BRIEF COMMUNICATIONS

A New Estimate of the Montgomery Constant in the Theory of Zeros Distribution of the Riemann Zeta-Function in the Critical Stripe

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Abstract—The new value $\alpha = \frac{91}{116}$ is obtained for the constant α such that for all $\sigma \geq \alpha$ the “density hypothesis” is valid for the Riemann zeta-function in the critical stripe. The previous value $\alpha = \frac{11}{14}$ was obtained by M. Jutila in 1977.

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In 1971, Montgomery [1] proved the existence of a constant $a \in [0.5; 1)$ such that for all $\sigma > a$ the “density hypothesis” holds, i.e., the following estimate is valid:

$$N(\sigma, T) \ll_{\varepsilon} T^{2(1-\sigma)+\varepsilon}, \quad T \rightarrow +\infty.$$

Here $\varepsilon > 0$ is arbitrarily small and the value $N(\sigma, T)$ is defined as the number of complex values p lying in the rectangle $1 > \operatorname{Re} p \geq \sigma$, $|\operatorname{Im} p| \leq T$ and satisfying the equality $\zeta(p) = 0$, where $\zeta(s)$ is the Riemann zeta-function.

Problems related to the distribution of points p satisfying the latter equality and called zeros of the Riemann zeta-function play a key role in the study of asymptotic properties of mean values of arithmetic functions, first of all this concerns problems related to distributions of primes in the natural series. In particular, the hypothetical estimate

$$N(\sigma, T) \ll T^{2(1-\sigma)+\varepsilon}, \quad \forall \sigma \geq 0.5,$$

called the “density hypothesis” implies

$$p_{n+1} - p_n \ll_{\varepsilon} p_n^{0.5+\varepsilon},$$

where $\{p_n\}$ is the series of all prime numbers enumerated in their sequence order in the natural series.

Obviously, a theorem stating the validity of the density hypothesis is equivalent to the assertion that the number $a = 0.5$ can be taken as the Montgomery constant.

At the same time, it was proved in [1] that $a \leq 0.9$. Many noted scientists improved this estimate in subsequent years. The following values were obtained: $a = \frac{5}{6}$ (M. N. Huxley, 1972), $a = \frac{43}{54}$ (M. Jutila, 1972), $a = \frac{21}{26}$ (K. Ramachandra, 1975), $a = 0.8059 \dots$ (F. Forti and C. Viola, 1973), $a = 0.8$ (M. N. Huxley, 1975), $a = \frac{11}{14}$ (M. Jutila, 1977, see [2, p. 294]). The last result of Jutila $a = \frac{11}{14}$ remained the best until now.

In this paper we indicate a new value for a . The following assertion is valid.

Theorem. *The estimate $N(\sigma, T) \ll_{\varepsilon} T^{2(1-\sigma)+\varepsilon}$, $T \rightarrow +\infty$ holds for all $\sigma \geq a = \frac{91}{116}$.*

Since $\frac{91}{116} < \frac{11}{14}$, then the theorem formulated above gives some improvement to the result of M. Jutila.

The proof of the theorem is based on the method from [3] and is its development. The most essential new element in considerations consists in application of a nontrivial estimate of the trigonometric sum similar to the sum W [2, p. 41] and further correction of the choice of the parameter n .

REFERENCES

1. H. L. Montgomery, *Topics in Multiplicative Number Theory*. LNM 227. (Springer Verlag, Berlin, Heidelberg, N.Y., 1971).
2. A. Ivic, *The Riemann Zeta-Function. The Theory of the Riemann Zeta-Function with Applications*. (John Wiley & Sons, N.Y., 1985).
3. I. F. Avdeev, *Estimation of the Number of Nontrivial Zeros of the Riemann Zeta-Function*. (Izd-vo OGU, Orel, 2007) [in Russian].

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