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REAL AND COMPLEX ANALYSIS

Orientation-Dependent Chord Length Distribution in a Convex Quadrilateral

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Abstract—This work contributes to the research devoted to the recognition of a convex body by probabilistic characteristics of its lower-dimensional sections. In this paper, for any convex quadrilateral, five orientation-dependent characteristics are introduced and explicitly evaluated per direction. In terms of these characteristics, simple explicit representations of the orientation-dependent chord length distribution function and the covariogram are obtained not only for an arbitrary convex quadrilateral but also for any right prism based on it.

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1. INTRODUCTION

Inferring properties of an unknown convex body $\mathbf{D} \subset \mathbb{R}^n$ with a non-empty interior from its chord length measurements is one of the fundamental problems in geometric tomography. Although it is known that the body cannot be characterized by its chord length distribution (see [1]), there are positive results when the distribution function is known for each separate direction. Such a function is known as an *orientation-dependent chord length distribution function (ODCLD)*.

On the other hand, the problem of finding the ODCLD function is equivalent to the problem of finding the function

$$C_{\mathbf{D}}(x) = L_n(\mathbf{D} \cap \{\mathbf{D} + x\}), \quad x \in \mathbb{R}^n,$$

where $\mathbf{D} + x = \{\mathcal{P} + x : \mathcal{P} \in \mathbf{D}\}$ and $L_n(\cdot)$ is the n-dimensional Lebesgue measure in $\mathbb{R}^n$. This function is called the *covariogram* of $\mathbf{D}$.

The hypothesis [2] that $\mathbf{D}$ can be determined from its covariogram was rejected when $n \geq 4$ (see [4, 5]) and confirmed when $\mathbf{D}$ is a planar convex domain (see [6]), or a three-dimensional convex polytope (see [7]). Since then, numerous papers have been published with the objective of achieving an explicit form of the ODCLD function or the covariogram for a specific body $\mathbf{D} \subset \mathbb{R}^n$. In particular, when $n = 2, 3$, the research includes the articles [8] and [9], where $\mathbf{D}$ is a triangle or a parallelogram, [10] and [11], where $\mathbf{D}$ is a regular polygon, an ellipse, or a prism with a triangular or elliptical base. The most recent research in this direction is reflected in [12, 13], and [14], where the ODCLD function and the covariogram are found for some quadrilateral prisms and their rectangular or trapezoidal bases.

This paper focuses on finding an explicit representation of the ODCLD function for an arbitrary convex quadrilateral. The quadrilateral is closed: it contains its interior points and the boundary.

The necessary terminology and characteristics of the quadrilateral to build the ODCLD function are provided in Sections 2 and 3. Particularly, we extend there the concept of a φ-diameter for a polygon introduced by David Mount [3], and then define supplementary measures for a standard image (defined in section 2) of a convex quadrilateral. Readers, already familiar with the concept of X-ray (refer to Chapter 1 of [4]), may benefit while contemplating the origins and significance of the newly introduced

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orientation-dependent characteristics. To determine the ODCLD function, acquiring orientation-dependent X-rays is sufficient (see, for example, [15]). These X-rays, which exhibit convex functions with up to three graph pieces for a convex quadrilateral, can be accurately determined using φ-diameters and supplementary φ-measures as necessary parameters.

The main synthetic results are presented in Section 4, where the ODCLD function and the covariogram of a convex quadrilateral are found in terms of the lengths of orientation-dependent diameters and supplementary measures. As an application, in the last section, the analogs of those results are established for quadrilateral prisms. All orientation-dependent computations are processed in Section 5.

2. A STANDARD IMAGE OF A QUADRILATERAL

In a Cartesian plane, for any convex quadrilateral $\mathbf{D}$ there are points $B(b,0)$, $b>0$, $A\in\{(x,y):x\geq 0, y>0\}$, and $C\in\{(x,y):x>0,y>0\}$ such that $\mathbf{D}$ is congruent to the quadrilateral $OACB$, where O is the origin of coordinates. We will call such a quadrilateral an *image* of $\mathbf{D}$. The side OB will be called the *base*, the sides OA and BC will be called *legs*, α and β will stand for the inclination angles (measured anticlockwise from the positive direction of x-axis) of the legs OA and BC, respectively. If $\alpha\leq\beta$ then the quadrilateral $OABC$ will be called *a standard image* of $\mathbf{D}$.

Proposition 2.1. *Every convex quadrilateral* $\mathbf{D}$ *has a standard image.*

Proof. Let $OACB$ be an image of $\mathbf{D}$. Then let θ_A and θ_C be the internal angles at the vertices A and C, respectively. If $\beta<\alpha$ then $\theta_A+\theta_C<\pi$.

If $\theta_A<\frac{\pi}{2}$, consider the Euclidean transformation $\mathcal{T}$ that rotates the plane clockwise about the origin by α and then reflects it on the x-axis. Then $OA'C'B'$ becomes a standard image of $\mathbf{D}$, where $A'=\mathcal{T}(B)$, $B'=\mathcal{T}(A)$, and $C'=\mathcal{T}(C)$. Indeed, if α' and β' are the corresponding inclination angles of the legs OA' and $B'C'$, then

$$\alpha'=\alpha\leq\frac{\pi}{2}<\pi-\theta_A=\beta'.$$

If $\theta_C<\frac{\pi}{2}$, let $\mathcal{T}$ be the translation by $\overrightarrow{CO}$ followed by the clockwise rotation by $\alpha+\theta_A$ about O. Denoting $A'=\mathcal{T}(B)$, $B'=\mathcal{T}(A)$, and $C'=\mathcal{T}(O)$ we again obtain a standard image of $\mathbf{D}$ since

$$\alpha'=\theta_C<\pi-\theta_A=\beta'.$$

□

In addition to the length of the base, b and inclination angles of legs, α and β, we introduce two more parameters for $OACB$, a standard image of $\mathbf{D}$. Let α_0 and β_0 be the inclination angles of the diagonals OC and BA, respectively. Obviously,

$$\alpha_0<\alpha\leq\beta<\beta_0,$$

and any standard image is determined by the five parameters $b,\alpha_0,\alpha,\beta,\beta_0$. We will utilize the notation

$$\mathbf{D}_s=[b,\alpha_0,\alpha,\beta,\beta_0]$$

for a standard image. For example, a rectangle with sides of lengths 1 and $\sqrt{3}$ has two standard images, $\mathbf{D}_s^{(1)}=[1,\frac{\pi}{3},\frac{\pi}{2},\frac{\pi}{2},\frac{2\pi}{3}]$ and $\mathbf{D}_s^{(2)}=[\sqrt{3},\frac{\pi}{6},\frac{\pi}{2},\frac{\pi}{2},\frac{5\pi}{6}]$.

The values $\alpha_0,\alpha,\beta,\beta_0$ determine another parameter γ, the inclination angle of AC. It is easy to check that

$$\tan\gamma=\frac{\cot\alpha+\cot\beta-\cot\alpha_0-\cot\beta_0}{\cot\alpha\cot\beta-\cot\alpha_0\cot\beta_0}.$$

We classify the standard images into two categories based on the value of γ. Due to convexity of $\mathbf{D}$, either $0\leq\gamma<\alpha_0$, or $\beta_0<\gamma<\pi$. If the first inequality occurs, we will call the standard image to be of Type 1, otherwise, of Type 2. For example, a right-angled trapezoid has five standard images, where three of them are of Type 1, and two are of Type 2. Any parallelogram has only standard images of Type 1, whereas any kite with three congruent obtuse internal angles permits only standard images of Type 2.

3. ORIENTATION-DEPENDENT DIAMETERS AND SUPPLEMENTARY MEASURES

Let $\mathbf{D}_s$ be a standard image of a convex quadrilateral $\mathbf{D} \subset \mathbb{R}^2$. Consider the vector

$$\phi = (\cos\varphi, \sin\varphi) \in \mathbb{S}^1,$$

and let l_φ be the subspace of $\mathbb{R}^2$ spanned by ϕ. By $\phi^\perp$ we denote the orthogonal complement of l_φ. For any $y \in \phi^\perp$, let $l_\varphi + y$ be the line which is parallel to ϕ and passes through y. Denote

$$\chi(l_\varphi + y) = L_1\big((l_\varphi + y) \cap \mathbf{D}_s\big).$$

If the line $l_\varphi + y$ intersects $\mathbf{D}_s$, then we will say that it makes a chord in $\mathbf{D}_s$ of length $\chi(l_\varphi + y)$. Denote

$$\Pi^x_{\mathbf{E}}(\varphi) = \{y \in \Pi_{\mathbf{E}}(\varphi) : \chi(l_\varphi + y) \le x\},$$

where $\Pi_{\mathbf{E}}(\varphi)$ is the orthogonal projection of $\mathbf{E} \subset \mathbb{R}^2$ onto $\phi^\perp$. Assuming that y is uniformly distributed over $\Pi_{\mathbf{D}_s}(\varphi)$, the chord length distribution function in direction ϕ for $\mathbf{D}_s$ is defined by

$$F_{\mathbf{D}_s}(x,\varphi) = \frac{L_1\big(\Pi^x_{\mathbf{D}_s}(\varphi)\big)}{b_{\mathbf{D}_s}(\varphi)}, \tag{3.1}$$

where $b_{\mathbf{D}_s}(\varphi) = L_1(\Pi_{\mathbf{D}_s}(\varphi))$.

Hereinafter, since $l_{\varphi-\pi} = l_\varphi$, we will assume $\varphi \in [0,\, \pi)$.

To determine the distribution function $F_{\mathbf{D}_s}(x,\varphi)$ we need the quantities (introduced in [13])

$$x_0(\varphi) = \min_{y\in\phi_v^\perp} \chi(l_\varphi + y) \quad \text{and} \quad x_1(\varphi) = \max_{y\in\phi_v^\perp} \chi(l_\varphi + y),$$

where $\phi_v^\perp$ is the set of vectors $y \in \phi^\perp$ so that the line $l_\varphi + y$ passes through a vertex of $\mathbf{D}_s$ and makes a chord of positive Lebesgue measure there. The quantity $x_1(\varphi)$ coincides with

$$x_{\max}(\varphi) = \max_{y\in\Pi_{\mathbf{D}_s}(\varphi)} \chi(l_\varphi + y),$$

and any chord of length $x_{\max}(\varphi)$ is known as a *φ-diameter* of $\mathbf{D}_s$ (see [3]). In this paper, where convenient, we will call it a *first-order φ-diameter* of $\mathbf{D}_s$. Any chord of length $x_0(\varphi)$ will be called a *second-order φ-diameter* of $\mathbf{D}_s$.

Below, in addition to $x_0(\varphi)$ and $x_1(\varphi)$, we aim to introduce three more orientation-dependent characteristics $\ell_0(\varphi)$, $\ell(\varphi)$, and $\ell_1(\varphi)$ of the standard image $\mathbf{D}_s = [b, \alpha_0, \alpha, \beta, \beta_0]$. These characteristics will be non-negative continuous functions and will satisfy to $b_{\mathbf{D}_s}(\varphi) = \ell_0(\varphi) + \ell(\varphi) + \ell_1(\varphi)$ for all $\varphi \in [0,\pi)$. We will call them *supplementary φ-measures* of $\mathbf{D}_s$.

Case 1: *$\mathbf{D}_s$ has no parallel sides*. We have $\gamma > 0$ and $\alpha < \beta$. Then for any φ the first and the second-order φ-diameters are unique. Let them be $(l_\varphi + y_1) \cap \mathbf{D}_s$ and $(l_\varphi + y_0) \cap \mathbf{D}_s$, respectively. If $\varphi \ne \alpha_0$ and $\varphi \ne \beta_0$ then $y_0 \ne y_1$. In the case when $y_0,\, y_1 \in int\big(\Pi_{\mathbf{D}_s}(\varphi)\big)$, they partition $\Pi_{\mathbf{D}_s}(\varphi)$ into three segments: the middle segment, the side-segment adjacent to y_0, and the other side-segment adjacent to y_1. We denote the lengths of those segments by $\ell(\varphi)$, $\ell_0(\varphi)$, and $\ell_1(\varphi)$, respectively. If $y_0 \in \partial\Pi_{\mathbf{D}_s}(\varphi)$ or $y_1 \in \partial\Pi_{\mathbf{D}_s}(\varphi)$, we define correspondingly $\ell_0(\varphi) = 0$ or $\ell_1(\varphi) = 0$.

When $\varphi = \alpha_0$ or $\varphi = \beta_0$, the first and the second-order φ-diameters coincide. We extend the definitions of ℓ, ℓ_0, and ℓ_1 preserving their continuous dependence on φ:

$$\ell(\alpha_0) = \ell(\beta_0) = |y_0 - y_1| = 0, \quad \ell_0(\alpha_0) = \lim_{\varphi\to\alpha_0} \ell_0(\varphi),$$

$$\ell_0(\beta_0) = \lim_{\varphi\to\beta_0} \ell_0(\varphi), \quad \ell_1(\alpha_0) = \lim_{\varphi\to\alpha_0} \ell_1(\varphi), \quad \ell_1(\beta_0) = \lim_{\varphi\to\beta_0} \ell_1(\varphi).$$

Case 2: *$\mathbf{D}_s$ has exactly one pair of parallel sides.*

Subcase 2.1: Let $\gamma = 0$ and $\alpha < \beta$. Uniqueness of the first and the second-order φ-diameters takes place if and only if $\varphi \in [0, \alpha_0] \cup [\beta_0, \pi)$. If $\varphi \ne \alpha_0$ and $\varphi \ne \beta_0$, we define y_0, y_1, and then $\ell(\varphi)$, $\ell_0(\varphi)$, $\ell_1(\varphi)$ the same way we did it in Case 1. The values at α_0 and β_0 are defined below:

$$\ell(\alpha_0) = \ell(\beta_0) = 0,$$

$$\ell_0(\alpha_0) = \ell_0(\alpha_0-), \quad \ell_0(\beta_0) = \ell_0(\beta_0+), \quad \ell_1(\alpha_0) = \ell_1(\alpha_0-), \quad \ell_1(\beta_0) = \ell_1(\beta_0+).$$

The case $\alpha_0 < \varphi < \beta_0$ yields $x_0(\varphi) = x_1(\varphi)$, so we face infinitely many φ-diameters. Here by $\ell(\varphi)$ we denote the distance between the two farthest φ -diameters, $(l_\varphi + y_0) \cap \mathbf{D}_s$ and $(l_\varphi + y_1) \cap \mathbf{D}_s$. Using these vectors y_0 and y_1, we define $\ell(\varphi)$, $\ell_0(\varphi)$, $\ell_1(\varphi)$ again by the algorithm provided in Case 1.

Subcase 2.2: Now let $\gamma > 0$ and $\alpha = \beta$. The first and the second-order φ-diameters are unique if and only if $\varphi \in [\alpha_0, \beta_0]$. For $\varphi \in (\alpha_0, \beta_0)$ we define y_0, y_1, and then $\ell(\varphi)$, $\ell_0(\varphi)$, $\ell_1(\varphi)$ by the algorithm of Case 1. For the boundary values we define

$$\ell(\alpha_0) = \ell(\beta_0) = 0,$$

$$\ell_0(\alpha_0) = \ell_0(\alpha_0+), \quad \ell_0(\beta_0) = \ell_0(\beta_0-), \quad \ell_1(\alpha_0) = \ell_1(\alpha_0+), \quad \ell_1(\beta_0) = \ell_1(\beta_0-).$$

If $\varphi \notin [\alpha_0, \beta_0]$ then $x_0(\varphi) = x_1(\varphi)$, so $\mathbf{D}_s$ has infinitely many φ-diameters. We define $\ell(\varphi)$, $\ell_0(\varphi)$, $\ell_1(\varphi)$ the same way as we did it in Subcase 2.1 for $\varphi \in (\alpha_0, \beta_0)$.

Case 3: $\mathbf{D}_s$ *has two pairs of parallel sides.* In a parallelogram, $x_0(\varphi) = x_1(\varphi)$ holds for any value of φ. We define

$$\ell(\varphi) = |y_0 - y_1|,$$

and

$$\ell_0(\varphi) = \ell_1(\varphi) = \frac{b_{\mathbf{D}_s}(\varphi) - \ell(\varphi)}{2},$$

where $(l_\varphi + y_0) \cap \mathbf{D}_s$ and $(l_\varphi + y_1) \cap \mathbf{D}_s$ are the two farthest φ - diameters of $\mathbf{D}_s$.

4. REPRESENTATION OF THE ORIENTATION-DEPENDENT CHORD LENGTH DISTRIBUTION FUNCTION AND THE COVARIOGRAM

The following theorem represents the function introduced in (3.1) in terms of the lengths of orientation-dependent diameters and supplementary measures.

Theorem 4.1. *Let* $\mathbf{D}_s$ *be a standard image of a convex quadrilateral* $\mathbf{D}$ *and* $0 \le \varphi < \pi$. *If* x_1, x_0 *are the lengths of respectively the first and the second-order* φ*-diameters, and* ℓ_0, ℓ, ℓ_1 *are the supplementary* φ*-measures of* $\mathbf{D}_s$*, then*

$$F_{\mathbf{D}_s}(x, \varphi) = \frac{1}{\ell_0 + \ell + \ell_1} \begin{cases} 0, \quad \text{if} \quad x < 0 \\ \left(\dfrac{\ell_0}{x_0} + \dfrac{\ell_1}{x_1}\right) x, \quad \text{if} \quad 0 \le x < x_0(\varphi) \\ \ell_0 + \dfrac{x - x_0}{x_1 - x_0}\ell + \dfrac{x}{x_1}\ell_1, \quad \text{if} \quad x_0(\varphi) \le x < x_1(\varphi) \\ \ell_0 + \ell + \ell_1, \quad \text{if} \quad x \ge x_1(\varphi) \end{cases}. \tag{4.1}$$

Proof. The statement is obvious when $x < 0$ or $x \ge x_1$. Below we assume $0 \le x < x_1$.

Case A: φ *is such that* $x_0(\varphi) < x_1(\varphi)$. Let $(l_\varphi + y_1) \cap \mathbf{D}_s$ and $(l_\varphi + y_0) \cap \mathbf{D}_s$ be the first and second-order φ - diameters of $\mathbf{D}_s$. If y_0, $y_1 \in int\big(\Pi_{\mathbf{D}_s}(\varphi)\big)$, then the mentioned diameters partition $\mathbf{D}_s$ into two triangles $\mathbf{T}_0(\varphi)$, $\mathbf{T}_1(\varphi)$, and a trapezoid $\mathbf{T}(\varphi)$, where $\mathbf{T}_0$ is based on the second-order diameter and has a height of length ℓ_0, $\mathbf{T}_1$ is based on the first-order diameter and has a height of length ℓ_1, and the trapezoid $\mathbf{T}$ is based on the mentioned diameters and has a height of length ℓ. Then

$$L_1\big(\Pi^x_{\mathbf{D}_s}(\varphi)\big) = \sum_{i=0}^{1} L_1\big(\Pi^x_{\mathbf{T}_i}(\varphi)\big) + L_1\big(\Pi^x_{\mathbf{T}}(\varphi)\big). \tag{4.2}$$

If $0 \le x < x_0$, then $\Pi^x_{\mathbf{T}}(\varphi) = \varnothing$ and

$$L_1\big(\Pi^x_{\mathbf{T}_i}(\varphi)\big) = \frac{x}{x_i} L_1\big(\Pi_{\mathbf{T}_i}(\varphi)\big) = \frac{x}{x_i}\ell_i.$$

If $x_0 \le x < x_1$, then

$$L_1\big(\Pi^x_{\mathbf{T}_0}(\varphi)\big) = L_1\big(\Pi_{\mathbf{T}_0}(\varphi)\big) = \ell_0,$$

$$L_1\big(\Pi^x_{\mathbf{T}_1}(\varphi)\big) = \frac{x}{x_1} L_1\big(\Pi_{\mathbf{T}_1}(\varphi)\big) = \frac{x}{x_1}\ell_1,$$

and

$$L_1\big(\Pi^x_{\mathbf{T}}(\varphi)\big) = \frac{x-x_0}{x_1-x_0} L_1\big(\Pi_{\mathbf{T}}(\varphi)\big) = \frac{x-x_0}{x_1-x_0}\ell.$$

Now according to (3.1) and (4.2), we obtain

$$F_{\mathbf{D}_s}(x,\varphi) = \frac{1}{b_{\mathbf{D}_s}(\varphi)}\left(\frac{x}{x_0}\ell_0 + \frac{x}{x_1}\ell_1\right), \quad \text{for} \quad 0 \le x < x_0, \tag{4.3}$$

and

$$F_{\mathbf{D}_s}(x,\varphi) = \frac{1}{b_{\mathbf{D}_s}(\varphi)}\left(\ell_0 + \frac{x-x_0}{x_1-x_0}\ell + \frac{x}{x_1}\ell_1\right), \quad \text{for} \quad x_0 \le x < x_1. \tag{4.4}$$

Formula (4.2) works for such values of φ that imply $y_i \notin int\big(\Pi_{\mathbf{D}_s}\big)$ for $i=0$ or $i=1$. In this case, $\mathbf{T}_i$ turns into the segment $(l_\varphi + y_i)\cap \mathbf{D}_s$, and yields

$$L_1\big(\Pi^x_{\mathbf{T}_i}(\varphi)\big) = L_1\big(\Pi_{\mathbf{T}_i}(\varphi)\big) = \ell_i(\varphi) = 0.$$

Since $l_i(\varphi)$ has been defined as a continuous function, the formulas (4.3) and (4.4) remain valid.

Case B: *φ is such that $x_0(\varphi) = x_1(\varphi)$.* Consider $(l_\varphi + y_1)\cap \mathbf{D}_s$ and $(l_\varphi + y_0)\cap \mathbf{D}_s$, the two farthest φ — diameters of $\mathbf{D}_s$. If $y_0 \ne y_1$ and they both belong to $int\big(\Pi_{\mathbf{D}_s}(\varphi)\big)$ then $\mathbf{D}_S$ will be partitioned into the two triangles $\mathbf{T}_0(\varphi)$, $\mathbf{T}_1(\varphi)$, and the trapezoid $\mathbf{T}(\varphi)$ defined in Case A. If $y_0 = y_1$ or $y_i \notin int\big(\Pi_{\mathbf{D}_s}\big)$ for $i=0$ or $i=1$, then $\mathbf{T}$, or correspondingly, $\mathbf{T}_i$, turns into the segment $(l_\varphi + y_i)\cap \mathbf{D}_s$. In all these scenarios the formula (4.2) does operate, and since the functions $\ell_i(\varphi)$ are continuous, it implies (4.3). □

Corollary 4.1. The function $F_{\mathbf{D}_s}(\cdot,\varphi)$ is continuous on the real axis if and only if the φ-diameter of $\mathbf{D}_s$ is unique. If for some φ, the φ-diameter of $\mathbf{D}_s$ is not unique then $F_{\mathbf{D}_S}(\cdot,\varphi)$ holds a jump discontinuity at $x_{\max}(\varphi)$. The jump is equal to

$$\frac{\ell}{\ell_0+\ell+\ell_1}.$$

Proof. A φ-diameter is unique if and only if $x_0(\varphi) < x_1(\varphi)$, or $x_0(\varphi) = x_1(\varphi)$ but $\ell(\varphi) = 0$. Due to (4.1), this is equivalent to the continuity of $F_{\mathbf{D}_S}(\cdot,\varphi)$.

If a φ-diameter is not unique, then $x_0(\varphi) = x_1(\varphi) = x_{\max}(\varphi)$ and $\ell(\varphi) > 0$. Hence, $F_{\mathbf{D}_s}(x_{\max}(\varphi)+,$ $\varphi) = 1$ whereas $F_{\mathbf{D}_s}(x_{\max}(\varphi)-,\varphi) = \frac{\ell_0+\ell_1}{\ell_0+\ell+\ell_1} = 1 - \frac{\ell}{\ell_0+\ell+\ell_1}$. □

From now on, the notation $C_E(t,\varphi)$ will be used for the covariogram $C_E(t\phi)$, where $\mathbf{E}\subset\mathbb{R}^2$ and $t \ge 0$. Further in the text, $\|\mathbf{E}\|$ will stand for the area of $\mathbf{E}$.

Theorem 4.2. *Let $\mathbf{D}_s$ be a standard image of a convex quadrilateral $\mathbf{D}$ and $0 \le \varphi < \pi$. If x_1, x_0 are the lengths of respectively the first and the second-order φ-diameters, and ℓ_0, ℓ, ℓ_1 are the supplementary φ-measures of $\mathbf{D}_s$, then $C_{\mathbf{D}_s}(t,\varphi) =$*

$$\begin{cases} \dfrac{x_0\ell_0 + (x_0+x_1)\ell + x_1\ell_1}{2} - (\ell_0+\ell+\ell_1)t + \dfrac{1}{2}\left(\dfrac{\ell_0}{x_0} + \dfrac{\ell_1}{x_1}\right)t^2, & if \quad 0 \le t < x_0 \\ \dfrac{1}{2}\left(\dfrac{\ell_1}{x_1} + \dfrac{\ell}{x_1-x_0}\right)(x_1-t)^2, \quad if \quad x_0 \le t < x_1 \\ 0, \quad if \quad t \ge x_1 \end{cases}.$$

Proof. The case $t \ge x_1$ is obvious so below we assume $0 \le t < x_1$.

Due to the Matheron formula [2], p. 86, we have

$$\frac{\partial C_{\mathbf{D}_s}(t,\varphi)}{\partial t} = -L_1\left(\Big\{y \in \phi^\perp : L_1\left(\mathbf{D}_s \cap (l_\varphi + y)\right) \ge t\Big\}\right),$$

which can be rewritten in terms of the orientation-dependent chord length distribution function as

$$\frac{\partial C_{\mathbf{D}_s}(t,\varphi)}{\partial t} = -b_{\mathbf{D}_s}(\varphi)\cdot[1-F_{\mathbf{D}_s}(t,\varphi)].$$

Integration of both parts of the last formula yields

$$C_{\mathbf{D}_s}(t,\varphi) = C_{\mathbf{D}_s}(0,\varphi) - b_{\mathbf{D}_s}(\varphi)\cdot t + b_{\mathbf{D}_s}(\varphi)\cdot\int\limits_0^t F_{\mathbf{D}_s}(u,\varphi)du,\ t\geq 0. \tag{4.5}$$

Since

$$C_{\mathbf{D}_s}(0,\varphi) = \|\mathbf{D}_s\| = \frac{x_0\ell_0 + (x_0+x_1)\ell + x_1\ell_1}{2},$$

$$b_{\mathbf{D}_s}(\varphi) = \ell_0(\varphi) + \ell(\varphi) + \ell_1(\varphi),$$

and

$$\int\limits_0^t \left(\frac{\ell_0}{x_0} + \frac{\ell_1}{x_1}\right)udu = \frac{1}{2}\left(\frac{\ell_0}{x_0} + \frac{\ell_1}{x_1}\right)t^2,$$

then the required form of $C_{\mathbf{D}_s}(t,\varphi)$, where $0\leq t<x_0$, immediately follows from (4.5) and Theorem 4.1.

If $x_0\leq t<x_1$, then we use the corresponding part of Theorem 4.1 in (4.5):

$$C_{\mathbf{D}_s}(t,\varphi) = C_{\mathbf{D}_s}(0,\varphi) - b_{\mathbf{D}_s}(\varphi)\cdot t + \frac{1}{2}\left(\frac{\ell_0}{x_0} + \frac{\ell_1}{x_1}\right)x_0^2 + \int\limits_{x_0}^t \ell_0 + \frac{u-x_0}{x_1-x_0}\ell + \frac{u}{x_1}\ell_1 du.$$

Computation of the integral followed by the regrouping of similar terms produces

$$\begin{aligned} C_{\mathbf{D}_s}(t,\varphi) &= \frac{x_1^2\ell}{2(x_1-x_0)} + \frac{x_1\ell_1}{2} - \left(\frac{x_1\ell}{x_1-x_0} + \ell_1\right)\cdot t + \frac{1}{2}\left(\frac{\ell_1}{x_1} + \frac{\ell}{x_1-x_0}\right)\cdot t^2 \\ &= \frac{1}{2}\left(\frac{\ell_1}{x_1} + \frac{\ell}{x_1-x_0}\right)(x_1-t)^2. \end{aligned}$$

5. COMPUTATION OF ORIENTATION-DEPENDENT DIAMETERS AND SUPPLEMENTARY MEASURES

For a standard image $\mathbf{D}_s = [b,\alpha_0,\alpha,\beta,\beta_0]$, we denote

$$\Lambda = \{\alpha,\ \beta\},\quad \Delta = \{\alpha_0,\ \beta_0\},\quad \Sigma = \{0,\ \alpha,\ \gamma,\ \beta\},$$

which are the sets of the inclination angles of the legs, diagonals, and the sides of $\mathbf{D}_s$, respectively. For any $\varphi\in[0,\ \pi)$, we define the functions $X_\varphi : \Lambda\times\Delta\times\Sigma\setminus\{\varphi\}\longrightarrow\mathbb{R}$ and $L_\varphi : (\Lambda\times\Delta)\cup(\Delta\times\Lambda)\longrightarrow\mathbb{R}$ by

$$X_\varphi(x,\ y,\ z) = \frac{b\sin x\sin(y-z)}{\sin(y-x)\sin(z-\varphi)},$$

$$L_\varphi(x,\ y) = \frac{b\sin(x-\varphi)\sin y}{\sin(x-y)}.$$

Theorem 5.1. *Let $\mathbf{D}_s = [b,\alpha_0,\alpha,\beta,\beta_0]$ be a standard image of Type 1 of a convex quadrilateral $\mathbf{D}$. If x_1, x_0 are the lengths of respectively the first and the second-order φ-diameters of $\mathbf{D}_s$, then*

i. $x_0(\varphi) = X_\varphi(\alpha,\ \beta_0,\ \beta)$ *and* $x_1(\varphi) = X_\varphi(\beta,\ \alpha_0,\ \beta)$, *for* $0\leq\varphi<\gamma$;

ii. $x_0(\varphi) = X_\varphi(\beta,\ \alpha_0,\ \alpha)$ *and* $x_1(\varphi) = X_\varphi(\beta,\ \alpha_0,\ \beta)$, *for* $\gamma\leq\varphi<\alpha_0$;

iii. $x_0(\varphi) = X_\varphi(\beta,\, \alpha_0,\, \gamma)$ *and* $x_1(\varphi) = X_\varphi(\beta,\, \alpha_0,\, 0)$, *for* $\alpha_0 \le \varphi < \alpha$;

iv. $x_0(\varphi) = -X_\varphi(\alpha,\, \beta_0,\, 0)$ *and* $x_1(\varphi) = X_\varphi(\beta,\, \alpha_0,\, 0)$, *for* $\alpha \le \varphi < \beta$;

v. $x_0(\varphi) = -X_\varphi(\alpha,\, \beta_0,\, 0)$ and $x_1(\varphi) = -X_\varphi(\alpha,\, \beta_0,\, \gamma)$, *for* $\beta \le \varphi < \beta_0$;

vi. $x_0(\varphi) = -X_\varphi(\alpha,\, \beta_0,\, \beta)$ *and* $x_1(\varphi) = -X_\varphi(\alpha,\, \beta_0,\, \alpha)$, *for* $\beta_0 \le \varphi < \pi$.

Proof. Let the quadrilateral $\mathbf{D}_s = OACB$ not have any pair of parallel sides. The lengths of the diagonals AB and OC are

$$d_{AB} = \frac{b\sin\alpha}{\sin(\beta_0 - \alpha)} \quad \text{and} \quad d_{OC} = \frac{b\sin\beta}{\sin(\beta - \alpha_0)}. \tag{5.1}$$

We denote $\Pi_{\{A\}}(\varphi) = y_A$, $\Pi_{\{C\}}(\varphi) = y_C$, and $\Pi_{\{B\}}(\varphi) = y_B$. Then the first and the second-order φ-diameters of $\mathbf{D}_s$ are, respectively, equal to

$$\begin{cases} l_\varphi \cap \mathbf{D}_s \quad \text{and} \quad (l_\varphi + y_A) \cap \mathbf{D}_s, & \text{if} \quad 0 \le \varphi < \gamma \\ l_\varphi \cap \mathbf{D}_s \quad \text{and} \quad (l_\varphi + y_C) \cap \mathbf{D}_s, & \text{if} \quad \gamma \le \varphi < \alpha_0 \\ (l_\varphi + y_C) \cap \mathbf{D}_s \quad \text{and} \quad l_\varphi \cap \mathbf{D}_s, & \text{if} \quad \alpha_0 \le \varphi < \alpha \\ (l_\varphi + y_C) \cap \mathbf{D}_s \quad \text{and} \quad (l_\varphi + y_A) \cap \mathbf{D}_s, & \text{if} \quad \alpha \le \varphi < \beta \\ (l_\varphi + y_B) \cap \mathbf{D}_s \quad \text{and} \quad (l_\varphi + y_A) \cap \mathbf{D}_s, & \text{if} \quad \beta \le \varphi < \pi \end{cases}. \tag{5.2}$$

To compute $x_0(\varphi)$ we initially assume that the chosen direction ϕ is not parallel to any side or a diagonal of $\mathbf{D}_s$, which means $\varphi \notin \Delta \cup \Sigma$. This allows us to determine uniquely the triangle, where one of its sides is the second-order diameter of $\mathbf{D}_s$ and another side is the diagonal that shares an endpoint with the mentioned diameter. In that triangle, the internal angles that occurred in front of the second-order diameter and in front of the corresponding diagonal, are respectively equal to

$$\beta_0 - \beta \quad \text{and} \quad \beta - \varphi, \quad \text{if} \quad 0 < \varphi < \gamma; \quad \alpha - \alpha_0 \quad \text{and} \quad \pi - \alpha + \varphi, \quad \text{if} \quad \gamma < \varphi < \alpha_0;$$

$$\alpha_0 - \gamma \quad \text{and} \quad \pi - \varphi + \gamma, \quad \text{if} \quad \alpha_0 < \varphi < \alpha; \quad \pi - \beta_0 \quad \text{and} \quad \varphi, \quad \text{if} \quad \alpha < \varphi < \beta;$$

$$\pi - \beta_0 \quad \text{and} \quad \varphi, \quad \text{if} \quad \beta < \varphi < \beta_0; \quad \beta_0 - \beta \quad \text{and} \quad \pi - \varphi + \beta, \quad \text{if} \quad \beta_0 < \varphi < \pi.$$

Since $x_0 \in C[0, \pi)$, by (5.1), (5.2) and the Law of sines we conclude

$$x_0(\varphi) = \begin{cases} d_{AB}\dfrac{\sin(\beta_0 - \beta)}{\sin(\beta - \varphi)} = X_\varphi(\alpha,\, \beta_0,\, \beta), & \text{if} \quad 0 \le \varphi < \gamma \\ d_{OC}\dfrac{\sin(\alpha - \alpha_0)}{\sin(\alpha - \varphi)} = X_\varphi(\beta,\, \alpha_0,\, \alpha), & \text{if} \quad \gamma \le \varphi < \alpha_0 \\ d_{OC}\dfrac{\sin(\alpha_0 - \gamma)}{\sin(\varphi - \gamma)} = X_\varphi(\beta,\, \alpha_0,\, \gamma), & \text{if} \quad \alpha_0 \le \varphi < \alpha \\ d_{AB}\dfrac{\sin\beta_0}{\sin\varphi} = -X_\varphi(\alpha,\, \beta_0,\, 0), & \text{if} \quad \alpha \le \varphi < \beta_0 \\ d_{AB}\dfrac{\sin(\beta_0 - \beta)}{\sin(\varphi - \beta)} = -X_\varphi(\alpha,\, \beta_0,\, \beta), & \text{if} \quad \beta_0 \le \varphi < \pi \end{cases}. \tag{5.3}$$

To prove the required identities for $x_1(\varphi)$, we assume again that $\varphi \notin \Delta \cup \Sigma$. Consider the triangle, where one of its sides is the first-order diameter of $\mathbf{D}_s$ and another side is the diagonal that shares an endpoint with the mentioned diameter. In this case, the internal angles of the triangle that occurred in front of the first-order diameter and in front of the corresponding diagonal, are respectively equal to

$$\beta - \alpha_0 \quad \text{and} \quad \pi - \beta + \varphi, \quad \text{if} \quad 0 < \varphi < \gamma \quad \text{or} \quad \gamma < \varphi < \alpha_0;$$

$$\alpha_0 \quad \text{and} \quad \pi - \varphi, \quad \text{if} \quad \alpha_0 < \varphi < \alpha \quad \text{or} \quad \alpha < \varphi < \beta;$$

$$\pi - \beta_0 + \gamma \quad \text{and} \quad \varphi - \gamma, \quad \text{if} \quad \beta < \varphi < \beta_0; \quad \beta_0 - \alpha \quad \text{and} \quad \pi - \varphi + \alpha, \quad \text{if} \quad \beta_0 < \varphi < \pi.$$

As $x_1 \in C[0,\pi)$, we obtain

$$x_1(\varphi) = \begin{cases} d_{OC}\dfrac{\sin(\beta-\alpha_0)}{\sin(\beta-\varphi)} = X_\varphi(\beta,\,\alpha_0,\,\beta), & \text{if} \quad 0 \le \varphi < \alpha_0 \\ d_{OC}\dfrac{\sin\alpha_0}{\sin\varphi} = X_\varphi(\beta,\,\alpha_0,\,0), & \text{if} \quad \alpha_0 \le \varphi < \beta \\ d_{AB}\dfrac{\sin(\beta_0-\gamma)}{\sin(\varphi-\gamma)} = -X_\varphi(\alpha,\,\beta_0,\,\gamma), & \text{if} \quad \beta \le \varphi < \beta_0 \\ d_{AB}\dfrac{\sin(\beta_0-\alpha)}{\sin(\varphi-\alpha)} = -X_\varphi(\alpha,\,\beta_0,\,\alpha), & \text{if} \quad \beta_0 \le \varphi < \pi \end{cases}. \tag{5.4}$$

It remains to notice that the formulas (5.3) and (5.4) also hold if $\gamma = 0$ or $\alpha = \beta$. □

Theorem 5.2. *Let $\mathbf{D}_s = [b,\alpha_0,\alpha,\beta,\beta_0]$ be a standard image of Type 1 of a convex quadrilateral $\mathbf{D}$. If ℓ_0, ℓ and ℓ_1 are the supplementary φ-measures of $\mathbf{D}_s$, then*

$$\ell_0(\varphi) = \begin{cases} L_\varphi(\alpha,\,\beta_0) - L_\varphi(\alpha_0,\,\beta), & \textit{if} \quad 0 \le \varphi < \gamma \\ L_\varphi(\alpha_0,\,\beta) - L_\varphi(\alpha,\,\beta_0), & \textit{if} \quad \gamma \le \varphi < \alpha_0 \quad \textit{or} \quad \beta_0 \le \varphi < \pi \\ -L_\varphi(\alpha,\,\beta_0), & \textit{if} \quad \alpha_0 \le \varphi < \alpha \\ L_\varphi(\alpha,\,\beta_0), & \textit{if} \quad \alpha \le \varphi < \beta_0 \end{cases}, \tag{5.5}$$

$$\ell(\varphi) = \begin{cases} -L_\varphi(\alpha,\,\beta_0), & \textit{if} \quad 0 \le \varphi < \gamma \\ -L_\varphi(\alpha_0,\,\beta), & \textit{if} \quad \gamma \le \varphi < \alpha_0 \\ L_\varphi(\alpha_0,\,\beta), & \textit{if} \quad \alpha_0 \le \varphi < \alpha \\ L_\varphi(\alpha_0,\,\beta) - L_\varphi(\alpha,\,\beta_0), & \textit{if} \quad \alpha \le \varphi < \beta \\ L_\varphi(\beta_0,\,\alpha), & \textit{if} \quad \beta \le \varphi < \beta_0 \\ -L_\varphi(\beta_0,\,\alpha), & \textit{if} \quad \beta_0 \le \varphi < \pi \end{cases}, \tag{5.6}$$

$$\ell_1(\varphi) = \begin{cases} b\sin\varphi, & \textit{if} \quad 0 \le \varphi < \alpha_0 \quad \textit{or} \quad \beta_0 \le \varphi < \pi \\ L_\varphi(\beta,\,\alpha_0), & \textit{if} \quad \alpha_0 \le \varphi < \beta \\ -L_\varphi(\beta,\,\alpha_0), & \textit{if} \quad \beta \le \varphi < \beta_0 \end{cases}. \tag{5.7}$$

Proof. First of all, we notice that

$$L_1(\Pi_{\mathbf{E}}(\varphi)) = L_1(\mathbf{E})\sin|\varepsilon-\varphi|, \tag{5.8}$$

for any line segment $\mathbf{E} \subset \mathbb{R}^2$, $L_1(\mathbf{E}) < \infty$ inclined by $\varepsilon \in [0,\,\pi)$. When $\mathbf{E}$ is a diagonal of $\mathbf{D}_s$, then $L_1(\mathbf{E})$ can be read from (5.1). If $\mathbf{E}$ is a leg, we use either of the notations

$$s_{OA} = \frac{b\sin\beta_0}{\sin(\beta_0-\alpha)} \quad \text{and} \quad s_{CB} = \frac{b\sin\alpha_0}{\sin(\beta-\alpha_0)} \tag{5.9}$$

for its length.

Let us first prove (5.6). For φ, being in either of the six intervals

$$[0,\gamma),\ [\gamma,\alpha_0),[\alpha_0,\alpha),\ [\alpha,\beta),\ [\beta,\beta_0),\ [\beta_0,\pi),$$

the corresponding six-term sequence of the quantity $\ell(\varphi)$ becomes

$$L_1(\Pi_{OA}(\varphi)),\ L_1(\Pi_{OC}(\varphi)),\ L_1(\Pi_{OC}(\varphi)),\ L_1(\Pi_{OC}(\varphi)) - L_1(\Pi_{OA}(\varphi)),$$

$$L_1(\Pi_{AB}(\varphi)),\ L_1(\Pi_{AB}(\varphi)).$$

Since the inclination angles of OA, OC, and AB are, respectively, α, α_0, and β_0, formulas (5.8), (5.9), (5.1) yield $\ell(\varphi) =$

$$\begin{cases} s_{OA}\sin|\alpha-\varphi| = \dfrac{b\sin\beta_0}{\sin(\beta_0-\alpha)}\sin(\alpha-\varphi) = -L_\varphi(\alpha,\,\beta_0), & \text{if} \quad 0\le\varphi<\gamma \\ d_{OC}\sin|\alpha_0-\varphi| = \dfrac{b\sin\beta}{\sin(\beta-\alpha_0)}\sin(\alpha_0-\varphi) = -L_\varphi(\alpha_0,\,\beta), & \text{if} \quad \gamma\le\varphi<\alpha_0 \\ d_{OC}\sin|\alpha_0-\varphi| = \dfrac{b\sin\beta}{\sin(\beta-\alpha_0)}\sin(\varphi-\alpha_0) = L_\varphi(\alpha_0,\,\beta), & \text{if} \quad \alpha_0\le\varphi<\alpha \\ d_{OC}\sin|\alpha_0-\varphi| - s_{OA}\sin|\alpha-\varphi| = L_\varphi(\alpha_0,\,\beta) - L_\varphi(\alpha,\,\beta_0), & \text{if} \quad \alpha\le\varphi<\beta \\ d_{AB}\sin|\beta_0-\varphi| = \dfrac{b\sin\alpha}{\sin(\beta_0-\alpha)}\sin(\beta_0-\varphi) = L_\varphi(\beta_0,\,\alpha), & \text{if} \quad \beta\le\varphi<\beta_0 \\ d_{AB}\sin|\beta_0-\varphi| = \dfrac{b\sin\alpha}{\sin(\beta_0-\alpha)}\sin(\varphi-\beta_0) = -L_\varphi(\beta_0,\,\alpha), & \text{if} \quad \beta_0\le\varphi<\pi \end{cases}.$$

Similarly, $\ell_0(\varphi) =$

$$\begin{cases} d_{OC}\sin|\alpha_0-\varphi| - s_{OA}\sin|\alpha-\varphi| = L_\varphi(\alpha,\,\beta_0) - L_\varphi(\alpha_0,\,\beta), & \text{if} \quad 0\le\varphi<\gamma \\ s_{OA}\sin|\alpha-\varphi| - d_{OC}\sin|\alpha_0-\varphi| = L_\varphi(\alpha_0,\,\beta) - L_\varphi(\alpha,\,\beta_0), & \text{if} \quad \gamma\le\varphi<\alpha_0 \\ s_{OA}\sin|\alpha-\varphi| = -L_\varphi(\alpha,\,\beta_0), & \text{if} \quad \alpha_0\le\varphi<\alpha \\ s_{OA}\sin|\alpha-\varphi| = L_\varphi(\alpha,\,\beta_0), & \text{if} \quad \alpha\le\varphi<\beta \\ s_{OA}\sin|\alpha-\varphi| = L_\varphi(\alpha,\,\beta_0), & \text{if} \quad \beta\le\varphi<\beta_0 \\ d_{OC}\sin|\alpha_0-\varphi| - s_{OA}\sin|\alpha-\varphi| = L_\varphi(\alpha_0,\,\beta) - L_\varphi(\alpha,\,\beta_0), & \text{if} \quad \beta_0\le\varphi<\pi \end{cases},$$

and

$$\ell_1(\varphi) = \begin{cases} b\sin|0-\varphi| = b\sin\varphi, & \text{if} \quad 0\le\varphi<\gamma \\ b\sin|0-\varphi| = b\sin\varphi, & \text{if} \quad \gamma\le\varphi<\alpha_0 \\ s_{CB}\sin|\beta-\varphi| = L_\varphi(\beta,\,\alpha_0), & \text{if} \quad \alpha_0\le\varphi<\alpha \\ s_{CB}\sin|\beta-\varphi| = L_\varphi(\beta,\,\alpha_0), & \text{if} \quad \alpha\le\varphi<\beta \\ s_{CB}\sin|\beta-\varphi| = -L_\varphi(\beta,\,\alpha_0), & \text{if} \quad \beta\le\varphi<\beta_0 \\ b\sin|0-\varphi| = b\sin\varphi, & \text{if} \quad \beta_0\le\varphi<\pi \end{cases},$$

which are equivalent to (5.5) and (5.7), respectively. □

Corollary 5.1. If a standard image $\mathbf{D}_s = [b, \alpha_0, \alpha, \beta, \beta_0]$ of a convex quadrilateral is of Type 1 then

$$b_{\mathbf{D}_s}(\varphi) = \begin{cases} L_\varphi(\beta,\,\alpha_0), & \text{if} \quad 0\le\varphi<\gamma \\ L_\varphi(\beta_0,\,\alpha), & \text{if} \quad \gamma\le\varphi<\alpha \\ b\sin\varphi, & \text{if} \quad \alpha\le\varphi<\beta \\ L_\varphi(\alpha_0,\,\beta), & \text{if} \quad \beta\le\varphi<\pi \end{cases}. \tag{5.10}$$

Proof. Since $b_{\mathbf{D}_s}(\varphi) = \ell_0(\varphi) + \ell(\varphi) + \ell_1(\varphi)$, we substitute $\ell_0(\varphi)$, $\ell(\varphi)$, and $\ell_1(\varphi)$ by their corresponding expressions from (5.5), (5.6), and (5.7). To reach (5.10), it remains to check the identity $L_\varphi(x, y) + L_\varphi(y, x) = b\sin\varphi$ over the domain of L_φ. □

The proofs of the following results for a standard image of Type 2 are omitted since they are similar to the ones provided for Type 1.

Theorem 5.3. *Let* $\mathbf{D}_s = [b, \alpha_0, \alpha, \beta, \beta_0]$ *be a standard image of Type 2 of a convex quadrilateral* $\mathbf{D}$. *If* x_1, x_0 *are the lengths of respectively the first and the second-order* φ*-diameters of* $\mathbf{D}_s$*, then*

i. $x_0(\varphi) = X_\varphi(\beta,\,\alpha_0,\,\alpha)$ *and* $x_1(\varphi) = X_\varphi(\beta,\,\alpha_0,\,\beta)$, *for* $0 \le \varphi < \alpha_0$;

ii. $x_0(\varphi) = X_\varphi(\beta,\,\alpha_0,\,0)$ *and* $x_1(\varphi) = X_\varphi(\beta,\,\alpha_0,\,\gamma)$, *for* $\alpha_0 \le \varphi < \alpha$;

iii. $x_0(\varphi) = X_\varphi(\beta,\, \alpha_0,\, 0)$ *and* $x_1(\varphi) = -X_\varphi(\alpha,\, \beta_0,\, 0)$, *for* $\alpha \le \varphi < \beta$;

iv. $x_0(\varphi) = -X_\varphi(\alpha,\, \beta_0,\, \gamma)$ *and* $x_1(\varphi) = -X_\varphi(\alpha,\, \beta_0,\, 0)$, *for* $\beta \le \varphi < \beta_0$;

v. $x_0(\varphi) = -X_\varphi(\alpha,\, \beta_0,\, \beta)$ *and* $x_1(\varphi) = -X_\varphi(\alpha,\, \beta_0,\, \alpha)$, *for* $\beta_0 \le \varphi < \gamma$;

vi. $x_0(\varphi) = -X_\varphi(\beta,\, \alpha_0,\, \alpha)$ *and* $x_1(\varphi) = -X_\varphi(\alpha,\, \beta_0,\, \alpha)$, *for* $\gamma \le \varphi < \pi$.

Theorem 5.4. *Let* $\mathbf{D}_s = [b, \alpha_0, \alpha, \beta, \beta_0]$ *be a standard image of Type 2 of a convex quadrilateral* $\mathbf{D}$. *If* ℓ_0, ℓ *and* ℓ_1 *are the supplementary* φ*-measures of* $\mathbf{D}_s$*, then*

$$\ell_0(\varphi) = \begin{cases} L_\varphi(\beta_0,\, \alpha) - L_\varphi(\beta,\, \alpha_0), & \textit{if} \quad 0 \le \varphi < \alpha_0 \quad \textit{or} \quad \beta_0 \le \varphi < \gamma \\ L_\varphi(\beta,\, \alpha_0), & \textit{if} \quad \alpha_0 \le \varphi < \beta \\ -L_\varphi(\beta,\, \alpha_0), & \textit{if} \quad \beta \le \varphi < \beta_0 \\ L_\varphi(\beta,\, \alpha_0) - L_\varphi(\beta_0,\, \alpha), & \textit{if} \quad \gamma \le \varphi < \pi \end{cases},$$

$$\ell(\varphi) = \begin{cases} -L_\varphi(\alpha_0,\, \beta), & \textit{if} \quad 0 \le \varphi < \alpha_0 \\ L_\varphi(\alpha_0,\, \beta), & \textit{if} \quad \alpha_0 \le \varphi < \alpha \\ L_\varphi(\beta_0,\, \alpha) - L_\varphi(\beta,\, \alpha_0), & \textit{if} \quad \alpha \le \varphi < \beta \\ L_\varphi(\beta_0,\, \alpha), & \textit{if} \quad \beta \le \varphi < \beta_0 \\ -L_\varphi(\beta_0,\, \alpha), & \textit{if} \quad \beta_0 \le \varphi < \gamma \\ -L_\varphi(\beta,\, \alpha_0), & \textit{if} \quad \gamma \le \varphi < \pi \end{cases},$$

$$\ell_1(\varphi) = \begin{cases} b\sin\varphi, & \textit{if} \quad 0 \le \varphi < \alpha_0 \quad \textit{or} \quad \beta_0 \le \varphi < \pi \\ -L_\varphi(\alpha,\, \beta_0), & \textit{if} \quad \alpha_0 \le \varphi < \alpha \\ L_\varphi(\alpha,\, \beta_0), & \textit{if} \quad \alpha \le \varphi < \beta_0 \end{cases}.$$

Corollary 5.2. If a standard image $\mathbf{D}_s = [b, \alpha_0, \alpha, \beta, \beta_0]$ of a convex quadrilateral is of Type 2 then

$$b_{\mathbf{D}_s}(\varphi) = \begin{cases} L_\varphi(\beta_0,\, \alpha), & \text{if} \quad 0 \le \varphi < \alpha \\ b\sin\varphi, & \text{if} \quad \alpha \le \varphi < \beta \\ L_\varphi(\alpha_0,\, \beta), & \text{if} \quad \beta \le \varphi < \gamma \\ L_\varphi(\alpha,\, \beta_0), & \text{if} \quad \gamma \le \varphi < \pi \end{cases}.$$

6. ORIENTATION-DEPENDENT CHORD LENGTH DISTRIBUTION FUNCTION AND THE COVARIOGRAM OF A CONVEX QUADRILATERAL PRISM

Denote by $\mathbf{D}_s^h$ the right prism $\{(x, y, z) : (x, y) \in \mathbf{D}_s,\ 0 < z \le h\}$, where $\mathbf{D}_s$ is a standard image of a convex quadrilateral. For a vector

$$\omega = (\cos\varphi\cos\theta,\, \sin\varphi\cos\theta,\, \sin\theta) \in \mathbb{S}^2,$$

let $\omega^\perp$ be the orthogonal complement of $\{t\omega \,:\, t \in \mathbb{R}\}$ in $\mathbb{R}^3$, and $\Pi_{\mathbf{D}_s^h}(\varphi, \theta)$ be the orthogonal projection of $\mathbf{D}_s^h$ onto the plane $\omega^\perp$.

We define the chord length distribution function in direction ω for $\mathbf{D}_s^h$ by

$$F_{\mathbf{D}_s^h}(t, \varphi, \theta) = \frac{L_2\{y \in \Pi_{\mathbf{D}_s^h}(\varphi, \theta) : \chi(l_{(\varphi,\theta)} + y) \le t\}}{b_{\mathbf{D}_s^h}(\varphi, \theta)},$$

where $l_{(\varphi,\theta)} + y$ is the line that passes through $y \in \omega^\perp$ and has direction vector ω,

$$\chi(l_{(\varphi,\theta)} + y) = L_1\big((l_{(\varphi,\theta)} + y) \cap \mathbf{D}_s^h\big),$$

and

$$b_{\mathbf{D}_s^h}(\varphi,\theta)=L_2(\Pi_{\mathbf{D}_s^h}(\varphi,\theta)).$$

As $\{z\in\mathbb{R}^3:\ z=\frac{h}{2}\}$ is a plane of symmetry of $\mathbf{D}_s^h$, we notice that $F_{\mathbf{D}_s^h}(t,\varphi,\theta)=F_{\mathbf{D}_s^h}(t,\varphi-\pi,\theta)$, for $\varphi\in[\pi,2\pi)$ and $F_{\mathbf{D}_s^h}(t,\varphi,\theta)=F_{\mathbf{D}_s^h}(t,\varphi,-\theta)$. Based on this observation, from now on we will assume that $\varphi\in[0,\ \pi)$ and $\theta\in[0,\ \frac{\pi}{2}]$.

Denote

$$x_{\max}(\varphi,\theta)=\max_{y\in\Pi_{\mathbf{D}_s^h}(\varphi,\theta)}\chi(l_{(\varphi,\theta)}+y).$$

It is easy to check that

$$x_{\max}(\varphi,\theta)=\begin{cases}\frac{x_{\max}(\varphi)}{\cos\theta}, & \text{if}\quad 0\le\theta\le\tan^{-1}\frac{h}{x_{\max}(\varphi)}\\ \frac{h}{\sin\theta}, & \text{if}\quad \tan^{-1}\frac{h}{x_{\max}(\varphi)}<\theta\le\frac{\pi}{2}\end{cases}. \tag{6.1}$$

Theorem 6.1. *For a $\varphi\in[0,\ \pi)$, let x_1 and x_0 be the lengths of the first and the second-order φ-diameters of $\mathbf{D}_s$, respectively. Let $\ell_0,\ \ell,\ \ell_1$ be the supplementary φ-measures of $\mathbf{D}_s$, and denote $b_{\mathbf{D}_s}=\ell_0+\ell+\ell_1$. Then, for the direction $\omega=(\cos\varphi\cos\theta,\ \sin\varphi\cos\theta,\ \sin\theta)$, $0\le\theta\le\frac{\pi}{2}$ and the prism $\mathbf{D}_s^h$, the following statements take place:*

(a) *If $\tan^{-1}\frac{h}{x_0}<\theta\le\frac{\pi}{2}$ and $0\le t<x_{\max}(\varphi,\theta)$, or $0\le\theta\le\tan^{-1}\frac{h}{x_0}$ and $0\le t<x_0\sec\theta$, then*

$$F_{\mathbf{D}_s^h}(t,\varphi,\theta)=\frac{a_1t+a_2t^2}{\|\mathbf{D}_s\|\sin\theta+b_{\mathbf{D}_s}h\cos\theta}, \tag{6.2}$$

where

$$a_1=h\left(\frac{\ell_0}{x_0}+\frac{\ell_1}{x_1}\right)\cos^2\theta+b_{\mathbf{D}_s}\sin 2\theta,\quad a_2=-\frac{3}{2}\left(\frac{\ell_0}{x_0}+\frac{\ell_1}{x_1}\right)\sin\theta\cos^2\theta;$$

(b) *If $0\le\theta\le\tan^{-1}\frac{h}{x_0}$ and $x_0\sec\theta\le t<x_{\max}(\varphi,\theta)$, then $x_0<x_1$ and*

$$F_{\mathbf{D}_s^h}(t,\varphi,\theta)=\frac{c_0+c_1t+c_2t^2}{\|\mathbf{D}_s\|\sin\theta+b_{\mathbf{D}_s}h\cos\theta}, \tag{6.3}$$

where

$$c_0=(h\cos\theta+\frac{x_0}{2}\sin\theta)\left(\ell_0-\frac{\ell x_0}{x_1-x_0}\right),$$

$$c_1=(h\cos^2\theta+x_1\sin 2\theta)\left(\frac{\ell}{x_1-x_0}+\frac{\ell_1}{x_1}\right),\quad c_2=-\frac{3}{2}\sin\theta\cos^2\theta\left(\frac{\ell}{x_1-x_0}+\frac{\ell_1}{x_1}\right).$$

Proof. Using the formula (see [11]) that establishes a relation between the orientation-dependent chord length distribution functions of a cylinder and its base, for $0\le t<x_{\max}(\varphi,\theta)$ we obtain

$$F_{\mathbf{D}_s^h}(t,\varphi,\theta)=\frac{b_{\mathbf{D}_s}\cos\theta}{\|\mathbf{D}_s\|\sin\theta+b_{\mathbf{D}_s}h\cos\theta}$$
$$\times\left[(h-t\sin\theta)F_{\mathbf{D}_s}(t\cos\theta,\varphi)+2t\sin\theta-\sin\theta\int_0^t F_{\mathbf{D}_s}(u\cos\theta,\varphi)du\right]. \tag{6.4}$$

(a) By (6.1), the inequality $\tan\theta>\frac{h}{x_0}$ implies $x_{\max}(\varphi,\theta)=\frac{h}{\sin\theta}$, and then

$$t\cos\theta<\frac{h}{\tan\theta}<x_0,$$

for any $t\in[0,\ x_{\max}(\varphi,\theta))$.

If $\tan\theta \le \frac{h}{x_0}$ but $0 \le t < x_0 \sec\theta$, the inequality $t\cos\theta < x_0$ still holds. Therefore, by Theorem 4.1, we substitute $F_{\mathbf{D}_s}(t\cos\theta,\varphi)$ and $F_{\mathbf{D}_s}(u\cos\theta,\varphi)$ in (6.4) by

$$\frac{1}{b_{\mathbf{D}_s}}\left(\frac{\ell_0}{x_0}+\frac{\ell_1}{x_1}\right)t\cos\theta \quad \text{and} \quad \frac{1}{b_{\mathbf{D}_s}}\left(\frac{\ell_0}{x_0}+\frac{\ell_1}{x_1}\right)u\cos\theta,$$

respectively. Computation of the integral in (6.4) followed by combining the like terms results in (6.2).

(b) Let now $\tan\theta \le \frac{h}{x_0}$ but $x_0\sec\theta \le t < x_{\max}(\varphi,\theta)$. Then $x_0 < x_1$, otherwise it will contradict to (6.1). Theorem 4.1 yields

$$F_{\mathbf{D}_s}(t\cos\theta,\varphi)=\frac{1}{b_{\mathbf{D}_s}}\left(\ell_0+\frac{t\cos\theta-x_0}{x_1-x_0}\ell+\frac{t\cos\theta}{x_1}\ell_1\right), \tag{6.5}$$

and

$$\int\limits_0^t F_{\mathbf{D}_s}(u\cos\theta,\varphi)du$$

$$=\int\limits_0^{x_0\sec\theta} F_{\mathbf{D}_s}(u\cos\theta,\varphi)du+\int\limits_{x_0\sec\theta}^{t} F_{\mathbf{D}_s}(u\cos\theta,\varphi)du=\frac{1}{b_{\mathbf{D}_s}}$$

$$\times\left[\int\limits_0^{x_0\sec\theta}\left(\frac{\ell_0}{x_0}+\frac{\ell_1}{x_1}\right)u\cos\theta du+\int\limits_{x_0\sec\theta}^{t}\left(\ell_0+\frac{u\cos\theta-x_0}{x_1-x_0}\ell+\frac{u\cos\theta}{x_1}\ell_1\right)du\right]. \tag{6.6}$$

To reach (6.3), it remains to evaluate (6.6), substitute its value along with (6.5) into (6.4), and simplify. □

Corollary 6.1. Let

$$\mu(\varphi,\theta)=L_2\Big(\{y\in\Pi_{\mathbf{D}_s^h}(\varphi,\theta):\chi(l_{(\varphi,\theta)}+y)=x_{\max}(\varphi,\theta)\}\Big).$$

The function $F_{\mathbf{D}_s^h}(\cdot,\varphi,\theta)$ is continuous on the real axis if and only if $\mu(\varphi,\theta)=0$. Otherwise, if $\mu(\varphi,\theta)>0$ for some pair (φ,θ), then $F_{\mathbf{D}_s^h}(\cdot,\varphi,\theta)$ has a jump discontinuity at $x_{\max}(\varphi,\theta)$. The jump is equal to

$$\frac{\mu(\varphi,\theta)}{\|\mathbf{D}_s\|\sin\theta+b_{\mathbf{D}_s}h\cos\theta}.$$

Proof. For any (φ,θ), the continuity of $F_{\mathbf{D}_s^h}(\cdot,\varphi,\theta)$ at $t=0$ immediately follows from (6.2). The continuity at $t=x_0\sec\theta$ also takes place. Careful calculations show that the expressions in (6.2) and (6.3) coincide when $t=x_0\sec\theta$. Thus, the only discontinuity may occur at $t=x_{\max}(\varphi,\theta)$.

Since

$$F_{\mathbf{D}_s^h}(x_{\max}(\varphi,\theta)-,\varphi,\theta)=\frac{L_2\{y\in\Pi_{\mathbf{D}_s^h}(\varphi,\theta):\chi(l_{(\varphi,\theta)}+y)<x_{\max}(\varphi,\theta)\}}{b_{\mathbf{D}_s^h}(\varphi,\theta)}$$
$$=1-\frac{\mu(\varphi,\theta)}{b_{\mathbf{D}_s^h}(\varphi,\theta)},$$

the continuity at $x_{\max}(\varphi,\theta)$ holds if and only if $\mu(\varphi,\theta)=0$. The jump is equal to $\frac{\mu(\varphi,\theta)}{b_{\mathbf{D}_s^h}(\varphi,\theta)}=\frac{\mu(\varphi,\theta)}{\|\mathbf{D}_s\|\sin\theta+b_{\mathbf{D}_s}h\cos\theta}$. □

Remark 6.1. One can verify that $\mu(\varphi,0)=h\cdot\ell(\varphi)$, so we rediscover Corollary 4.1. For the other extreme, $\mu(\varphi,\frac{\pi}{2})=\|\mathbf{D}_s\|$ holds. The jump in this case is the highest possible, 1. We do not aim to compute $\mu(\varphi,\theta)$ for other directions.

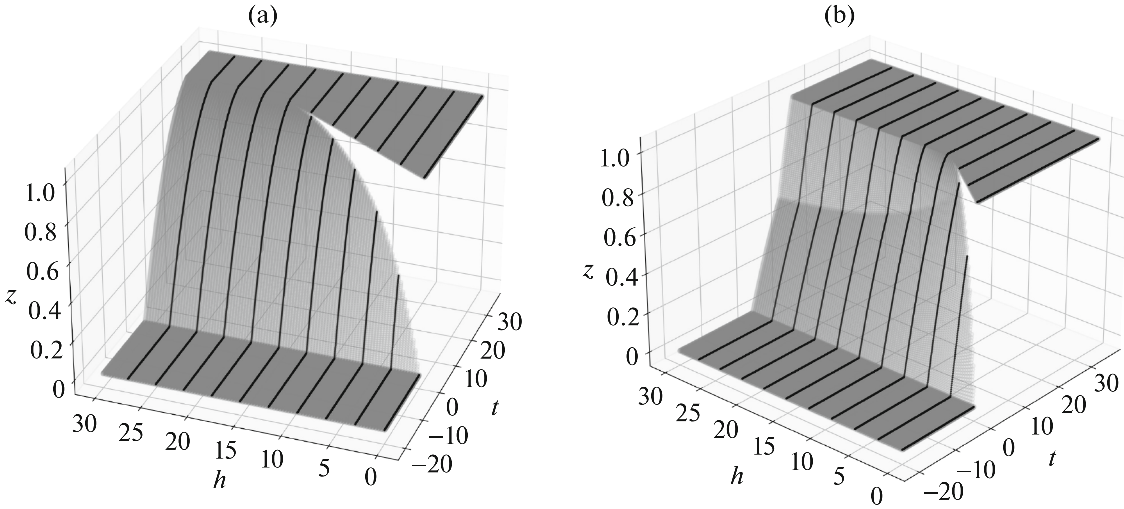

Fig. 1. Examples of orientation-dependent chord length distribution functions in right prisms $\mathbf{D}_s^h$ with base $\mathbf{D}_s = [10, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}]$. (a) Represents the surface $z(t,h) = F_{\mathbf{D}_s^h}(t, \frac{\pi}{6}, \frac{\pi}{3})$, (b) represents the surface $z(t,h) = F_{\mathbf{D}_s^h}(t, \frac{\pi}{2}, \frac{\pi}{4})$.

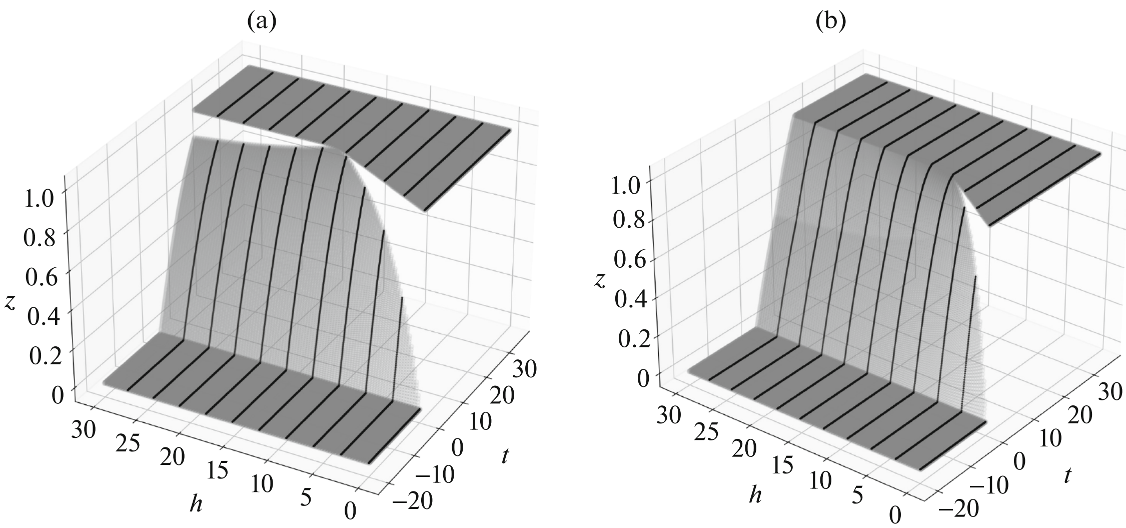

Fig. 2. Examples of orientation-dependent chord length distribution functions in right prisms $\mathbf{D}_s^h$ with base $\mathbf{D}_s = [10, \frac{\pi}{6}, \frac{\pi}{4}, \frac{2\pi}{3}, \pi - \tan^{-1}\frac{\sqrt{3}}{4-\sqrt{3}}]$. (a) Represents the surface $z(t,h) = F_{\mathbf{D}_s^h}(t, \frac{\pi}{2}, \frac{2\pi}{5})$, (b) represents the surface $z(t,h) = F_{\mathbf{D}_s^h}(t, \frac{9\pi}{10}, \frac{2\pi}{7})$.

In order to visualize the possible breaks in continuity and smoothness of the ODCLD function, we plot the function $z(t,h) = F_{\mathbf{D}_s^h}(t,\varphi,\theta)$ for a given pair (φ,θ) and different values of the height h. As an example, in Figure 1, this is done for the prism based on the kite $\mathbf{D}_s = [10, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}]$, where $\varphi = \frac{\pi}{6}$, $\theta = \frac{\pi}{3}$, and then $\varphi = \frac{\pi}{2}$, $\theta = \frac{\pi}{4}$.

Each of the highlighted curves on the surface represents the graph of the ODCLD function for the prism of a given height. Figure 2 is created by the same logic for the prisms with a trapezoidal base $\mathbf{D}_s = [10, \frac{\pi}{6}, \frac{\pi}{4}, \frac{2\pi}{3}, \pi - \tan^{-1}\frac{\sqrt{3}}{4-\sqrt{3}}]$.

Theorem 6.2. *For a $\varphi \in [0,\pi)$, let x_1 and x_0 be the lengths of the first and the second-order φ-diameters of $\mathbf{D}_s$, respectively. Let ℓ_0, ℓ, ℓ_1 be the supplementary φ-measures of $\mathbf{D}_s$, and denote $b_{\mathbf{D}_s} = \ell_0 + \ell + \ell_1$. Then, for the direction $\omega = (\cos\varphi\cos\theta, \sin\varphi\cos\theta, \sin\theta)$, $0 \le \theta \le \frac{\pi}{2}$, the covariogram $C_{\mathbf{D}_s^h}(t\omega) = C_{\mathbf{D}_s^h}(t,\varphi,\theta)$ of the prism $\mathbf{D}_s^h$ has the following representation:*

(a) *If* $\tan^{-1}\frac{h}{x_0} < \theta \le \frac{\pi}{2}$ *and* $0 \le t < x_{\max}(\varphi,\theta)$, *or* $0 \le \theta \le \tan^{-1}\frac{h}{x_0}$ *and* $0 \le t < x_0 \sec\theta$, *then*

$$C_{\mathbf{D}_s^h}(t,\varphi,\theta) = \left(\|\mathbf{D}_s\| - b_{\mathbf{D}_s}\cos\theta\cdot t + \frac{1}{2}\left(\frac{\ell_0}{x_0} + \frac{\ell_1}{x_1}\right)\cos^2\theta\cdot t^2\right)(h - \sin\theta\cdot t);$$

(b) *If* $0 \le \theta \le \tan^{-1}\frac{h}{x_0}$ *and* $x_0 \sec\theta \le t < x_{\max}(\varphi,\theta)$, *then* $x_0 < x_1$ *and*

$$C_{\mathbf{D}_s^h}(t,\varphi,\theta) = \frac{1}{2}\left(\frac{\ell}{x_1 - x_0} + \frac{\ell_1}{x_1}\right)(x_1 - \cos\theta\cdot t)^2(h - \sin\theta\cdot t).$$

Proof. Let $0 \le t < x_{\max}(\varphi,\theta)$. Since

$$\mathbf{D}_s^h \cap \left(\mathbf{D}_s^h + t\omega\right) = (\mathbf{D}_s \cap \{\mathbf{D}_s + (t\cos\theta)\phi\}) \times [t\sin\theta, h],$$

we obtain

$$C_{\mathbf{D}_s^h}(t\omega) = L_2(\mathbf{D}_s \cap \{\mathbf{D}_s + (t\cos\theta)\phi\}) \cdot (h - t\sin\theta),$$

and then

$$C_{\mathbf{D}_s^h}(t,\varphi,\theta) = C_{\mathbf{D}_s}(t\cos\theta,\varphi)(h - t\sin\theta). \tag{6.7}$$

The proof now follows from (6.7) and Theorem 4.2. □

Remark 6.2. Taking $\theta = 0$, it is easy to check that all the results obtained in Section 4 are coherent with the results presented in the current section.

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CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

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