Calculation of the Shaping of a Space Umbrella Antenna During Strong Bending of Radial Rods Connected Along Parallels by Tensile Cables

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Abstract—A cyclically symmetric umbrella antenna is considered, the frame of which consists of flexible inextensible radial rods connected in nodes along parallels by tensile cables. In the initial transport position, the multilink rods are packed in packages oriented in the direction of the system axis. After the packing ties are removed, the rods are deployed in radial planes under the action of elastic springs connecting the links, and are fixed in rectilinear positions at a given angle with respect to the axis, at which all cables connecting the same type of rod nodes take the form of regular polygons, while remaining loose. Further, under the action of the force of a damping hydraulic cylinder with pre-compressed springs, the root parts of all rods are slowly turned to the stops. In the final position, the radial rods, connected at the nodes by tensioned cables, take a curved shape. The tensile stiffnesses of the cables are determined so that the radial and axial coordinates of the nodes of the curved rods coincide with the coordinates of the points of the given surface of revolution.

A model of strong bending of a flexible inextensible rod is constructed taking into account the unknown radial reactions of tensioned cables acting on it at the nodes. The links of the rod are considered as “cantilever” elements connected in series at the nodes in local coordinate systems, which can make large displacements and turns. The bending of each element is described by two specified functions, the shrinkage of the element due to bending is taken into account in a quadratic approximation. The obtained nonlinear deformation equations of the system, taking into account the geometric connections at the nodes, are solved by the method of successive approximations with respect to the unknown reactions of the cables. The obtained values of the reactions are then used to determine the required tensile stiffness of the cables at the given coordinates of the nodes.

As an example of the calculation, a parabolic antenna is considered for various numbers of radial rods and components of links. The estimates of the accuracy of the proposed computational model of the antenna shaping are carried out.

Keywords: space umbrella antennas, elastic rods, tensile cables, shaping, strong bending, nonlinear problems, finite element method

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1. INTRODUCTION

Large composite space antennas are launched into space in a folded state and deployed under conditions of vacuum and zero gravity.

The design and calculation of folding umbrella-type antennas are discussed in the book [1]. An overview of modern transformable antenna designs is presented in [2, 3]. Application of the finite element method and commercial software systems for numerical modeling of statics and dynamics of large space antennas is considered in [4–8]. Nonlinear equations of the dynamics of the deployment of a flat system of flexible inextensible rods connected in series by elastic hinges with stops, folded in the initial state into a packing, were obtained in [9, 10], where analytical expressions are given in the form of equations for all coefficients of equations in generalized coordinates.

The predetermined deformed shape of the antenna after deployment and damping of vibrations is obtained when designing a structure as a statically indeterminate system by determining the internal forces

Fig. 1. Diagram of the antenna in its folded state (a) and in final state (b).

required for stable equilibrium of this form of force. For this, it is required to solve the inverse geometrically nonlinear problem of deformation of the system with constraints and to determine the necessary geometric and stiffness parameters of some of its adjustable elements. In [11], an algorithm was developed for the numerical solution of a geometrically nonlinear problem of the space umbrella antenna shaping in the form of a shallow parabolic surface of revolution with cyclically symmetrically arranged flexible radial rods connected in parallel by tensile tether elements. In this paper, this approach is generalized for an antenna with a non-sloping surface of revolution. In this case, when solving the problem of shaping due to deformation of flexible radial rods, their strong bending with large displacements and angles of rotation of the elements is considered.

2. STATEMENT OF THE PROBLEM

The diagram of the proposed cyclically symmetric space antenna of the umbrella type with n radial rods 1, each of which consists of m links, is shown in Fig.1 in the folded state (a) and in the final deformed state (b). The opening and shaping of the antenna is as follows. On the signal, the holding connection between the body of the damping hydraulic cylinder 2 and the rod 3 is removed and a slow movement of the rod begins under the action of the pre-compressed springs 4, due to which, with the help of cables 5 and levers 6, the packing of rods 1 is rotated in radial planes. With some deviation of the packages, the bonds 7 are broken and the multi-link rods with elastic hinged joints are deployed with fixation on the stops in a straight-line position. It is believed that in some given deflected position (1', Fig. 1b) of the straightened rods (at an angle β_0 between the X axis and the rod axis) due to the choice of the initial lengths of the cable sections 8 connecting in the parallel planes the corresponding nodes $k = 1, 2, \dots, m$ rods, these sections become rectilinear (without sagging), but still not tensioned. In this case, all cables $k = 1, 2, \dots, m$ will have the form of regular n -gons. It should be noted that after fast (dynamic) opening of the packs of elastically connected links, each of the n rods will become rectilinear in a deflected angle position with some spread in time and angles β_n . The deflections of the rods at a slow (quasi-static) stroke of the rod 3 of the damping hydraulic cylinder will be leveled out under the action of the cables connecting them $k = 1, 2, \dots, m$, since in all n elements of each of these cables the forces must be the same with the cyclic symmetry of the system.

When turning the levers 6 to the stop, the cables will stretch, and the rods will bend in the radial planes under the action of the reactions of the cables at the nodes $k = 1, 2, \dots, m$. The final curved shape 1 of the rod with stretched cables 8 is shown in Fig. 1b. The antenna canvas is connected to the rods at separate points, including the nodes; when folded, it is located in the space between the packs of the rods. When opening and shaping the composite structure of the antenna, the reactions of the canvas are not taken into account.

Fig. 2. Finite element model of strong bending of a radial rod.

The axisymmetric surface of the antenna has a given shape $Y = F(X)$, Fig. 1b. When designing and calculating a compound cyclic symmetric antenna structure, it is required that the coordinates of the nodal points $k = 1, 2, \dots, m$ of curved radial rods coincide with the corresponding coordinates of the antenna surface $X_k, Y_k = F(X_k)$, $k = 0, 1, \dots, m$. The central section C–0 after turning the lever 6 to the stop is considered absolutely rigid and profiled in the form $Y = F(X)$; $X = Y = 0$ at point C and $X_0 = a_0$, $Y_0 = F(a_0)$, $\theta_0 = \arctan(F'(a_0))$ at point 0. The elastic part of the rod consists of m links of approximately the same length a_k , $k = 1, 2, \dots, m$ (for the convenience of folding them into packings). For given values of a_k , the coordinates of the nodes of the curved inextensible rod X_k , $k = 1, 2, \dots, m$ are calculated sequentially from the relations

$$a_k = \int_{X_{k-1}}^{X_k} \sqrt{1 + F'^2} dX, \quad k = 1, 2, \dots, m$$

using the iteration method. Flexural rigidity of each link EI_k within its length a_k is considered constant.

To describe the strong bending of the rod, each of its links $k = 1, 2, \dots, m$ will be considered as a finite element (FE) in the local coordinate system xy , which at the node $k - 1$ is rigidly connected with the $(k - 1)$ th FE, Fig.2. Coordinates and the angle of inclination of the rod axis at the node $k - 1$ are equal to X_{k-1} , Y_{k-1} , θ_{k-1} . The relative transverse displacement $v(x)$ and the rotation angle $\vartheta(x)$ of the k -th FE as a cantilever rod of length a_k are written in the form [11]

$$\begin{aligned} v(x) &= v_k \varphi_k(x) + \vartheta_k \psi_k(x), \quad \vartheta(x) = v'(x), \\ \varphi_k(x) &= 3 \frac{x^2}{a_k^2} - 2 \frac{x^3}{a_k^3}, \quad \psi_k(x) = \frac{x^3}{a_k^2} - \frac{x^2}{a_k}, \end{aligned} \quad (2.1)$$

where the prime denotes the derivative with respect to the x coordinate. Longitudinal displacement u_k of the FE end, due to strong bending provided that the rod is inextensible ($\epsilon = u' + 0.5v'^2 = 0$), taking into account (2.1), is written in the form

$$u_k = -\frac{1}{2} \int_0^{a_k} (v')^2 dx = -\frac{1}{2} \left(\frac{6}{5a_k} v_k^2 - \frac{1}{10} v_k \vartheta_k + \frac{a_k}{15} \vartheta_k^2 \right). \quad (2.2)$$

Potential energy of the bending of the rod

$$\Pi = \sum_{k=1}^m \Pi_k, \quad \Pi_k = \frac{1}{2} \frac{12EI_k}{a_k^3} \left(v_k^2 - a_k v_k \vartheta_k + \frac{a_k^2}{3} \vartheta_k^2 \right). \quad (2.3)$$

The displacements and angles of rotation at the nodes of the rod $k = 1, 2, \dots, m$ are determined as follows

$$\begin{aligned} X_k &= a_0 + \sum_{i=1}^k [(a_i + u_i) \cos \theta_{i-1} - v_i \sin \theta_{i-1}], \\ Y_k &= v_0 + \sum_{i=1}^k [(a_i + u_i) \sin \theta_{i-1} + v_i \cos \theta_{i-1}], \\ \theta_k &= \vartheta_0 + \sum_{i=1}^k \vartheta_i. \end{aligned} \quad (2.4)$$

The variation of the work of forces R_k on displacements δX_k (Fig. 2) is written in the form

$$-\delta A = \sum_{k=1}^m R_k \delta X_k = \sum_{k=1}^m R_k \sum_{i=1}^k \left(\xi_i + y_i \sum_{j=1}^{i-1} \delta \vartheta_j \right) \quad (2.5)$$

where, taking into account (2.2) and (2.4), the following notations are introduced:

$$\begin{aligned} \xi_i &= (p_i c_{i-1} - q_i) \delta v_i + q_i c_{i-1} \delta \vartheta_i, \quad y_i = -[(a_i + u_i) s_{i-1} + v_i c_{i-1}], \\ p_i &= -\frac{1}{5} \left(\frac{6}{a_i} v_i - \frac{1}{4} \vartheta_i \right), \quad q_i = -\frac{1}{5} \left(\frac{a_i}{3} \vartheta_i - \frac{1}{4} v_i \right), \quad u_i = \frac{1}{2} (p_i v_i + q_i \vartheta_i), \\ c_i &= \cos \theta_i, \quad s_i = \sin \theta_i. \end{aligned}$$

The double and triple sums in (2.5) transform as

$$\begin{aligned} \sum_{k=1}^m R_k \sum_{i=1}^k \xi_i &= \sum_{k=1}^m \xi_k \sum_{i=k}^m R_i, \\ \sum_{k=1}^m R_k \sum_{i=1}^k y_i \sum_{j=1}^{i-1} \delta \vartheta_j &= \sum_{k=1}^{m-1} \delta \vartheta_k \sum_{i=k+1}^m y_i \sum_{j=i}^m R_j. \end{aligned}$$

Then we obtain

$$-\delta A = \sum_{k=1}^m \left[\delta v_k (c_{k-1} p_k - s_{k-1}) \sum_{i=k}^m R_i + \delta \vartheta_k \left(q_k c_{k-1} \sum_{i=k}^m R_i + (1 - \delta_{km}) \sum_{i=k+1}^m y_i \sum_{j=i}^m R_j \right) \right], \quad (2.6)$$

where δ_{km} is the Kronecker symbol ($\delta_{km} = 0$ for $k \neq m$, $\delta_{km} = 1$ for $k = m$).

The equilibrium equations of a curved rod in generalized coordinates v_k and ϑ_k , $k = 1, 2, \dots, m$, are obtained on the basis of the principle of possible displacements $\delta \Pi - \delta A = 0$ and, taking into account (2.3), and (2.6), are written in the form

$$\begin{aligned} \frac{12EI_k}{a_k^3} \left(v_k - \frac{a_k}{2} \vartheta_k \right) + (c_{k-1} p_k - s_{k-1}) \sum_{i=k}^m R_i &= 0, \\ \frac{12EI_k}{a_k^3} \left(-\frac{a_k}{2} v_k + \frac{a_k^2}{3} \vartheta_k \right) + q_k c_{k-1} \sum_{i=k}^m R_i + (1 - \delta_{km}) \sum_{i=k+1}^m y_i \sum_{j=i}^m R_j &= 0. \end{aligned} \quad (2.7)$$

$k = 1, 2, \dots, m.$

Let us reduce Eqs. (2.7) to dimensionless form. For this, we introduce the following notation:

$$\begin{aligned} \bar{v}_k &= \frac{v_k}{a_k}, \quad \bar{u}_k = \frac{u_k}{a_k} = \frac{1}{2} (p_k \bar{v}_k + \bar{q}_k \vartheta_k), \\ p_k &= -\frac{1}{5} \left(6\bar{v}_k - \frac{1}{4} \vartheta_k \right), \quad \bar{q}_k = -\frac{1}{5} \left(\frac{1}{3} \vartheta_k - \frac{1}{4} \bar{v}_k \right), \\ \xi_k^v &= c_{k-1} p_k - s_{k-1}, \quad \xi_k^\vartheta = c_{k-1} \bar{q}_k, \quad \eta_i = \frac{y_i}{a_i} = -[(1 + \bar{u}_i) s_{i-1} + \bar{v}_i c_{i-1}]. \end{aligned} \quad (2.8)$$

Taking into account (2.8), Eqs. (2.7) are written in the form

$$\begin{aligned} \frac{12EI_k}{a_k^2} \left(\bar{v}_k - \frac{1}{2} \vartheta_k \right) + \xi_k^v \sum_{i=k}^m R_i &= 0, \\ \frac{12EI_k}{a_k^2} \left(-\frac{1}{2} \bar{v}_k + \frac{1}{3} \vartheta_k \right) + \xi_k^\vartheta \sum_{i=k}^m R_i + (1 - \delta_{km}) \sum_{i=k+1}^m \eta_i \sum_{j=i}^m R_j &= 0, \\ k &= 1, 2, \dots, m. \end{aligned}$$

We transform the double sum in the second equation:

$$\begin{aligned} \sum_{i=k+1}^m \eta_i \sum_{j=i}^m R_j &= \sum_{i=k+1}^m R_i \sum_{j=k+1}^i \eta_j = \sum_{i=k+1}^m R_i \zeta_{ki}, \\ \zeta_{ki} &= \sum_{j=k+1}^i \eta_j = \begin{cases} 0, & i < k+1 \\ \eta_{k+1} + \eta_{k+2} + \dots + \eta_i, & i \geq k+1. \end{cases} \end{aligned}$$

As a result, Eqs. (2.7) for the generalized coordinates $\bar{v}_k$ and ϑ_k , $k = 1, 2, \dots, m$, will have the form

$$\begin{aligned} \frac{12EI_k}{a_k^2} \left(\bar{v}_k - \frac{1}{2} \vartheta_k \right) + \xi_k^v \sum_{i=k}^m R_i &= 0, \\ \frac{12EI_k}{a_k^2} \left(-\frac{1}{2} \bar{v}_k + \frac{1}{3} \vartheta_k \right) + \xi_k^\vartheta \sum_{i=k}^m R_i + (1 - \delta_{km}) \sum_{i=k+1}^m R_i \zeta_{ki} &= 0, \\ k &= 1, 2, \dots, m. \end{aligned} \quad (2.9)$$

For their numerical solution, let us write Eqs. (2.9) in matrix form depending on the displacement vectors $\mathbf{V} = \{\bar{v}_k\}$, rotation angles $\mathbf{\Omega} = \{\vartheta_k\}$, and cable reactions $\mathbf{R} = \{R_k\}$:

$$\begin{aligned} 12\mathbf{G} \left(\mathbf{V} - \frac{1}{2} \mathbf{\Omega} \right) + \mathbf{A} \mathbf{R} &= 0, \\ 12\mathbf{G} \left(-\frac{1}{2} \mathbf{V} + \frac{1}{3} \mathbf{\Omega} \right) + \mathbf{B} \mathbf{R} &= 0, \end{aligned} \quad (2.10)$$

where the stiffness matrix $\mathbf{G}$ and nonlinear matrices $\mathbf{A}$ and $\mathbf{B}$ of order m have the form:

$$\begin{aligned} \mathbf{G} &= \begin{bmatrix} \ddots & & & & \\ & \ddots & & & \\ & & \frac{EI_k}{a_k^2} & & \\ & & & \ddots & \\ & & & & \ddots \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \xi_1^v & \xi_1^v & \dots & \xi_1^v & \xi_1^v \\ 0 & \xi_2^v & \dots & \xi_2^v & \xi_2^v \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \xi_{m-1}^v & \xi_{m-1}^v \\ 0 & 0 & \dots & 0 & \xi_m^v \end{bmatrix}, \\ \mathbf{B} &= \begin{bmatrix} \xi_1^\vartheta & \xi_1^\vartheta & \dots & \xi_1^\vartheta & \xi_1^\vartheta \\ 0 & \xi_2^\vartheta & \dots & \xi_2^\vartheta & \xi_2^\vartheta \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \xi_{m-1}^\vartheta & \xi_{m-1}^\vartheta \\ 0 & 0 & \dots & 0 & \xi_m^\vartheta \end{bmatrix}, \\ &+ \begin{bmatrix} 0 & \zeta_{12} & \dots & \zeta_{1,m-1} & \zeta_{1,m} \\ 0 & 0 & \dots & \zeta_{2,m-1} & \zeta_{2,m} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \zeta_{m-1,m} \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \end{aligned} \quad (2.11)$$

3. ALGORITHM FOR SOLVING THE PROBLEM

The shaping of the surface of revolution (or, in this case, a curved radial rod of the frame, on which a soft shell is superimposed with some tension) according to the equation $Y = F(X)$ can be carried out within the considered elements by choosing the reactions R_k of tensioned cables located in the planes of parallels, or due to bending stiffnesses EI_k of the elements of radial rods, or by changing both at the same time. The solution to such a nonlinear problem is not unique, and in some areas of variation of the indicated parameters, it may not exist.

Here, we consider the case when the equation of a given form $Y = F(X)$ is satisfied at points $k = 1, 2, \dots, m$ due to the controlled reactions of the cables R_k only along the coordinates X_k and $Y_k = F(X_k)$, and the tilt angles θ_k at these points remain free, i.e. condition $\theta_k = \arctan(F'(X_k))$ is not required. The stiffnesses EI_k of the elements of the bending rod are considered given.

The equivalent displacement v_k of the curved rod, which exactly corresponds to the given shape along the $X_k, Y_k = F(X_k)$ coordinates, is determined from the kinematic relations:

$$v_k = -(X_k - X_{k-1}) \cos \theta_{k-1} + (Y_k - Y_{k-1}) \sin \theta_{k-1}, \quad k = 1, 2, \dots, m, \quad (3.1)$$

where $\theta_k = \vartheta_0 + \sum_{i=1}^k \vartheta_i$, $\vartheta_0 = \arctan(F'(X_0))$; the angles ϑ_i at $i = 1, 2, \dots, m$ are considered unknown along with the reactions R_k at $k = 1, 2, \dots, m$.

As a result, dimensionless displacements $\bar{u}_k, \bar{v}_k$ and parameters (2.8) included in Eqs. (2.9) and, accordingly, in nonlinear matrices **A**, **B** (2.11), depend on unknown angles $\vartheta_i, i = 1, 2, \dots, m$. Thus, when the kinematic relations (3.1) are satisfied, the main unknowns will be ϑ_k and $R_k, k = 1, 2, \dots, m$, i.e. vectors $\mathbf{\Omega} = \{\vartheta_k\}$ and $\mathbf{R} = \{R_k\}$, for the definition of which we have equations (2.10). For the convenience of solving the system of two matrix equations (2.10) by eliminating the vector **R**, we reduce to one equation containing only unknown parameters $\vartheta_1, \vartheta_2, \dots, \vartheta_m$:

$$\mathbf{G}\mathbf{\Omega} = \left(\frac{1}{2}\mathbf{A} + \mathbf{B}\right) \left(\frac{1}{3}\mathbf{A} + \frac{1}{2}\mathbf{B}\right)^{-1} \mathbf{G}\mathbf{V} = \mathbf{P}. \quad (3.2)$$

This equation, taking into account the system of relations (3.1), is solved by the method of successive approximations according to the scheme $\mathbf{G}\mathbf{\Omega}^{(r+1)} = \mathbf{P}^{(r)}$, where $r = 0, 1, \dots$ is the number of the approximation by a set of unknown parameters $\vartheta_1, \vartheta_2, \dots, \vartheta_m$; $\mathbf{P}^{(r)}$ is a vector (the right side of Eq. (3.2)) depending on $\vartheta_1^{(r)}, \vartheta_2^{(r)}, \dots, \vartheta_m^{(r)}$. In the initial (zero) approximation at $r = 0$ for the angles ϑ_k we will use the exact values of $\vartheta_k^{(0)} = \vartheta_k^0$, corresponding to the given shape of the curved rod $Y = F(X)$ at points k , which are defined as $\vartheta_k^0 = \theta_k^0 - \vartheta_0 - \sum_{i=1}^{k-1} \vartheta_i^0$ at $\theta_k^0 = \arctan(F'(X_k)), k = 1, 2, \dots, m$.

After obtaining a solution converging with a given accuracy for the vector $\mathbf{\Omega} = \{\vartheta_k\}$, the vector of the reactions of the cables **R** is calculated from the equation that is obtained from system (2.10) by eliminating the vector **GΩ**:

$$\mathbf{R} = -\left(\frac{1}{3}\mathbf{A} + \frac{1}{2}\mathbf{B}\right)^{-1} \mathbf{G}\mathbf{V}. \quad (3.3)$$

In the initial state, after the system is opened, the radial rod 1' is rectilinear and deflected with respect to the X axis by a certain angle β_0 , Fig. 1b. The coordinates of the point $k' = 1', 2', \dots, m'$ of the rod in the initial state are equal to $X_{k'} = L_{k'} \cos \beta_0$, where $L_{k'} = (a_0 - \delta_0) + \sum_{i=1}^k a_i$ is the distance between the hinge connecting the hydraulic cylinder 2 with the pivot arm 6 (Fig. 1a), and point k' . The length of the k -th cable in the form of a regular n -gon connecting the k -th points of all n radial rods in parallel is chosen so that in the initial state it is unstretched and has no sagging. After turning the bracket 6 to the stop (the angle of inclination of the rod at the root point 0 becomes equal to ϑ_0), the k -th cable will stretch with a force $T_k = F_k \sigma(\epsilon_k)$, where F_k is the cross-sectional area, $\sigma(\epsilon_k)$ is the tensile stress in the k -th cable. The relative elongation of the k -th cable in the final tensioned state will be

Fig. 3. Longitudinal compression of an inextensible rod (Euler elastic).

$$\epsilon_k = \frac{X_k}{X_{k'}} - 1, \quad (3.4)$$

where X_k is the known coordinate of node k in the final deformed state (i.e., the coordinate of a given shape), and $X_{k'} = L_k \cdot \cos \beta_0$ is the coordinate of node k' in the initial undeformed position of the rod.

The reaction of the k -th cable at the k -th node of the rod is determined from the equilibrium equation of the node

$$R_k = 2T_k \sin \frac{\pi}{n} = 2F_k \sigma(\epsilon_k) \sin \frac{\pi}{n}. \quad (3.5)$$

In the case of linear elastic deformation of the cables $\sigma(\epsilon_k) = E_T \epsilon_k$ at $\epsilon_k > 0$ and $\sigma(\epsilon_k) = 0$ at $\epsilon_k \leq 0$, where E_T is the modulus of elasticity of the cable material. From the found values of R_k , it is possible to determine the required areas F_k or tensile stiffness $E_T F_k$ of the cables to obtain the shape $Y_k = F(X_k)$ of curved radial rods.

4. STRONG BENDING OF THE CANTILEVER ROD

To verify the developed FE-model of strong bending of a rod described by Eqs. (2.9), let us consider a rod of length l with constant bending stiffness EI , which is fixed at the end $s = 0$ and loaded at the end $s = l$ by a longitudinal compressive force R_l exceeding the critical loss force stability (Euler elastica). The differential equations of bending of an inextensible rod (Fig. 3) are written in the form

$$\frac{dX}{ds} = \cos \theta, \quad \frac{dY}{ds} = \sin \theta, \quad \frac{d\theta}{ds} = \frac{R_l}{EI} (Y_l - Y), \quad (4.1)$$

where

$$Y_l = Y(l), \quad R_l > \frac{\pi^2 EI}{4 l^2}.$$

Initial conditions for $s = 0$: $X = 0$, $Y = 0$, $\theta = 0$.

Comparisons of solutions of differential equations (4.1) with solutions of the system of algebraic equations (2.9) for the FE model for three different values of the parameter $\nu = R_l l^2 / EI$: 2.5, 3.0, 3.5. We used the following values of the initial (zero) approximations $\bar{Y}_l = Y_l / l < 1$ for “zeroing in” the numerical solutions of Eqs. (4.1) by the method of successive approximations in combination with the Adams method: $\bar{Y}_l = 0.25$ and $\bar{Y}_l = 0.40$ at $\nu = 2.5$; $\bar{Y}_l = 0.50$ and $\bar{Y}_l = 0.75$ at $\nu = 3.0$; $\bar{Y}_l = 0.65$ and $\bar{Y}_l = 0.90$ at $\nu = 3.5$ (two initial values of $\bar{Y}_l$ for one ν lead to the same result). The results of solutions converging with an accuracy of 10^{-9} for the rotation angles $\theta(s/l)$ at points $s/l = 1/4, 1/2, 3/4, 1.0$ are given in Table 1 in lines I.

Table 1. Verification of the obtained FE-model of strong bending of the rod

ν	Var.	$\bar{Y}_l$	$\theta(1/4)$	$\theta(1/2)$	$\theta(3/4)$	$\theta(1)$
2.5	I	0.2037	0.1241	0.2290	0.2990	0.3236
	II	0.1879	0.1141	0.2107	0.2752	0.2978
	III	0.1995	0.1214	0.2241	0.2926	0.3166
3.0	I	0.6636	0.4825	0.8806	1.1368	1.2245
	II	0.6678	0.4839	0.8844	1.1423	1.2304
	III	0.6640	0.4824	0.8806	1.1369	1.2246
3.5	I	0.7704	0.6503	1.1739	1.5002	1.6095
	II	0.7772	0.6532	1.1823	1.5116	1.6213
	III	0.7713	0.6506	1.1747	1.5014	1.6107

When using the FE model, the rod was divided into $m = 4$ and $m = 8$ identical “cantilever” FE ($a_k = l/m$, $EI_k = EI$, $R_k = 0$ at $k = 1, 2, \dots, m-1$). In this case, the results of solving Eqs. (4.1) for $\theta(s/l)$ were used as a zero approximation in solving Eqs. (2.9) by the method of successive approximations. The obtained solutions for the considered cases (ν, m) are given in Table. 1 (lines II and III for $m = 4$ and $m = 8$, respectively).

Figures 4a–4c show the rod bending shapes in dimensionless coordinates $\bar{X} = X/l$, $\bar{Y} = Y/l$ for values of $\nu = R_0 l^2/EI$ equal to 2.5, 3.0, 3.5, respectively.

Comparisons of the results of solving Eqs. (4.1) and (2.9) show that the FE model of strong bending of the rod has a sufficiently high accuracy.

5. CALCULATION EXAMPLES

Let us consider a parabolic antenna with a shape $Y = \lambda X^2$ at three values of the parameter λ : $1/15$, $1/10$, $1/5$; the number of radial rods $n = 24$. The angles of inclination of the rods in the initial rectilinear position (Fig. 1b), in which the sections of the cables connecting the k -th nodes of the rods are also rectilinear and non-tensioned, are taken equal to $\beta_0 = 60^\circ$ at $\lambda = 1/15$ and $\lambda = 1/10$, $\beta_0 = 70^\circ$ for $\lambda = 1/5$. The length of the central non-deformable section is $a_0 = 0.25$ m ($\delta_0 = 0$). The maximum antenna diameter is about 20 m.

Calculations of the strong bending of radial antenna rods under the action of reactions of stretched cables in the final deformed state of a cyclically symmetric system are performed. The results were obtained by solving nonlinear algebraic equations (3.2), and (3.3) by the method of successive approximations with an accuracy of 10^{-9} relative to unknowns θ_k , R_k taking into account the coupling conditions (3.1), which ensure the exact fulfillment of the given antenna shape at the nodes along the coordinates X_k ,

Fig. 4. Rod bending shapes for values of ν , equal to 2.5 (a), 3.0 (b), 3.5 (c).

Table 2. Initial data and results of calculation of antenna parameters at $m = 4$

λ	k	EI_k	θ_k^0	θ_k	R_k	ε_k	$E_T F_k$
$\frac{1}{15}$	1	1978.3237	0.3986	0.3885	47.8979	0.9467	193.8079
	2	1643.2336	0.6526	0.6483	41.6127	0.8361	190.6416
	3	1213.0318	0.8143	0.8099	6.1016	0.7195	32.4871
	4	671.2248	0.9214	0.9019	17.6858	0.6136	110.4073
$\frac{1}{10}$	1	1978.3237	0.5503	0.5342	96.4922	0.8935	413.6997
	2	1643.2336	0.8197	0.8182	54.8316	0.7167	293.0548
	3	1213.0318	0.9651	0.9608	5.4540	0.5634	37.0847
	4	671.2248	1.0545	1.0391	13.6527	0.4397	118.9434
$\frac{1}{5}$	1	1978.3237	0.8352	0.8165	217.5199	1.5098	551.8955
	2	1643.2336	1.0604	1.0634	54.1653	1.1016	188.3499
	3	1213.0318	1.1626	1.1585	1.7656	0.8357	8.0929
	4	671.2248	1.2224	1.2129	8.6875	0.6491	51.2688

$Y_k, k = 1, 2, \dots, m$. In this case, the values of θ_k^0 of a given shape were used as initial approximations for θ_k , i.e., $\theta_k^{(0)} = \theta_k^0$.

Two design cases are considered for each of the 3 antennas differing in the value of λ :

(1) The radial rod of the antenna is divided into 4 elastic links ($m = 4, k = 1, 2, 3, 4$) $a_k = 3$ m long; given bending stiffnesses of these links $EI_k, \text{Pa} \cdot \text{m}^4$ and design angles of inclination at nodes of a given shape $\theta_k^0 = \arctan(2\lambda X_k)$ are presented in Table 2. The shapes of the radial rod in the initial undeformed state (dashed line) and in the required final deformable state (solid line) for $m = 4$ at $\lambda = 1/15, 1/10, 1/5$ are shown in Fig. 5a–5c, respectively, in coordinates X, Y, m .

(2) The radial rod is divided into 6 elastic links ($m = 6, k = 1, 2, \dots, 6$) $a_k = 2$ m long; for this case, the values of $EI_k, \text{Pa} \cdot \text{m}^4$ and angles θ_k^0 are presented in Table 3. The shapes of the radial rod in the initial and required final states at $\lambda = 1/15, 1/10, 1/5$ are shown in coordinates X, Y, m in Figs. 6a, b, and c, respectively.

The obtained values of θ_k, R_k, N , as well as the relative tensile deformations of the cables ε_k and the cable stiffness in tension $E_T F_k (\text{Pa} \cdot \text{m}^2)$ required to obtain a given antenna shape, which are determined by the equation $R_k = 2E_T F_k \varepsilon_k \sin(\pi/n)$, are given in Table 2 and Table 3, respectively, for cases $m = 4$ and $m = 6, k = 1, 2, \dots, m$.

Table 4 and Table 5 for the cases $m = 4$ and $m = 6$, respectively, compare the accuracy of the solution by comparing the bending moments $M_k, \text{N} \cdot \text{m}, k = 1, 2, \dots, m - 1$, at the nodes of the rod in the final

Fig. 5. Radial rod shapes at start and end positions for $n = 24, m = 4$ and $\lambda = 1/15$ (a), $\lambda = 1/10$ (b), $\lambda = 1/5$ (c).

Table 3. Initial data and results of calculation of antenna parameters at $m = 6$

λ	k	EI_k	θ_k^0	θ_k	R_k	ε_k	$E_T F_k$
$\frac{1}{15}$	1	7656.4383	0.2876	0.2822	260.5498	0.9728	1025.9801
	2	6684.4295	0.4961	0.4961	105.2135	0.9113	442.2760
	3	6013.2047	0.6526	0.6458	151.509	0.8352	694.9330
	4	4204.6526	0.7681	0.7717	10.4413	0.7567	52.8544
	5	4204.6526	0.8547	0.8465	5.4236	0.6820	30.4614
	6	1631.6882	0.9214	0.9109	66.9765	0.6131	418.4988
$\frac{1}{10}$	1	7656.4383	0.4116	0.3987	592.7990	0.9427	2408.7411
	2	6013.2047	0.6609	0.6611	326.3337	0.8311	1504.1060
	3	5193.4296	0.8197	0.8157	105.8771	0.7147	567.4419
	4	4204.6526	0.9250	0.9277	2.8618	0.6095	17.9877
	5	4204.6526	0.9993	0.9927	10.1289	0.5179	74.9241
	6	1631.6882	1.0545	1.0459	50.4612	0.4386	440.6749
$\frac{1}{5}$	1	7656.4383	0.6849	0.6686	1225.6470	1.6655	2819.0376
	2	6684.4295	0.9357	0.9416	428.9441	1.3402	1226.0534
	3	6684.4295	1.0604	1.0561	43.6339	1.0934	152.8741
	4	5193.4296	1.1352	1.1358	36.4977	0.9070	154.1510
	5	4204.6526	1.1856	1.1817	12.6620	0.7617	63.6785
	6	1631.6882	1.2224	1.2165	30.3713	0.6449	180.4132

deformed state: I are calculated from the equilibrium equation of the cut-off part with the found reactions

$R_k: M_k = \sum_{i=k+1}^m R_k(Y_i - Y_k)$; II are the mean values at the node found when solving a nonlinear problem in displacements by the finite element method: $M_k = 1/2 (M_k(a_k) + M_{k+1}(0))$, where $M_k(a_k) = 2EI_k/a_k (-3\bar{\vartheta}_k + 2\vartheta_k)$, $M_{k+1}(0) = 2EI_{k+1}/a_{k+1} (3\bar{\vartheta}_{k+1} - 2\vartheta_{k+1})$. The comparison results show the high accuracy obtained by solving the nonlinear algebraic equations of the FE-model by the method of successive approximations.

CONCLUSIONS

A constructive scheme is proposed and a mathematical model is developed for calculating the formation of a cyclically symmetric space umbrella antenna formed by a system of flexible multi-link radial rods connected in parallel at certain nodal points by tensile cables. The bending of the rods in the radial plane, taking into account the reactions of the cables, is created by turning the root parts of the rods by the force

Fig. 6. Radial rod shapes at start and end positions for $n = 24$, $m = 6$ and $\lambda = 1/15$ (a), $\lambda = 1/10$ (b), $\lambda = 1/5$ (c).

Table 4. The results of calculating bending moments at the nodes of the rod at $m = 4$

λ	Var.	$k = 1$	$k = 2$	$k = 3$
$\frac{1}{15}$	I	188.3897	88.6112	40.5765
$\frac{1}{15}$	II	193.7100	90.1060	40.9970
$\frac{1}{10}$	I	222.0288	79.5639	34.7384
$\frac{1}{10}$	II	230.6572	80.9606	34.9822
$\frac{1}{5}$	I	211.5091	52.4085	24.2377
$\frac{1}{5}$	II	225.4776	53.1092	24.3086

Table 5. The results of calculating bending moments at the nodes of the rod at $m = 6$

λ	Var.	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$\frac{1}{15}$	I	835.2111	573.4939	317.5385	209.1970	104.0202
$\frac{1}{15}$	II	848.8874	579.8688	319.7085	209.9830	104.5593
$\frac{1}{10}$	I	1026.0813	512.9911	283.3631	185.9495	86.4142
$\frac{1}{10}$	II	1055.8608	520.2044	284.7586	186.4288	86.7087
$\frac{1}{5}$	I	1295.6631	485.1380	277.4441	135.6802	56.7180
$\frac{1}{5}$	II	1351.8396	490.7088	278.1707	135.9047	56.8058

of the slowly moving rod of the damping hydraulic cylinder under the action of pre-compressed springs. The model of strong bending of an inextensible antenna rod is built using “cantilever” finite elements (links) that allow large displacements and turns as rigid bodies and relative elastic displacements with a two-term approximation along the length of the element, taking into account its shrinkage due to bending in a quadratic approximation. The obtained nonlinear equations are solved by the method of successive approximations with respect to the unknown reactions of the cables at the given values of the bending stiffnesses of the rod elements and the given radial and axial coordinates of the nodes, which exactly correspond to the shape of the simulated antenna. Based on the found values of the reactions of the cables, taking into account their known displacements in the nodes, the tensile stiffnesses of the cables necessary to ensure the given antenna shape are determined.

The accuracy of the developed model is estimated by comparison with the numerical solution of the problem of strong bending of a cantilever rod under the action of a longitudinal compressive force exceeding the critical buckling force of the rod.

As an example, calculations were performed for a parabolic antenna with $n = 24$ radial rods and with $m = 4$ and $m = 8$ of their constituent links (respectively, cables).

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