equal to the area of the protected region around the new point divided by the total area of the target square, L^2 (see Fig. 1a). However, if the protected region around the new point overlaps the similar region near a previous point or overhangs the edge of the target square, the amount of

overlap must be added back into the probability expression. As the number of points increases, the average amount of overlap between each new point and existing points must also increase until almost no new territory is covered by the protected regions around the last few points. Because of the complications introduced by these overlapping regions, an exact expression for the probability is impossible to find. The purpose of this paper is to derive and to justify a workable approximation to such an expression.

DERIVATION OF THEORETICAL EXPRESSION

We sought a simple expression approximating the relation between the probability of a successful draw, $P(n)$, and the number of the point to be located, n , on a target square of side, L . It is clear that a linear approximation in n is insufficient, because the probability of overlap of the protected regions around adjacent points increases as the number of points increases. We therefore begin with the general quadratic expression.

$$P(n) = An^2 + Bn + C,$$

where $P(n)$ is the probability of successfully situating the n^{th} point on a given random draw, and A , B , and C are constants to be determined.

Since the first point can be placed with a probability of one, we write our first condition as

$$\begin{aligned} P(1) &= A + B + C \\ &= 1. \end{aligned} \tag{1}$$

To find the second condition, we consider the probability of failure of placing the second point, $\overline{P(2)}$. This is given simply by the ratio of the mean effective protected area around the first point, m , to the total area of the target square, L^2 . Then

$$\begin{aligned} P(2) &= 1 - \overline{P(2)} \\ &= 1 - m/L^2. \end{aligned}$$

To find m , we must examine the positioning possibilities of the first point. If it lands at least a distance, D , away from any edge of the target square (the area denoted as I in Fig. 1b), then its entire protected region [area = $(2D)^2$] will fall within the target square. Should the first point fall within D of any side, but not of two sides

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Fig. 1. (a) Illustration of city block protected regions about points. (b) Illustration of restrictions on placement of second point. (c) Illustration of maximum packing density. (d) Illustration of minimal packing density.

(Areas II in Fig. 1b), then up to half of its protected region may fall outside the target square. The mean effective protected region is only $(3/4)(2D)^2$ in this case. Finally, if the first point falls in any corner square (Areas III in Fig. 1b), the mean effective protected region falls to $(9/16)(2D)^2$. (See Appendix I for a derivation of these expressions.). The total mean protected region, m , is simply the sum of these three areas multiplied by their respective probabilities of occurrence, i.e.,

$$\begin{aligned}
 m &= (2D)^2 [(L-2D)(L-2D)]/L^2 \quad (\text{contributed by I}) \\
 &+ (3/4)(2D)^2 [4(L-2D)(D)]/L^2 \quad (\text{contributed by II}) \\
 &+ (9/16)(2D)^2 [4(D)(D)]/L^2 \quad (\text{contributed by III}) \\
 &= [(2D)^2/L^2] [(L-2D)^2 + 3D(L-2D) + (9/4)D^2] \\
 &= [(2D)^2/L^2] [L^2 - DL + D^2/4].
 \end{aligned}$$

Then,

$$\begin{aligned}
 P(2) &= 1 - m/L^2 \\
 &= 1 - 4(D/L)^2 [1 - D/L + (1/4)(D/L)^2]
 \end{aligned}$$

$$= 1 - 4(D/L)^2 + 4(D/L)^3 - (D/L)^4.$$

This gives our second condition,

$$\begin{aligned}
 P(2) &= 4A + 2B + C \\
 &= 1 - 4(D/L)^2 + 4(D/L)^3 - (D/L)^4. \quad (2)
 \end{aligned}$$

We now postulate the existence of a point, n_0 , at which the probability, $P(n_0)$, goes to zero. That is, after the $(n_0 - 1)^{\text{st}}$ point is situated, the target square is completely filled. This yields the third condition,

$$\begin{aligned}
 P(n_0) &= A n_0^2 + B n_0 + C \\
 &= 0. \quad (3)
 \end{aligned}$$

An upper and a lower limit can be found for n_0 by considering the maximum and minimum packing densities allowable on the filled square. The maximum density occurs when every protected region completely overlaps with its neighbors (see Fig. 1c). n_0 is then found from the figure as follows:

$$(n_0)_{\text{max}} - 1 = (L/D + 1)^2$$

or

$$(n_o)_{\max} = (L/D + 1)^2 + 1.$$

The minimum density occurs when no overlap exists between neighboring protected regions (Fig. 1d). Then,

$$(n_o)_{\min} - 1 = (L/2D)^2$$

or

$$(n_o)_{\min} = (L/2D)^2 + 1.$$

Hence,

$$(L/2D)^2 + 1 \leq n_o \leq (L/D + 1)^2 + 1.$$

It is possible to estimate a "median value" for n_o from the above expression, and, by substituting into Eq. 3, to find A, B, and C from the three constraint conditions given. We have chosen, however, to introduce another condition, namely, that at n_o , $dP/dn = 0$, and then to find A, B, C, and n_o from the four constraint conditions. The fourth condition becomes

$$\begin{aligned} (dP/dn) |_{n=n_o} &= 2An_o + B \\ &= 0. \end{aligned} \quad (4)$$

This can be assumed to be true because the probability of overlap increases as the number of points increases. The protected region of the last point is likely therefore to cover little new area, but rather to overlap previous protected areas almost completely.

The four constraint equations numbered above constitute four equations in the four unknown constants, A, B, C, and n_o . From Eqs. 1, 3, and 4, we can write A, B, and C in terms of n_o . We find the resultant probability relation to be

$$P(n) = [(n_o - n)/(n_o - 1)]^2$$

and from Eq. 2, we find an expression for n_o :

$$n_o = [1 + f(D) + \sqrt{1 - f(D)}] / f(D),$$

where

$$f(D) = 4(D/L)^2 - 4(D/L)^3 + (D/L)^4.$$

The negative-term root of Eq. 2 was rejected as a solution for n_o , because it allows $n > n_o$ over the range of interest and is therefore a nonphysical solution.

CORROBORATIVE DATA

To test the derived formula, it proved easier to measure the inverse of the probability,

$$\begin{aligned} Y(n) &= [P(n)]^{-1} \\ &= [(n_o - 1)/(n_o - n)]^2 \end{aligned}$$

which represents the expected value of the number of attempts necessary to situate the n^{th} point.

A program was written for the PDP-8 computer, which generated random points on a 1,024 by 1,024 matrix, tested the protection of previous points, and counted unsuccessful tries.

The average number of drawings required for the indicated D levels are plotted on the ordinate of Fig. 2. The abscissa represents the n^{th} dot, expressed relative to n_o . The horizontal scaling is adjusted between successive curves such that the distance from $n = 0$ to $n = n_o$ is independent of D. The abscissa scale for each successive curve is successively displaced leftward 0.20 along the abscissa.

Each run terminated when the number of searches for a given point surpassed 4,095. The graphs do not include any point for which the number of searches surpassed 4,095 during any run. The solid line on each graph represents the curve calculated according to the equation for $Y(n)$ above, which approaches n_o (dashed line) asymptotically.

Because the field of random points generated by the computer is discrete and not continuous, some error is introduced into the point search. For example, the values of D used in the calculation of n_o were the same integral values used in the computer search. But there is an uncertainty of 1 unit in those values, because they are constrained to be integers. The effect of this uncertainty was minimized by maintaining $D \gg 1$. The limitation of the point search to a finite matrix could not, of course, be corrected and is simply noted here as a possible source of some error.

The agreement of the curves to the points is considered to be very good except at extremely low probabilities of achieving a successful drawing, i.e., where the average number of drawings > 100 .

AN ALTERNATIVE SOLUTION¹

In the editorial review of this paper, a referee generously suggested an alternative solution for generating random-dot patterns which does not require the testing of any random draw against previous draws. The alternative method, therefore, may be faster than the method outlined in the earlier sections.

With the entire display defined by a p-bit digital-to-analog converter, there are 2^p horizontal lines, 2^p vertical lines, and 2^{2p} possible intersections for random dots. With $2D = 2^0 = 1$, all intersections are admissible, and the drawing of the first n points from a random permutation of 2^{2p} possible points will define the required dot pattern.

With $2D = 2^1 = 2$, a horizontal and a vertical grid based upon $(p - 1)$ bits will satisfy the protection feature. And, in general, with $2D = 2^k$, a grid based upon $(p - k)$ bits will satisfy the protection feature. The selection of the first n points from a random permutation of $2^{2(p-k)}$ points will define the required random-dot pattern.

One beauty of this alternative procedure is that the

Fig. 2. The average number of successive random drawings required to satisfy several city block protection levels upon a 1,024 by 1,024 square. The lower abscissa is the n^{th} point within the matrix, scaled so that the distance is normalized, relative to n_0 , and is independent of D . The abscissa scale is appropriate to the rightmost curve. The abscissa scale for each successive curve is displaced successively leftward 0.20 along the abscissa. Associated with each curve is D , the city block protection distance, and n_0 . The probability of a successful random drawing for the n^{th} dot, $P(n)$, is the reciprocal of the average number of drawings required for a successful draw, $Y(n)$. Each point represents an average over 40 runs. At smaller values of D , a progressively greater number of points are deleted in order to avoid visual overlap. The vertical dashed lines represent $n/n_0 = 1.0$.

random permutation is accomplished through fewer and fewer points as the protection feature is made larger and larger.

Are there any disadvantages of the alternative procedure? We can think of only two. First, the grid size becomes coarser and coarser with larger and larger protection regions, resulting in the appearance of matrix-type patterns in selected areas of the display. Second, with a small computer, the alternative method may be impossible and/or may actually become slower than the method outlined in the earlier section. For example, with a 10-bit D-to-A converter, with $2D = 1$, one must randomly permute 2^{20} or 10^6 points. It is difficult to accomplish this in a small computer with a limited memory. For small protection regions and for a small number of points, one might be willing to employ each possible x and y value only once. This would involve two random permutations among $2^{(p-k)}$ x and y positions. Separate permutation of x and y positions is, of course, unworkable when $n > 2^{(p-k)}$, i.e., when n is greater than the square root of the number of possible points.

In short, if a grid-like appearance to segments of a

random-dot pattern is not unacceptable, the alternative procedure may be superior when a relatively small number of points is to be drawn with relatively large protected regions.

APPENDIX I Mean Protected Area Calculation

The expected value of the protected area evaluated over some region, S , is given by

$$E(A) = \int \int_S A(x,y)p(x,y) dx dy,$$

where $A(x,y)$ is the protected area as a function of the location of its center point, (x,y) , and $p(x,y)$ is the probability density function in S .

Consider a point in the singly cross-hatched region of Fig. 1b. Since the probability density function must be uniform, we have

$$p_{II}(x,y) = [D(L - 2D)]^{-1}.$$

The area of the protected region that falls inside the target square is

$$A(x,y) = 2D(x + D).$$

Hence,

$$\begin{aligned} E(A) &= \int_0^D dx \int_D^{L-D} dy [2D(x + D)] / [D(L - 2D)] \\ &= [2/(L - 2D)] (y)_D^{L-D} \int_0^D (x + D) dx \\ &= 2(x^2/2 + Dx)_0^D = 3D^2 = (3/4)(2D)^2. \end{aligned}$$

Inside the doubly cross-hatched region of Fig. 1b, we have

$$p_{III}(x,y) = (D)^{-2}$$

and

$$A(x,y) = (x + D)(y + D).$$

Hence,

$$\begin{aligned} E(A) &= \int_0^D dx \int_0^D dy [(x + D)(y + D)] / D^2 \\ &= (1/D^2)(x^2/2 + Dx)_0^D (y^2/2 + Dy)_0^D \\ &= [1/D^2] [(3/2)D^2]^2 = (9/4)D^2 = (9/16)(2D)^2. \end{aligned}$$

APPENDIX II

Circular Protection Regions

The analysis used to determine the probability expression, $P(n)$, and the limiting value, n_o , is general. With few modifications, the analysis can be employed with variously shaped protection regions. This section illustrates an extension from the square protection region with city block distance, D , to a circular protection region with radius D . Again, it is assumed that dots are painted upon a square display. In the interest of brevity, intermediate steps in the derivations are omitted.

Equations 1, 3, and 4 remain unchanged. They yield the previous result,

$$P(n) = [(n - n_o)/(n_o - 1)]^2.$$

To find n_o , we must examine $P(2)$ as before:

$$P(2) = 1 - (m/L^2),$$

where m is the mean effective protected area around the first point placed on the target square.

To find m , the target square is divided into four regions, identified as I, II, III, and IV in Fig. 3a. In Region I, the entire protection region will remain within the target square. Hence,

$$E_I(A) = \pi D^2.$$

Consider a point in one of the Regions II (Fig. 3b). The uniform probability density function is again

$$p_{II}(x,y) = [D(L - 2D)]^{-1}.$$

The area of the protected region that falls within the target square is

$$A_{II}(x,y) = \pi D^2 - D^2 \arccos(x/D) + x\sqrt{D^2 - x^2}.$$

Integrating,

$$\begin{aligned} E_{II}(A) &= \int_0^D dx \int_D^{L-D} dy p_{II}(x,y) A_{II}(x,y) \\ &= D^2 (\pi - 2/3). \end{aligned}$$

In Regions III (Fig. 3c),

$$p_{III}(x,y) = [D^2(1 - \pi/4)]^{-1}$$

and

$$\begin{aligned} A_{III}(x,y) &= \pi D^2 - D^2 [\arccos(x/D) + \arccos(y/D)] \\ &\quad + x\sqrt{D^2 - x^2} + y\sqrt{D^2 - y^2}. \end{aligned}$$

Integrating,

$$\begin{aligned} E_{III}(A) &= \int_0^D dx \int_0^D \frac{dy}{\sqrt{D^2 - x^2}} p_{III}(x,y) A_{III}(x,y) \\ &= D^2 [(\pi - \pi^2/8 - 4/3)/(1 - \pi/4)]. \end{aligned}$$

A similar analysis for one of the Regions IV (Fig. 3d), yields

$$p_{IV}(x,y) = [D^2(\pi/4)]^{-1}$$

and

$$\begin{aligned} A_{IV}(x,y) &= 3\pi D^2/4 - D^2 [\arccos(x/D) + \arccos(y/D)]/2 \\ &\quad + (x\sqrt{D^2 - x^2} + y\sqrt{D^2 - y^2})/2 + xy. \end{aligned}$$

Integrating again, we find

$$\begin{aligned} E_{IV}(A) &= \int_0^D dx \int_0^{\sqrt{D^2 - x^2}} dy p_{IV}(x,y) A_{IV}(x,y) \\ &= D^2 (\pi^2 + 1)/2\pi. \end{aligned}$$

These expressions for $E(A)$ in Regions I, II, III, and IV can be combined to find m , as before:

$$\begin{aligned} mL^2 &= (L - 2D)^2 E_I(A) + 4D(L - 2D) E_{II}(A) \\ &\quad + 4D^2(1 - \pi/4) E_{III}(A) + 4(\pi D^2/4) E_{IV}(A) \\ &= D^2 [\pi L^2 - (8/3)LD + (1/2)D^2]. \end{aligned}$$

The derived expression for $P(2)$ can now be written as

$$\begin{aligned} P(2) &= 1 - (m/L^2) \\ &= 1 - \pi(D/L)^2 + (8/3)(D/L)^3 - (1/2)(D/L)^4. \end{aligned}$$

And finally, the expression for n_o can be found, in the same manner as before:

Fig. 3. Illustration of circular protection regions about points (a) and of restrictions on placement within Region II (b), Region III (c), and Region IV (d).

$$n_o = [1 + g(D) + \sqrt{1 - g(D)}] / g(D)$$

where

$$g(D) = \pi(D/L)^2 - (8/3)(D/L)^3 + (1/2)(D/L)^4.$$

When the analysis of this section is applied to the D levels of Fig. 2, the equivalent n_o levels for $L = 1,024$ are: 670, 302, 172, 79.1, 45.9, 21.9, and 13.2, respectively. As anticipated, there is a gain in the mean limiting number of dots with the circular protection region, relative to the square protection region. Over the range examined, n_o for the circular protection region is 1.21 to 1.26 times that for the square protection region.

Lastly, we have extended the examination to diamond-shaped regions with x and y diagonals of length $2D$. Paralleling the development for the other shaped regions, we express

$$n_o = [1 + h(D) + \sqrt{1 - h(D)}] / h(D),$$

where

$$h(D) = 2(D/L)^2 - (4/3)(D/L)^3 + (1/6)(D/L)^4.$$

Since the area of the city block square protection region is $4D^2$ and the area of the diamond protection

region is $2D^2$ in the form of a rotated square, the coefficients of $f(D)$ and $h(D)$ should be in the ratio of 2 to 1, and they are for first term. The higher ordered coefficients reflect edge effects and do not maintain the 2-to-1 ratio. When the analysis of the diamond protection region is applied to the D levels of Fig. 2, the equivalent n_o levels for $L = 1,024$ are: 1,046, 470, 268, 122, 70.2, 32.9, and 19.6, respectively. As anticipated, these levels are higher than those for the square and circular protection regions. The mean limiting number of dots with the diamond protection region is 1.80 to 1.95 times that for the square protection region.

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NOTE

1. The authors are indebted to Professor E. Carterette, University of California at Los Angeles, for the alternative procedure described in this section and for pointing out the attempt of Kendall and Moran (1963) to achieve an analytical solution for the case of a circular protection region (see Appendix II).

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