Estimates of Polynomials of some g -Adic Numbers

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Abstract—A g -adic estimate of a polynomial is obtained in numbers from $\mathbb{Q}_g$ well-approximable by positive integers.

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Lower estimates of polynomials are obtained in numbers from $\mathbb{Q}_g$ well-approximable by positive integers. In the Archimedean case the transcendence measure of some Liouville numbers was obtained, e.g., in [1, §2].

Let $g \in \mathbb{N}$. If $g = p$ is a prime number, then $\mathbb{Q}_p$ is the field of p -adic numbers (see [2, Ch. 1, § 3]). If g is a composite number, then $\mathbb{Q}_g$ is a ring of g -adic numbers where the pseudo-normalization $|\cdot|_g$ is defined in the following way [2, Ch. 1, § 3]. Let $g \in \mathbb{N}$, $g \geq 2$, and also $a \neq 0$, $a = g^f \frac{r}{s}$, $(r, s) = (g, s) = 1$. By definition, assume $|a|_g = g^{-f}$.

Theorem. Let $\alpha = \sum_{n=0}^{\infty} a_n(g)g^{n!}$, where $a_n(g) \in \mathbb{N}$, $|a_n(g)|_g = 1$, $|a_n(g)| = \varepsilon_0(n)g^{n!}$, $\varepsilon_0(n) \rightarrow 0$, $n \rightarrow \infty$. Let $\varepsilon > 0$, $N \in \mathbb{N}$. There exists a number $H_0 = H_0(\varepsilon, N)$ such that for any nonzero polynomial $P(x)$, $P(x) \in \mathbb{Z}[x]$, $\deg(x) = N$, $H(P) = H$, the estimate $|P(\alpha)|_g \geq \frac{1}{N+1} H^{-N} H^{\frac{N \ln H}{\ln \ln H} (1+\varepsilon)}$ is valid for $H \geq H_0$.

Proof. Denote $\alpha_k = \sum_{n=0}^k a_n(g)g^{n!}$. By the Lemma on the absolute value of the principal term [3, Ch. 5, §23], the condition

$$\alpha_k > H + 1 \quad (1)$$

implies $P(\alpha_k) \neq 0$. In this case $|P(\alpha_k)| \prod_{p \in \mathbb{P}} |P(\alpha_k)|_p = 1$, where the product in the left-hand side is taken over all prime numbers p [4, Ch. 1, §2]. This inequality implies

$$|P(\alpha_k)|_g \geq \frac{1}{|P(\alpha_k)|} \geq \frac{1}{(N+1)H|\alpha_k|^N}. \quad (2)$$

The conditions of the theorem give the estimate $\alpha_k = g^{k!}(1 + \varepsilon_1(k))$, $\varepsilon_1(k) \rightarrow 0$, $k \rightarrow +\infty$. Therefore, inequality (1) follows from the inequality

$$g^{k!} > H. \quad (3)$$

According to the Tailor formula, we have

$$P(\alpha) = P(\alpha_k) + P'(\alpha_k)(\alpha - \alpha_k) + \dots + \frac{P^{(N)}(\alpha_k)}{N!}(\alpha - \alpha_k)^N. \quad (4)$$

The values of g -adic pseudo-normalizations do not exceed $g^{-(k+1)!}$ for all summands in the right-hand side of (4) except for $P(\alpha_k)$. Therefore, if

$$g^{-(k+1)!} < |P(\alpha_k)|_g, \quad (5)$$

then (4) implies $|P(\alpha)|_g = |P(\alpha_k)|_g$. Inequality (5) follows from the inequality $(k+1)! > Nk! + \log_g(N + 1)H + N \log_g(1 + \varepsilon_1(k))$ which, in its turn, follows from (1) for $k \geq k_0(N)$. Let k be the least number such that inequality (3) is valid, i.e., $g^{(k-1)!} \leq H < g^{k!}$. Then from (2) we get that the conditions of the theorem hold for $k \geq k_1(\varepsilon, N)$, i.e., for $H \geq H_0(\varepsilon, N)$.

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