Notes and Comment

The multiple dual-pair method

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A model for the multiple dual-pair method, a generalization of the traditional dual-pair (4IAX) paradigm, is given. This model is expressed in terms of normal and beta distributions. This generalization allows for the simultaneous estimation of the perceptual distances among three or more stimuli. This model has applications in cases in which multiple two-sample comparisons would be too time consuming and labor intensive. The theory discussed shows how unequal variances can be estimated on the basis of results from that method. Two numerical examples that illustrate the ability of the beta distribution-based model to retrieve the appropriate parameters are given. It is also shown how the traditional dual-pair model is a special case of the multiple dual-pair model.

Probabilistic models with applications in human psychophysics provide the capability to link the results of many different choice tasks to a common foundation. Experimentally, different tasks, such as m -alternative forced choice, the triangle or oddity method, the duo-trio method, categorical choice (including rating methods), same-different judgments, preferential choice, identification, and many others, may be modeled on the basis of the assumed existence of perceptual variance and a task-specific decision rule. The basis for these models was established in Thurstone's (1927) measurement contributions. The models may be unidimensional or multidimensional. Early contributions to multidimensional scaling and unfolding involved triads (Coombs, 1964, and Torgerson, 1958). These triads may be composed of stimuli (Torgerson's method) or an ideal point and two stimuli, as occurs in preferential choice unfolding, originating in an idea of Coombs. A formal connection between identification, preferential choice, and the method of triads was shown by Ennis (1993). In many of these models, the distribution of a unidimensional or multidimensional distance arises, and to achieve generality, it is useful to formulate the models with respect to this distribution.

The goal of the present paper is to develop a probabilistic model for the multiple dual-pair method to enlarge and enrich the repertoire of probabilistic psychophysical models. These methods have unique capabilities, which can be used to find solutions to many practical scaling problems.

In the traditional dual-pair paradigm, also called 4IAX, two stimuli, A and B, are compared. A subject is presented with two pairs of samples, one containing two identical stimuli (e.g., AA) and the other containing different stimuli, one of these being the same as that used in the identical pair (e.g., AB). The task of the subject is to select the pair containing the more different samples. A model has been published expressing the psychometric function in terms of normal distributions (Macmillan, Kaplan, & Creelman, 1977). An alternative Thurstonian model has been developed to describe the dual-pair psychometric function in terms of a singly noncentral beta distribution (Rousseau & Ennis, 2001). This model is more consistent with distance-based probabilistic multidimensional scaling.

The present paper introduces the multiple dual-pair method so that differences among more than two stimuli can be measured simultaneously. This generalization relates to the generalizations of the duo-trio (ABX) and triangular (oddity) models to Torgerson's (1958) and Richardson's (1938) methods of triads, respectively (Ennis, Mullen, & Frijters, 1988).

In many practical applications, similarities among more than two stimuli need to be scaled. A special case of this application arises when variants of a stimulus are not identical, such as when variants are manufactured at different locations. An experimental investigation of methods and models relevant to such situations has recently been conducted (Rousseau, O'Mahony, & Ennis, 2002).

In the multiple dual-pair method, at least three different stimuli are presented in each trial in two pairs—for example, AB/AC or AB/CD. The task of the subject is to select the pair with the more different samples. Appropriate sample rotation is used throughout the design (e.g., AC/BD, AD/BC, . . .). In this paper, a probabilistic model of the multiple dual-pair task is described. The model is expressed in two mathematical forms: The first one expresses the relation between response probabilities and the scaling parameters in terms of bivariate normal distributions, whereas the second relates these probabilities to a doubly noncentral beta distribution. In this article, they will be referred to as the *normal model* and the *beta distribution model*, respectively. It is worth noting that the models presented here assume that the respondents are unbiased (Kaplan, Macmillan, & Creelman, 1978; Macmillan & Creelman, 1991). The models relate to those of Torgerson and

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Table 1
Observed ($n = 100$) and Actual Decision Probabilities
for the Multiple Dual-Pair Paradigm When $\mu_1 = 0$, $\mu_2 = 0.5$,
 $\mu_3 = 1.3$, $\mu_4 = 3$, and All Perceptual Standard Deviations
are Equal to 1

First Pair	Second Pair					
	1,2	1,3	1,4	2,3	2,4	3,4
1,2	—	.59	.86	.52	.81	.65
	—	.60	.86	.53	.80	.66
1,3	—	—	.76	.47	.72	.65
	—	—	.79	.43	.71	.57
1,4	—	—	—	.11	.33	.27
	—	—	—	.16	.40	.26
2,3	—	—	—	—	.75	.63
	—	—	—	—	.77	.64
2,4	—	—	—	—	—	.39
	—	—	—	—	—	.35

Note—Observed values are in standard type; actual values are in bold.

Richardson (Ennis et al., 1988), which also considered unbiased observers. They also assume that the scaling of differences is unidimensional. The beta distribution form has the advantage of extendability to handle multidimensional representations.

Let $S_1, S_2, \dots, S_n$ represent n stimuli, with unidimensional intensities characterized by normal distributions. $x_1, x_2, \dots, x_n$ are the momentary psychological values, $\mu_1, \mu_2, \dots, \mu_n$ are their means, and $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2$ are their variances.

A tetrad used in the multiple dual-pair method is composed of four stimuli— S_1, S_2, S_3 , and S_4 . Let S_1 and S_2 be the first pair of samples, whereas S_3 and S_4 make up the second pair. Let ${}_{12}P_{34}$ denote the probability that S_1 and S_2 are more similar than S_3 and S_4 . This probability can be written as

$${}_{12}P_{34} = \Pr [|x_1 - x_2| < |x_3 - x_4|]. \quad (1)$$

This probability corresponds to this particular sample presentation. Many others (e.g., ${}_{12}P_{13}, {}_{13}P_{34}, \dots$) can and will be obtained during an experiment investigating differences among these stimuli.

The Normal Distribution Model

${}_{12}P_{34}$ can be expressed as

$${}_{12}P_{34} = \int_{-\infty}^0 \int_0^{\infty} f(z) dz + \int_0^{\infty} \int_{-\infty}^0 f(z) dz \quad (2)$$

$$f(z) = \frac{\exp\{-0.5[(z - \mu)'V^{-1}(z - \mu)]\}}{(2D)|V|^{1/2}}$$

with $\mu = (\mu_1 - \mu_2 - \mu_3 + \mu_4, \mu_1 - \mu_2 + \mu_3 - \mu_4)$ and

$$V = \begin{bmatrix} \sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \sigma_1^2 & \sigma_2^2 + \sigma_3^2 - \sigma_4^2 - \sigma_1^2 \\ \sigma_2^2 + \sigma_3^2 - \sigma_4^2 - \sigma_1^2 & \sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \sigma_1^2 \end{bmatrix}.$$

A proof of Equation 2 is given in Appendix A.

From Equation 2, the expression provided in Equation 5 by Macmillan et al. (1977) can be recovered. For the traditional dual-pair/4IAX, only two stimuli with equal vari-

ance are compared. Thus, if $\mu_1 = \mu_2 = \mu_3 = 0$, $\mu_4 = d'$ and $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2 = 1$,

$$P_c = {}_1\Phi({d' / 2})^2 + {}_1\Phi(-{d' / 2})^2. \quad (3)$$

A proof of Equation 3 is also provided in Appendix A.

The Beta Distribution Model

${}_{12}P_{34}$ can be expressed as

$${}_{12}P_{34} = G \left[1, \frac{\sigma_{34}^2}{\sigma_{12}^2}; \left(\frac{\mu_{12}}{\sigma_{12}} \right)^2, \left(\frac{\mu_{34}}{\sigma_{34}} \right)^2 \right], \quad (4)$$

where

$$\mu_{12} = \mu_1 - \mu_2, \mu_{34} = \mu_3 - \mu_4, \\ \sigma_{12}^2 = \sigma_1^2 + \sigma_2^2, \sigma_{34}^2 = \sigma_3^2 + \sigma_4^2,$$

and G is a doubly noncentral beta distribution (Johnson & Kotz, 1970a, 1970b) with noncentrality parameters

$$\left(\frac{\mu_{12}}{\sigma_{12}} \right)^2$$

and

$$\left(\frac{\mu_{34}}{\sigma_{34}} \right)^2$$

and argument

$$\frac{\sigma_{34}^2}{\sigma_{12}^2}.$$

A proof of Equation 4 is given in Appendix B.

From Ennis and Johnson (1993), the doubly noncentral beta distribution can be expressed as

$${}_{12}P_{34} = 0.5 - \frac{1}{\pi} \int_0^{\infty} \frac{\sin \theta(x)}{x \rho(x)} dx,$$

with

$$\theta(x) = 0.5 \left[\tan^{-1}(x) + \left(\frac{\mu_{12}}{\sigma_{12}} \right)^2 x (1 + x^2)^{-1} \tan^{-1} \left(\frac{\sigma_{34}^2}{\sigma_{12}^2} x \right) \right. \\ \left. - \left(\frac{\mu_{34}}{\sigma_{34}} \right)^2 x \left(1 + \left[\frac{\sigma_{34}^2}{\sigma_{12}^2} x \right]^2 \right)^{-1} \right]$$

and

$$\rho(x) = (1 + x^2)^{1/4} \left[1 + \left(\frac{\sigma_{34}^2}{\sigma_{12}^2} x \right)^2 \right]^{1/4} \exp \left\{ 0.5 \left[\left(\frac{\mu_{12}}{\sigma_{12}} \right)^2 \right. \right. \\ \left. \left. x^2 (1 + x^2)^{-1} + \left(\frac{\mu_{34}}{\sigma_{34}} \right)^2 \left(\frac{\sigma_{34}^2}{\sigma_{12}^2} x \right)^2 \right. \right. \\ \left. \left. \left(1 + \left[\frac{\sigma_{34}^2}{\sigma_{12}^2} x \right]^2 \right)^{-1} \right] \right\}.$$

Table 2
Population Means and Variances Estimated Using the Doubly Noncentral Beta
Distribution Model for the Multiple Dual Pair

	Population Mean	Population Mean Estimate	Variance of Estimate	95% Confidence Interval of Estimate
μ_1	0.0	0.00	—	—
μ_2	0.5	0.58	0.015	[0.34; 0.82]
μ_3	1.3	1.23	0.017	[0.97; 1.49]
μ_4	3.0	3.01	0.024	[2.71; 3.31]

Note— μ_1 is assigned to 0.0 and, therefore, has no variance. The means and their variances listed are those corresponding to the distance between the first stimulus and each of the other three.

It can be seen that Equation 4 is the generalization of the psychometric function for the dual-pair method. For the dual-pair, $\mu_1 = \mu_2 = \mu_3 = 0$, $\mu_4 = \delta$. Furthermore, all variances are the same and equal to 1. Therefore, Equation 4 becomes

$${}_{12}P_{34} = F\left[1, 1; \left(\frac{\delta}{\sqrt{2}}\right)^2\right],$$

where F is a singly noncentral beta distribution.

This is Equation 5 derived by Rousseau and Ennis (2001) for the psychometric function for the dual-pair paradigm.

Examples of Parameter Estimation for Four Stimuli

When four stimuli are used, a total of 15 different trials can be formed (excluding the case of pairs with identical stimuli), leading to a total of 15 independent probabilities, listed in the upper diagonal of Tables 1 and 3.

Equal Variance Example

Consider a hypothetical market study involving four noncarbonated orange beverages differing in sweetness. One hundred consumers are sampled from the population under study and are asked to estimate the 15 trials over three sessions involving 5 trials each. Presentation order is randomized and counterbalanced over all 100 subjects. In each trial, a subject is presented with two pairs of beverages and is asked to select the pair in which the samples are most different.

Let 0, 0.5, 1.3, and 3 be the means of the perceptual distributions of beverages 1, 2, 3, and 4, respectively. All the distribution standard deviations are identical and equal to 1. Table 1 gives the 15 probabilities for a sample of size 100 drawn at random from normal distributions with the appropriate parameters. Actual probabilities are also provided. The beta distribution model was used to fit the data in Table 1, using a gradient method (Dennis & Schnabel, 1983) and the method of maximum likelihood. Table 2 gives the estimates of the parameters, as well as their variances. Those variances can be used to perform some statistical test on the comparison of the estimates, using an approach described by Bi, Ennis, and O'Mahony (1997). The recovery of the parameters is not perfect owing to

sampling variability (only 100 trials were available for each probability). Chi-squared tests among the maximum likelihood solutions for the guessing, the specified, and the saturated models were also conducted. The guessing model assumes that no difference exists among the stimuli and, thus, that the best model to describe the data is a single distribution ($\mu_1 = \mu_2 = \mu_3 = \mu_4$). The specified model is one in which four distributions of equal variance are assumed. The saturated model represents the best fit that can be obtained from the data collected.

The χ^2 tests indicated that the specified model was significantly different from the guessing model ($p < .001$; the model with four distributions fit the data significantly better than the one with one distribution), whereas the specified and the saturated models were not significantly different ($p = .82$; the model with four distributions was not significantly different from the best fit of the data). This indicates that the model was successful at recovering the parameters and that there was no need to go to a higher hierarchical level model involving unequal variances.

Unequal Variance Example

Consider another study involving a comparison of the appearances of apples grown in four different locations. Apples in each location appear to exhibit differences in skin darkness. The experimenter suspects that variability

Table 3
Observed ($n = 1,000$) and Actual Decision Probabilities for the
Multiple Dual-Pair Paradigm When $\mu_1 = 0$, $\mu_2 = 0.5$, $\mu_3 =$
1.8, $\mu_4 = 2.5$, and $\sigma_1 = 1$, $\sigma_2 = 1.2$, $\sigma_3 = 1.4$, and $\sigma_4 = 0.8$

First pair	Second pair					
	1,2	1,3	1,4	2,3	2,4	3,4
1,2	—	.65	.78	.61	.69	.53
		.66	.77	.60	.69	.54
1,3	—	—	.62	.46	.53	.37
			.60	.46	.54	.38
1,4	—	—	—	.34	.41	.25
				.34	.41	.23
2,3	—	—	—	—	.57	.42
					.59	.40
2,4	—	—	—	—	—	.34
						.36

Note—Observed values are in standard type, actual values are in bold.

Table 4
Population Means and Variances Estimated Using the Doubly Noncentral
Beta Distribution Model for the Multiple Dual Pair

	Population Mean	Population Mean Estimate	Variance of Estimate	95% Confidence Interval of Estimate
μ_1	0.0	0.00	—	—
μ_2	0.5	0.48	0.014	[0.24, 0.71]
μ_3	1.8	1.77	0.119	[1.09, 2.44]
μ_4	2.5	2.51	0.155	[1.73, 3.28]
σ_1	1.0	1.00	—	—
σ_2	1.2	1.24	0.08	[0.69, 1.79]
σ_3	1.4	1.44	0.02	[1.16, 1.71]
σ_4	0.8	0.79	0.16	[0.00, 1.57]

Note— μ_1 is assigned to 0.0 and, therefore, has no variance. The same applies to the value of σ_1 . The means and their variances listed are those corresponding to the distance between the first stimulus and each of the other three.

ity within a type of apple might vary from one type to the next. A larger group of 1,000 subjects is used. In each trial, the subjects are required to indicate which pair is composed of products that are more different. Because of the limited fatigue effect involved in the evaluation, all 15 trials can be performed in one session.

Let 0, 0.5, 1.8, and 2.5 be the means of the perceptual distributions of Apples 1, 2, 3, and 4, respectively. Let 1, 1.2, 1.4, and 0.8 be the standard deviations of their perceptual distributions. Table 3 gives the 15 probabilities for a sample of size 1,000 drawn from normal distributions with mean and standard deviation structure as described above, as well as the actual values.

First, the beta distribution model, assuming equal standard deviation for all four distributions, was used. The specified model with four distributions was found to be significantly different from the saturated model (χ^2 , $df = 12$, $p = .003$). This indicated that a higher hierarchical model was needed to account for the data structure. The unequal variance beta distribution model was then used (see Table 4). In this case, the specified model with four distributions was not significantly different from the saturated model (χ^2 , $df = 9$, $p = .11$), whereas it was significantly different from the guessing model (χ^2 , $df = 15$, $p < .001$).

This indicated that the model with unequal variance was successful at recovering the parameters involved in the study. It also showed that the models could reliably identify the need for a higher hierarchical model when an unequal variance structure was involved.

Conclusion

The models presented here complement those developed for Torgerson's (1958) and Richardson's (1938) methods of triads. As with these two particular paradigms, the multiple dual pair allows the simultaneous estimation of perceptual distances among three or more stimuli. Furthermore, Equations 2 and 4 display the relationship between the bivariate

normal and the doubly noncentral beta distributions with a single degree of freedom.

REFERENCES

- BI, J., ENNIS, D. M., & O'MAHONY, M. (1997). How to estimate and use the variance of d' from difference tests. *Journal of Sensory Studies*, **12**, 87-104.
- COOMBS, C. H. (1964). *A theory of data*. New York: Wiley.
- DENNIS, J. E., & SCHNABEL, R. (1983). *Numerical methods for unconstrained optimization and nonlinear equations*. Englewood Cliffs, NJ: Prentice-Hall.
- ENNIS, D. M. (1993). A single multidimensional model for discrimination, identification and preferential choice. *Acta Psychologica*, **84**, 17-27.
- ENNIS, D. M., & JOHNSON, N. L. (1993). Noncentral and central chi-square, F and beta distribution functions as special cases of the distribution function of an indefinite quadratic form. *Communication in Statistics: Theory & Methods*, **22**, 897-905.
- ENNIS, D. M., MULLEN, K., & FRUITERS, J. E. (1988). Variants of the method of triads: Unidimensional Thurstonian models. *British Journal of Mathematical & Statistical Psychology*, **41**, 25-36.
- JOHNSON, N. L., & KOTZ, S. (1970a). *Distributions in statistics: Continuous univariate distributions—1*. New York: Wiley.
- JOHNSON, N. L., & KOTZ, S. (1970b). *Distributions in statistics: Continuous univariate distributions—2*. New York: Wiley.
- KAPLAN, H. L., MACMILLAN, N. A., & CREELMAN, C. D. (1978). Tables of d' for variable-standard discrimination paradigms. *Behavior Research Methods & Instrumentation*, **10**, 796-813.
- MACMILLAN, N. A., & CREELMAN, C. D. (1991). *Detection theory: A user's guide*. New York: Cambridge University Press.
- MACMILLAN, N. A., KAPLAN, H. L., & CREELMAN, C. D. (1977). The psychophysics of categorical perception. *Psychological Review*, **84**, 452-471.
- RICHARDSON, M. W. (1938). Multidimensional psychophysics. *Psychological Bulletin*, **35**, 659-660.
- ROUSSEAU, B., & ENNIS, D. M. (2001). A Thurstonian model for the dual pair (4IAX) discrimination method. *Perception & Psychophysics*, **63**, 1083-1090.
- ROUSSEAU, B., O'MAHONY, M., & ENNIS, D. M. (2002). *Methodological investigation of multiple stimulus evaluation in sensory discrimination testing*. Manuscript in preparation.
- THURSTONE, L. L. (1927). A law of comparative judgement. *Psychological Review*, **34**, 273-286.
- TORGERSOON, W. S. (1958). *Theory and methods of scaling*. New York: Wiley.

APPENDIX A

Derivation of a Bivariate Normal Distribution Model for the Multiple Dual-Pair Method

$${}_{12}P_{34} = \Pr [|x_1 - x_2| < |x_3 - x_4|] \quad (A1)$$

$$\begin{aligned} {}_{12}P_{34} &= \Pr [|x_1 - x_2| < (x_3 - x_4)] + \Pr [|x_1 - x_2| < (-x_3 + x_4)] \\ &= \Pr [(x_1 - x_2) < (x_3 - x_4) \text{ and } (-x_1 + x_2) < (x_3 - x_4)] + \Pr [(x_1 - x_2) < (-x_3 + x_4) \text{ and} \\ &\quad (-x_1 + x_2) < (-x_3 + x_4)] \end{aligned}$$

Let $r = x_1 - x_2$ and $s = x_3 - x_4$

$$r \sim N(\mu_1 - \mu_2; \sigma_1^2 + \sigma_2^2)$$

$$s \sim N(\mu_3 - \mu_4; \sigma_3^2 + \sigma_4^2)$$

$$\begin{aligned} {}_{12}P_{34} &= \Pr [r < s \text{ and } -r < s] + \Pr [r < -s \text{ and } -r < -s] \\ &= \Pr [r - s < 0 \text{ and } -(r + s) < 0] + \Pr [r + s < 0 \text{ and } -(r - s) < 0] \end{aligned}$$

Then

$$r - s \sim N(\mu_1 - \mu_2 - \mu_3 + \mu_4; \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)$$

$$r + s \sim N(\mu_1 - \mu_2 + \mu_3 - \mu_4; \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)$$

Let $z = (r - s, r + s)$

$$\mu = (\mu_1 - \mu_2 - \mu_3 + \mu_4, \mu_1 - \mu_2 + \mu_3 - \mu_4)$$

$$V = \begin{bmatrix} \sigma_2^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 & \sigma_1^2 + \sigma_2^2 - \sigma_3^2 - \sigma_4^2 \\ \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_4^2 & \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 \end{bmatrix}$$

$${}_{12}P_{34} = \int_{-\infty}^0 \int_0^{\infty} f(z) dz + \int_0^{\infty} \int_{-\infty}^0 f(z) dz, \quad (A2)$$

where

$$f(z) = \frac{\exp\{-0.5[(z - \mu)'V^{-1}(z - \mu)]\}}{(2D)|V|^{1/2}}.$$

In the traditional dual-pair/4IAX paradigm, two stimuli with equal variance are compared: $\mu_1 = \mu_2 = \mu_3 = 0$, $\mu_4 = \delta$, $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2 = \sigma^2 = 1$

$$\begin{aligned} P_c &= \Pr [(x_1 - x_2) < (x_3 - x_4) \text{ and } (-x_1 + x_2) < (x_3 - x_4)] + \Pr [(x_1 - x_2) < (-x_3 + x_4) \text{ and} \\ &\quad (-x_1 + x_2) < (-x_3 + x_4)] \\ &= \Pr [x_1 - x_2 - x_3 + x_4 < 0 \text{ and } -x_1 + x_2 - x_3 + x_4 < 0] + \Pr [x_1 - x_2 + x_3 - x_4 < 0 \text{ and} \\ &\quad -x_1 + x_2 + x_3 - x_4 < 0] \end{aligned}$$

$x_1 - x_2 - x_3 + x_4$ and $-x_1 + x_2 - x_3 + x_4$ are orthogonal and, thus, independently distributed, so that

$$\begin{aligned} P_c &= \Pr (x_1 - x_2 - x_3 + x_4 < 0) \Pr (-x_1 + x_2 - x_3 + x_4 < 0) \\ &\quad + \Pr (x_1 - x_2 + x_3 - x_4 < 0) \Pr (-x_1 + x_2 + x_3 - x_4 < 0). \end{aligned}$$

$N(\mu, \sigma^2)$ represents the normal function with mean and variance parameters μ and σ^2 , respectively, and $\Phi(x)$ represents the area under the standard normal curve from $-\infty$ to x .

Let

$$p = \Pr (x_1 - x_2 - x_3 + x_4 < 0) = \int_{-\infty}^0 N(\delta, 4) = \int_{-\infty}^{-\delta/2} N(0, 1) = \Phi(-\delta/2)$$

and

$$q = \Pr (-x_1 + x_2 - x_3 + x_4 < 0) = \int_{-\infty}^0 N(\delta, 4) = \int_{-\infty}^{-\delta/2} N(0, 1) = \Phi(-\delta/2).$$

APPENDIX A (Continued)

Thus,

$$\begin{aligned}
 P_c &= pq + (1-p)(1-q) \\
 &= [\Phi(-\delta/2)]^2 + [1 - \Phi(-\delta/2)][1 - \Phi(-\delta/2)] \\
 &= [\Phi(-\delta/2)]^2 + [\Phi(-\delta/2)]^2.
 \end{aligned} \tag{A3}$$

This is the expression derived by Macmillan et al. (1977) for the dual-pair method.

APPENDIX B
Derivation of a Doubly Noncentral Beta Distribution Model for the Multiple Dual-Pair Method

$${}_{12}P_{34} = \Pr [(x_1 - x_2)^2 < (x_3 - x_4)^2]$$

Let $z_1 = x_1 - x_2$ and $z_2 = x_3 - x_4$. z_1 is normally distributed with mean $\mu_1 - \mu_2 = \mu_{12}$ and variance $\sigma_1^2 + \sigma_2^2 = \sigma_{12}^2$, while z_2 is normally distributed with mean $\mu_3 - \mu_4 = \mu_{34}$ and variance $\sigma_3^2 + \sigma_4^2 = \sigma_{34}^2$.

Hence,

$$\begin{aligned}
 {}_{12}P_{34} &= \Pr (z_1^2 - z_2^2 < 0) \\
 &= \Pr [Z'AZ < 0],
 \end{aligned} \tag{B1}$$

where

$$Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, Z' = [z_1 \quad z_2], \text{ and } A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Z has a bivariate normal distribution with mean vector

$$\mu = \begin{bmatrix} \mu_{12} \\ \mu_{34} \end{bmatrix}$$

and variance covariance matrix

$$\begin{bmatrix} \sigma_{12}^2 & 0 \\ 0 & \sigma_{34}^2 \end{bmatrix}.$$

Standardizing Z , let

$$\begin{aligned}
 W &= \begin{bmatrix} \sigma_{12} & 0 \\ 0 & \sigma_{34} \end{bmatrix} (Z - \mu) \\
 &= L^{-1}(Z - \mu).
 \end{aligned}$$

Hence, $Z = LW + \mu$.

Substituting for Z in (B1),

$$\begin{aligned}
 {}_{12}P_{34} &= \Pr [Z'AZ < 0] \\
 &= \Pr \left\{ \left[\begin{pmatrix} \sigma_{12} & 0 \\ 0 & \sigma_{34} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} \mu_{12} \\ \mu_{34} \end{pmatrix} \right] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \left[\begin{pmatrix} \sigma_{12} & 0 \\ 0 & \sigma_{34} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} \mu_{12} \\ \mu_{34} \end{pmatrix} \right] < 0 \right\} \\
 &= \Pr \left\{ \begin{pmatrix} \sigma_{12}w_1 & + & \mu_{12} \\ \sigma_{34}w_2 & + & \mu_{34} \end{pmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} \sigma_{12}w_1 & + & \mu_{12} \\ \sigma_{34}w_2 & + & \mu_{34} \end{pmatrix} < 0 \right\} \\
 &= \Pr \left\{ \left[\sigma_{12}w_1 + \mu_{12} - (\sigma_{34}w_2 + \mu_{34}) \right] \begin{pmatrix} \sigma_{12}w_1 & + & \mu_{12} \\ \sigma_{34}w_2 & + & \mu_{34} \end{pmatrix} < 0 \right\} \\
 &= \Pr [(\sigma_{12}w_1 + \mu_{12})^2 - (\sigma_{34}w_2 + \mu_{34})^2 < 0] \\
 &= \Pr \left[\sigma_{12}^2 \left(w_1 + \frac{\mu_{12}}{\sigma_{12}} \right)^2 - \sigma_{34}^2 \left(w_2 + \frac{\mu_{34}}{\sigma_{34}} \right)^2 < 0 \right]
 \end{aligned}$$

APPENDIX B (Continued)

$$= \Pr \left[\frac{\left(w_1 + \frac{\mu_{12}}{\sigma_{12}} \right)^2}{\left(w_2 + \frac{\mu_{34}}{\sigma_{34}} \right)^2} < \frac{\sigma_{34}^2}{\sigma_{12}^2} \right]. \quad (\text{B2})$$

The term

$$\frac{\left(w_1 + \frac{\mu_{12}}{\sigma_{12}} \right)^2}{\left(w_2 + \frac{\mu_{34}}{\sigma_{34}} \right)^2}$$

follows a doubly noncentral beta distribution. From Ennis and Johnson (1993):

$$\Pr \left[\frac{\left(w_1 + \frac{\mu_{12}}{\sigma_{12}} \right)^2}{\left(w_2 + \frac{\mu_{34}}{\sigma_{34}} \right)^2} < \frac{\sigma_{34}^2}{\sigma_{12}^2} \right] = 0.5 - \frac{1}{\pi} \int_0^\infty \frac{\sin \theta(x)}{x \rho(x)} dx,$$

with

$$\theta(x) = 0.5 \left[\tan^{-1}(x) + \left(\frac{\mu_{12}}{\sigma_{12}} \right)^2 x (1+x^2)^{-1} \tan^{-1} \left(\frac{\sigma_{34}^2}{\sigma_{12}^2} x \right) - \left(\frac{\mu_{34}}{\sigma_{12}} \right) x \left(1 + \left[\frac{\sigma_{34}^2}{\sigma_{12}^2} x \right]^2 \right)^{-1} \right]$$

and

$$\rho(x) = (1+x^2)^{1/4} \left[1 + \left(\frac{\sigma_{34}^2}{\sigma_{12}^2} x \right)^2 \right]^{1/4} \exp \left\{ 0.5 \left[\left(\frac{\mu_{12}}{\sigma_{12}} \right)^2 x^2 (1+x^2)^{-1} + \left(\frac{\mu_{34}}{\sigma_{34}} \frac{\sigma_{34}^2}{\sigma_{12}^2} x \right)^2 \left(1 + \left[\frac{\sigma_{34}^2}{\sigma_{12}^2} x \right]^2 \right)^{-1} \right] \right\}.$$

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