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A Three State Hard-Core Model on a Cayley Tree

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Abstract

We consider a nearest-neighbor hard-core model, with three states , on a homogeneous Cayley tree of order k (with $k+1$ neighbors). This model arises as a simple example of a loss network with nearest-neighbor exclusion. The state $\sigma(x)$ at each node x of the Cayley tree can be 0, 1 and 2. We have Poisson flow of calls of rate λ at each site x, each call has an exponential duration of mean 1. If a call finds the node in state 1 or 2 it is lost. If it finds the node in state 0 then things depend on the state of the neighboring sites. If all neighbors are in state 0, the call is accepted and the state of the node becomes 1 or 2 with equal probability 1/2. If at least one neighbor is in state 1, and there is no neighbor in state 2 then the state of the node becomes 1. If at least one neighbor is in state 2 the call is lost. We focus on 'splitting' Gibbs measures for this model, which are reversible equilibrium distributions for the above process. We prove that in this model, $\forall\ \lambda > 0$ and $k \geq 1$, there exists a unique translation-invariant splitting Gibbs measure μ^*. We also study periodic splitting Gibbs measures and show that the above model admits only translation - invariant and periodic with period two (chess-board) Gibbs measures. We discuss some open problems and state several related conjectures.

1 Introduction

A Cayley tree $T^k = (V, L)$ of order $k \geq 1$ is defined as an infinite homogeneous tree, i.e., a graph without cycles, with exactly $k+1$ edges incident to each site. Here V is the set of sites and L is the set of edges. Fix a site x^0 (the origin) and set: $V_n = \{x \in V : \text{dist}\ (x^0, x) \leq n\}$, $W_n = \{x \in V : \text{dist}\ (x^0, x) = n\}$, where the distance between x, $y \in V$ is the number of edges in the shortest path $x \to y$. Call a site $x \in V$ even if dist (x^0, x) is even and odd if it is odd.

Our goal is to present results on the nature of a phase transition in the nearest-neighbor three state hard-core model. In this model one assigns, to each site x, values $\sigma(x) \in \{0, 1, 2\}$; values $\sigma(x) = 1, 2$ mean that site x is 'occupied' and $\sigma(x) = 0$ that x is 'vacant'.

A configuration σ on the tree is a collection $\{\sigma(x),\ x \in V\}$ considered also as a function $V \to \{0,1,2\}$. In a similar fashion one defines a configuration in V_n and W_n. We call σ an admissible configuration (on the tree, in V_n or W_n) if the sum $\sigma(x)+\sigma(y) \le 2\ \forall$ nearest-neighbor pair x, y (from V, V_n or W_n, respectively). Denote the set of admissible configurations by Ω (Ω_{V_n} and Ω_{W_n}). Ω is endowed by a natural topology in which it is a (totally disconnected) compact. For $\sigma_n \in \Omega_{V_n}$ we define : $\#\sigma_n = \sum_{x\in V_n} \mathbf{1}(\sigma_n(x) \ge 1)$ (the number of occupied sites in σ_n). Let $\mathbf{B}$ be the sigma-algebra generated by the cylinder subsets of Ω. Furthermore, $\forall\ n$, $\mathbf{B}_{V_n}$ stands for the sub-algebra of $\mathbf{B}$ generated by events $\{\sigma \in \Omega\colon\ \sigma\big|_{V_n} = \sigma_n\}$ where $\sigma_n\colon\ x \in V_n \mapsto \sigma_n(x)$ is an admissible configuration in V_n and $\sigma\big|_{V_n}$ the restriction of σ on V_n (the notation $\sigma\big|_{V\setminus V_n}$ and $\sigma\big|_{W_n}$ has a similar meaning).

Definition 1. A (three state) hard core Gibbs measure with fugacity $\lambda > 0$ is a probability measure μ on $(\Omega, \mathbf{B})$ such that, $\forall\ n$ and $\sigma_n \in \Omega_{V_n}$:

$$\mu\Big\{\sigma \in \Omega :\ \sigma\big|_{V_n} = \sigma_n\Big\} = \int_\Omega \mu(d\omega) P_n\big(\sigma_n \,\big|\, \omega_{W_{n+1}}\big), \tag{1.1}$$

where

$$P_n\big(\sigma_n \,\big|\, \omega_{W_{n+1}}\big) = \frac{1}{\Xi_n(\lambda; \omega|_{W_{n+1}})} \lambda^{\#\sigma_n} \mathbf{1}\left(\sigma_n \vee \omega\big|_{W_{n+1}} \in \Omega_{V_{n+1}}\right).$$

Here, and below, symbol $\vee$ means concatenation of configurations and $\Xi_n\left(\lambda; \omega\big|_{W_{n+1}}\right)$ is the partition function with the boundary condition $\omega\big|_{W_n}$:

$$\Xi_n\left(\lambda; \omega\big|_{W_{n+1}}\right) = \sum_{\widetilde{\sigma}_n \in \Omega_{V_n}} \lambda^{\#\widetilde{\sigma}_n} \mathbf{1}\left(\widetilde{\sigma}_n \vee \omega\big|_{W_{n+1}} \in \Omega_{V_{n+1}}\right). \tag{1.2}$$

Pictorially, this definition means that, given values $\sigma(x)$, $x \in W_n$, of the restriction of an admissible configuration $\sigma \in \Omega$ to W_n, (i) its restrictions $\sigma\big|_{V_n}$ inside V_n and $\sigma\big|_{V\setminus V_{n+1}}$ outside V_{n+1} are conditionally independent, and (ii) the conditional probability of $\sigma\big|_{V_n}$ is proportional to $\lambda^{\#\sigma|_{V_n}}$. For $k \ge 2$, this is an analogue of a Markov property (taking place for $k = 1$, when the tree is reduced to an integer lattice).

The rest of this section provides a review of basic properties of hard core Gibbs measures. They can be found in [8] (see in particular Ch. 12) and [23, 24]. The set of hard core Gibbs measures with a given fugacity is denoted by $\mathcal{G}$. The set of values of λ for which $\mathcal{G}$ is reduced to a single point is interpreted as a 'single-phase domain'; its complement referred to as the 'domain of phase transitions'. Set $\mathcal{G}$ is a Choquet simplex, i.e., a convex compact, in the topology of weak convergence, such that any $\mu \in \mathcal{G}$ can be uniquely written as an integral

$$\mu = \int_{\mathcal{E}} d\Pi(\nu)\nu.$$

Here $\mathcal{E} \subset \mathcal{G}$ is the set of extreme Gibbs measures, for which the probability of any tail event is 0 or 1, Π $(= \Pi_\mu)$ is a probability distribution on $\mathcal{E}$ (more precisely, on the Borel sigma-algebra in $\mathcal{E}$ induced by the above topology).

It is also interesting to consider translation-invariant and translation-periodic Gibbs measures, $\mathcal{G} \cap \mathcal{T}$ and $\mathcal{G} \cap \mathcal{P}$. Here sets $\mathcal{T}$ and $\mathcal{P}$ are formed, respectively, by translation-invariant and translation-periodic measures on $(\Omega, \mathbf{B})$.

Set $\mathcal{G}$ contains limiting points of measures $P\big(\ \cdot\ \big|\omega\big|_{W_n}\big)$, again in the topology of weak convergence. (Here we consider $P\big(\ \cdot\ \big|\omega\big|_{W_n}\big)$ as a measure on $(\Omega, \mathbf{B})$, defined in a standard way.) In particular, $\mathcal{G}$ contains a 'maximal' and a 'minimal' measure, $\mu_\pm$ such that every measure $\mu \in \mathcal{G}$ lies 'between' them (symbolically, $\mu_- \leq \mu \leq \mu_+$, see below). Clearly $\mathcal{G}$ is reduced to a single point iff $\mu_+ = \mu_-$. The maximality and minimality properties of $\mu_\pm$ with respect to $\mu \in \mathcal{G}$ are expressed in terms of integrals of (suitably defined) monotone functions $F : \Omega \to \mathbf{R}$ (the FKG inequalities). See, e.g., Refs [23, 24]).

The hard core model is interesting from the point of view of statistical mechanics [8] as well of combinatorics [3] and the theory of neuron networks ([11]). Here, we stress its application in communication networks. Consider the following model of a loss network with nearest-neighbor exclusion. Calls arrive in independent Poisson processes ξ_x, $x \in V$, of rate λ. If a call finds the node in state 1 or 2 it is lost. If it finds the node in state 0 then its fate depends on the state of the neighboring sites. If all neighbors are in state 0, the call is accepted and the state of the node becomes 1 or 2 with equal probability 1/2. If at least one neighbor is in state 1, but there is no neighbor in state 2, the state of the node becomes 1. Finally, if at least one of the neighbors is in state 2, the call is lost. Pictorially, state 2 means a high quality service that requires that all neighbors are silent. State 1 means a moderate quality service where neighbors can interfere (if they are also in state 1). Once accepted, the call is processed during an exponential time of rate one and then leaves the network.

The above description gives rise to a continuous-time Markov process $\mathbf{X}(t)$, $t \geq 0$, on space Ω. Due to a theorem by Dobrushin (see, [6]), any reversible equilibrium measure μ for $\mathbf{X}(t)$ belongs to $\mathcal{G}$, i.e. is hard core Gibbs measure with fugacity λ. Thus, our analysis leads to a description of various properties of reversible equilibrium distributions of the above loss network.

A more general model was considered in [4] (by using a different language), with three fugacities λ_0, λ_1 and λ_2. Results from [4] can be considered as complementary to the results of this paper. we comment on this in Remarks 1 and 2 following our Theorems 1 and 3.

2 Construction of splitting Gibbs measures

Following Ref. [22, 17] (and subsequent papers [23, 24], [1, 2, 7, 19, 20, 21]), we consider a special class $\mathcal{S} \subset \mathcal{G}$ of Gibbs measures. We call them splitting Gibbs measures, to emphasize the fact that, in addition to the aforementioned Markov property, they satisfy the following condition: given values $\sigma(x)$, $x \in V_n$, of an admissible configuration $\sigma \in \Omega$ over set V_n, its values $\sigma(y)$ at sites $y \in W_{n+1}$ are conditionally independent. [In [23, 24] such measures were called Markov chains (on a tree) and in [22] entrance laws.] A formal definition follows.

Write $x < y$ if the path from x^0 to y goes through y. Call vertex y a direct successor of x if $y > x$ and x, y are nearest neighbors. Denote by $S(x)$ the set of direct successors of x. Note that any vertex $x \neq x^0$ has k direct successors and x^0 has $k + 1$.

Let $z: \ x \mapsto z_x = (z_{0,x}, z_{1,x}, z_{2,x}) \in \mathbf{R}^3_+$ be a vector-valued function on V. Given $n = 1, 2, \ldots$, consider the probability distribution $\mu^{(n)}$ on Ω_{V_n} defined by

$$\mu^{(n)}(\sigma_n) = \frac{1}{Z_n} \lambda^{\#\sigma_n} \prod_{x \in W_n} z_{\sigma(x),x}. \tag{2.1}$$

Here, as before, Z_n is the corresponding partition function:

$$Z_n = \sum_{\widetilde{\sigma}_n \in \Omega_{V_n}} \lambda^{\#\widetilde{\sigma}_n} \prod_{x \in W_n} z_{\widetilde{\sigma}(x),x}.$$

We say that the probability distributions $\mu^{(n)}$ are compatible if $\forall \ n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$:

$$\sum_{\omega_n \in \Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n) \mathbf{1}(\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}) = \mu^{(n-1)}(\sigma_{n-1}). \tag{2.2}$$

In this case there exists a unique probability measure μ on $(\Omega, \mathbf{B})$ such that, $\forall \ n$ and $\sigma_n \in \Omega_{V_n}$, $\mu\left(\left\{\sigma\Big|_{V_n} = \sigma_n\right\}\right) = \mu^{(n)}(\sigma_n)$.

Definition 2. Measure μ defined by (2.1), (2.2) is called a splitting hard core Gibbs measure with fugacity λ, corresponding to function $z : x \in V \setminus \{x^0\} \mapsto z_x$. The set of such measures (for all possible choices of z) is denoted by $\mathcal{S}$.

The following statement describes conditions on z_x guaranteeing compatibility of distributions $\mu^{(n)}$.

Proposition 1. *Probability distributions $\mu^{(n)}$, $n = 1, 2, \ldots$, in (2.1) are compatible iff for any $x \in V$ the following system of equations holds:*

$$z'_{1,x} = \lambda \prod_{y \in S(x)} \frac{1 + z'_{1,y}}{1 + z'_{1,y} + z'_{2,y}}, \tag{2.3a}$$

$$z'_{2,x} = \lambda \prod_{y \in S(x)} \frac{1}{1 + z'_{1,y} + z'_{2,y}}. \tag{2.3b}$$

Here, and below, $z'_{i,x} = \lambda z_{i,x}/z_{0,x}, \quad i = 1, 2$.

Proof. Write:

$$\text{LHS of (2.2)} = \frac{1}{Z_n} \lambda^{\#\sigma_{n-1}} \prod_{x \in W_{n-1}} \prod_{y \in S(x)} \left(z_{0,y} + \mathbf{1}(\sigma_{n-1}(x) \in \{0,1\}) \lambda z_{1,y} + \mathbf{1}(\sigma_{n-1}(x) = 0) \lambda z_{2,y}\right). \tag{2.4}$$

Sufficiency. Suppose that (2.3) holds. It is equivalent to the representations

$$\prod_{y\in S(x)} (z_{0,y}+\lambda z_{1,y}+\lambda z_{2,y}) = a(x)z_{0,x},$$

$$\prod_{y\in S(x)} (z_{0,y}+\lambda z_{1,y}) = a(x)z_{1,x}, \quad \prod_{y\in S(x)} z_{0,y} = a(x)z_{2,x},$$

for some function $a(x)>0$, $x\in V$. Setting $A_n=\prod_{x\in W_n} a(x)$ and substituting (2.1) into (2.4), we get:

$$\text{RHS of (2.4)} = \frac{1}{Z_n}\lambda^{\#\sigma_{n-1}} \prod_{x\in W_{n-1}} z_{\sigma_{n-1}(x),x}a(x) = \frac{A_{n-1}}{Z_n}\lambda^{\#\sigma_{n-1}} \prod_{x\in W_{n-1}} z_{\sigma_{n-1}(x),x}.$$

We should have

$$\sum_{\sigma_{n-1}\in\Omega_{V_{n-1}}} \sum_{\omega_n\in\Omega_{W_n}} \mathbf{1}\left(\sigma_{n-1}\vee\omega_n\in\Omega_{V_n}\right)\mu^{(n)}(\sigma_{n-1}\vee\omega_n)=1,$$

hence $A_{n-1}/Z_n = 1/Z_{n-1}$, and (2.2) holds.

Necessity. Suppose that (2.2) holds; we want to prove (2.3). Substituting (2.1) in (2.2) and using (2.4), we obtain that $\forall\ \sigma_{n-1}\in\Omega_{V_{n-1}}$:

$$\frac{Z_{n-1}}{Z_n} \prod_{x\in W_{n-1}} \prod_{y\in S(x)} (z_{0,y}+\mathbf{1}(\sigma_{n-1}(x)\in\{0,1\})\lambda z_{1,y}+\mathbf{1}(\sigma_{n-1}(x)=0)\lambda z_{2,y}) = \prod_{x\in W_{n-1}} z_{\sigma_{n-1}(x),x}.$$

In particular, comparing a pair of configurations σ_{n-1}, σ'_{n-1} different at a single site $x\in W_{n-1}$ yields (2.3). ■

Proposition 2. *Any measure μ with local distributions $\mu^{(n)}$ satisfying (2.1), (2.3) belongs to $\mathcal{G}$.*

Proof. Straightforward. ■

Proposition 3. *$\mathcal{E}\subseteq\mathcal{S}$, i.e., any extreme Gibbs measure $\mu\in\mathcal{E}$ is splitting.*

Proof. See [8], Theorem 12.6. ■

3 Uniqueness of a translation-invariant splitting Gibbs measure

Without loss of generality, we set in future $z_{0,x}\equiv 1$ and $z_{i,x}=z'_{i,x}>0$, $i=1,2$. Then $\forall$ function $x\in V\mapsto z_x=(z_{1,x},z_{2,x})$ satisfying

$$z_{1,x}=\lambda\prod_{y\in S(x)}\frac{1+z_{1,y}}{1+z_{1,y}+z_{2,y}}, \tag{3.1a}$$

$$z_{2,x}=\lambda\prod_{y\in S(x)}\frac{1}{1+z_{1,y}+z_{2,y}}, \tag{3.1b}$$

there exists a unique hard core splitting Gibbs measure μ and vice versa. However, the analysis of solutions to (3.1) is rather tricky. It is natural to begin with translation-invariant solutions where $z_x = z$ is constant $\in \mathbf{R}^2_+$, $x \neq x^0$.

In this case we obtain, from (3.1), the following system of equations:

$$z_1 = \lambda\left(\frac{1+z_1}{1+z_1+z_2}\right)^k, \tag{3.2a}$$

$$z_2 = \lambda\left(\frac{1}{1+z_1+z_2}\right)^k. \tag{3.2b}$$

It is easy to see that

$$z_2 = z_1(1+z_1)^{-k}. \tag{3.3}$$

Using this equality from (3.2) we get

$$\lambda^{-1}z = \left(\frac{(1+z)^{k+1}}{z+(1+z)^{k+1}}\right)^k, \tag{3.4}$$

with $z = z_1$.

Proposition 4. *For any $\lambda > 0, k \geq 1$ the system of equations (3.2) has a unique positive solution (with $z_1, z_2 > 0$) .*

Proof. We shall prove that equation (3.4) has a unique positive solution. Denote

$$f(z) = f(z,k) = \left(\frac{(1+z)^{k+1}}{z+(1+z)^{k+1}}\right)^k.$$

We have $f(0) = 1$ and

$$f'(z) = \frac{k(1+z)^{k^2+k-1}}{(z+(1+z)^{k+1})^{k+1}}(kz-1).$$

For $0 < z < 1/k$, f decreases monotonically from 1 to $f(1/k) = ((k+1)^{k+1}((k+1)^{k+1} + k^k)^{-1})^k$. For $z > 1/k$, f is monotonically increasing to 1 as $z \to \infty$; thus there are at most three solutions of (3.4). On the other hand , it is easy to see that (3.4) has more than one solution if and only if there is more than one solution to the equation $zf'(z) = f(z)$ which is equivalent to

$$(1+z)^{k+2} = (k^2-1)z^2 - (k+1)z. \tag{3.5}$$

It is straightforward that equation (3.5) has no positive solutions. Thus (3.4) has a unique positive solution $z^* = z_1^*$. Using (3.3) we get unique z_2^*. ■

We then obtain

Theorem 1. $\forall\ \lambda > 0$, *the set $\mathcal{S} \cap \mathcal{T}$ of translation-invariant hard-core splitting Gibbs measures consists of a single point.*

The translation-invariant hard-core splitting Gibbs measure is denoted by μ^*.

Remark 1. The model considered in this paper, translating to the language of [4] has a constraint graph consisting of a three-node path 2-0-1, with loops on 0 and 1. This graph is called "wrench" in [4], where it is proved that there do exist multiple translation-invariant splitting (simple in [4]) Gibbs measures, for some fugacities λ_i on the three nodes. Theorem 1 shows that if fugacities are chosen as in this paper then there is always just one translation-invariant Gibbs measure. It is thus natural to ask: When the constraint graph H has three nodes, for which H in [4] is the analog of Theorem 1 true for fugacities chosen as in our case? Now we shall answer this question.

The other graphs in [4] with three nodes have node set, edges, and loops as follows:
the *pipe*: $\{0,1,2\}$; $\{0,1\}$, $\{1,2\}$; loop at 0,
the *hinge*: $\{0,1,2\}$; $\{0,1\}$, $\{0,2\}$; loops at $0,1$ and 2,
the *wand*: $\{0,1,2\}$; $\{0,1\}$, $\{0,2\}$; loops at 1 and 2.

By a similar argument one can see that the system of equations (3.2) has the following form for above constraint graphs:

$$(pipe)\qquad \begin{cases} z_1 = \lambda\left(\frac{1+z_2}{1+z_1}\right)^k, \\ z_2 = \lambda\left(\frac{z_1}{1+z_1}\right)^k. \end{cases} \tag{3.6}$$

$$(hinge)\qquad \begin{cases} z_1 = \lambda\left(\frac{1+z_1}{1+z_1+z_2}\right)^k, \\ z_2 = \lambda\left(\frac{1+z_2}{1+z_1+z_2}\right)^k. \end{cases} \tag{3.7}$$

$$(wand)\qquad \begin{cases} z_1 = \lambda\left(\frac{1+z_1}{z_1+z_2}\right)^k, \\ z_2 = \lambda\left(\frac{1+z_2}{z_1+z_2}\right)^k. \end{cases} \tag{3.8}$$

Case pipe. Assume $k=2$. Then from system (3.6) we have $z_1^3 = z_2(1+z_2)^2$ and

$$\lambda^{-1}z = f_{\rm pipe}(z), \quad \text{with} \quad f_{\rm pipe}(z) = \left(\frac{\sqrt[3]{z(1+z)^2}}{1+\sqrt[3]{z(1+z)^2}}\right)^2, \quad z = z_2$$

We have $f'_{\rm pipe}(z) > 0$. It is easy to see that the equation $zf'_{\rm pipe}(z) = f_{\rm pipe}(z)$ has the form $3(1+z)\sqrt[3]{z(1+z)^2} + 2 = 0$ which has no positive solutions. Thus in the pipe case for $k=2$ and $\forall\lambda > 0$ the splitting Gibbs measure is unique.

Of course , it is very natural to expect that this result is true for $k \geq 3$.

Case hinge. Subtracting from the first equation of system (3.7) the second one we get $z_1 = z_2$ and

$$(1+z_1+z_2)^k = \lambda\big((1+z_1)^{k-1} + ... + (1+z_2)^{k-1}\big), \quad \text{if} \quad z_1 \neq z_2. \tag{3.9}$$

For $z_1 = z_2 = z$ from system (3.7) we have

$$\lambda^{-1}z = f_{\rm hinge}(z) = \left(\frac{1+z}{1+2z}\right)^k. \tag{3.10}$$

It is easy to see that the function $f_{\text{hinge}}(z)$ is decreasing for $z > 0$ which implies that equation (3.10) has unique solution $z^* = z^*(k, \lambda)$ for any $\lambda > 0$.

If (3.9) is satisfied then we assume $k = 2$ and from (3.9) we have

$$1 + z_1 + z_2 = \frac{\lambda + \sqrt{\lambda^2 + 4\lambda}}{2}. \tag{3.11}$$

Using this equality from first equation of the system (3.7) we have (for $k = 2$)

$$z_1^{(1)} = \left(\frac{1 + \sqrt{1 - 4a^2}}{2a}\right)^2, \quad z_1^{(2)} = \left(\frac{1 - \sqrt{1 - 4a^2}}{2a}\right)^2, \tag{3.12}$$

if $\lambda > 9/4$ where $a = 2(\sqrt{\lambda} + \sqrt{\lambda + 4})^{-1}$. Using the second equation we also have $z_2^{(1)}, z_2^{(2)} \in \{z_1^{(1)}, z_1^{(2)}\}$. Since $z_1 \neq z_2$ we conclude that $z_1 = z_1^{(1)}, z_2 = z_1^{(2)}$ and $z_1 = z_1^{(2)}, z_2 = z_1^{(1)}$. It is easy to check that these solutions satisfies the condition (3.11).

Thus if $k = 2$, $\lambda > \frac{9}{4}$ then the system (3.7) has three solutions (z^*, z^*), $(z_1^{(1)}, z_1^{(2)})$, $(z_1^{(2)}, z_1^{(1)})$, where z^* is the unique solution of (3.10) and $z_1^{(i)}$, $i = 1, 2$ is defined in (3.12). Note that $z_1^{(1)} = \frac{1}{z_2^{(2)}}$. The value $\lambda = \lambda_{\text{cr}} = \frac{9}{4}$ is exactly the critical value for $k = 2$.

Case wand. This case is very similar to the case hinge and one can prove that if $k = 2$, $\lambda > 1$ then the system (3.8) has three solutions given by similar formulas of case hinge just replacing a with $a = 2(\sqrt{\lambda} + \sqrt{\lambda + 8})^{-1}$. Here also the value $\lambda_{\text{c}} = 1$ is the exact value of critical λ for $k = 2$.

Thus the uniqueness (Theorem 1) is true for pipe but is not true for cases hinge and wand.

Now we move to our model (the wrench graph case).

Proposition 5. *If $z_x = (z_{1,x}, z_{2,x})$ is a solution of (3.1) then $z_i^- \leq z_{i,x} \leq z_i^+$, for any $i = 1, 2$, $x \in V$, where $(z_1^-, z_1^+, z_2^-, z_2^+)$ is a solution of*

$$z_1^- = \lambda\left(\frac{1 + z_1^-}{1 + z_1^- + z_2^+}\right)^k, \tag{3.13a}$$

$$z_1^+ = \lambda\left(\frac{1 + z_1^+}{1 + z_1^+ + z_2^-}\right)^k, \tag{3.13b}$$

$$z_2^- = \lambda\left(\frac{1}{1 + z_1^+ + z_2^+}\right)^k, \tag{3.13c}$$

$$z_2^+ = \lambda\left(\frac{1}{1 + z_1^- + z_2^-}\right)^k. \tag{3.13d}$$

Proof. It is clear that $0 < z_{i,x} < \lambda$, $i = 1, 2$, $\forall x \in V$. We rewrite (3.1) as

$$z_{1,x} = \lambda \prod_{i=1}^{k} \frac{1 + z_{1,x_i}}{1 + z_{1,x_i} + z_{2,x_i}},$$

$$z_{2,x} = \lambda \prod_{i=1}^{k} \frac{1}{1 + z_{1,x_i} + z_{2,x_i}},$$

where $x_j, \quad j = 1, 2, ..., k$ are direct successors of x. Denote

$$f_1(u_1, ..., u_k, v_1, ..., v_k) = \lambda \prod_{i=1}^{k} \frac{1 + u_i}{1 + u_i + v_i},$$
$$f_2(u_1, ..., u_k, v_1, ..., v_k) = \lambda \prod_{i=1}^{k} \frac{1}{1 + u_i + v_i},$$

with $0 < u_j < \lambda, \quad 0 < v_j < \lambda$. It is not difficult to see that

$$\frac{\lambda}{(1+\lambda)^k} < f_1(u_1, ..., u_k, v_1, ..., v_k) < \lambda,$$
$$\frac{\lambda}{(1+2\lambda)^k} < f_2(u_1, ..., u_k, v_1, ..., v_k) < \lambda,$$

Thus for $z_{i,x}$ we get

$$\frac{\lambda}{(1+\lambda)^k} < z_{1,x} < \lambda, \quad \frac{\lambda}{(1+2\lambda)^k} < z_{2,x} < \lambda.$$

Now consider $f_i(u_1, ..., u_k, v_1, ..., v_k)$ with

$$\frac{\lambda}{(1+\lambda)^k} < u_j < \lambda, \quad \frac{\lambda}{(1+2\lambda)^k} < v_j < \lambda, \quad j = 1, ..., k.$$

Iterating this procedure we can obtain the following

$$z^-_{i,n} < z_{i,x} < z^+_{i,n}, \quad i = 1, 2,$$

where $z^{\pm}_{i,n}, \quad i = 1, 2, \quad n = 1, 2, ...$ satisfy

$$z^-_{1,n+1} = \lambda \left(\frac{1 + z^-_{1,n}}{1 + z^-_{1,n} + z^+_{2,n}} \right)^k,$$
$$z^+_{1,n+1} = \lambda \left(\frac{1 + z^+_{1,n}}{1 + z^+_{1,n} + z^-_{2,n}} \right)^k,$$
$$z^-_{2,n+1} = \lambda \left(\frac{1}{1 + z^+_{1,n} + z^+_{2,n}} \right)^k,$$
$$z^+_{2,n+1} = \lambda \left(\frac{1}{1 + z^-_{1,n} + z^-_{2,n}} \right)^k,$$

with $z^-_{1,1} = \frac{\lambda}{(1+\lambda)^k}$, $z^+_{1,1} = z^+_{2,1} = \lambda$ and $z^-_{2,1} = \frac{\lambda}{(1+2\lambda)^k}$. It is easy to see that $z^-_{i,n}, \quad i = 1, 2$ $(z^+_{i,n})$ are increasing (decreasing) and bounded sequences. Thus there exist $\lim_{n\to\infty} z^{\pm}_{i,n} = z^{\pm}_i, \quad i = 1, 2.$ ■

Proposition 6. *If $z = (z^-_1, z^+_1, z^-_2, z^+_2)$ a solution of (3.13) then $z^-_1 = z^+_1$ iff $z^-_2 = z^+_2$.*

Proof. From (3.13) we have

$$\begin{cases} z_1^- - z_1^+ = \lambda A((1+z_1^-)(z_2^- - z_2^+) + z_2^+(z_1^- - z_1^+)), \\ z_2^- - z_2^+ = \lambda B((z_2^- - z_2^+) + (z_1^- - z_1^+)), \end{cases} \tag{3.14}$$

where $A = A(z) > 0, \quad B = B(z) > 0$. Rewrite (3.14) in the following form:

$$\begin{cases} \lambda A(1+z_1^-)(z_2^- - z_2^+) + (\lambda A z_2^+ - 1)(z_1^- - z_1^+) = 0, \\ (\lambda B - 1)(z_2^- - z_2^+) + \lambda B(z_1^- - z_1^+) = 0, \end{cases} \tag{3.15}$$

If $z_1^- = z_1^+$ then from first equation of (3.15) we get $z_2^- = z_2^+$. If $z_2^- = z_2^+$ then from second equation of (3.15) we have $z_1^- = z_1^+$. ■

Corollary 1. *If system (3.13) has a unique solution, then system (3.1) also has a unique solution. Moreover this solution is* $z_x = (z_1^*, z_2^*), \quad x \in V$ *where* (z_1^*, z_2^*) *is the unique solution of (3.2).*

Now consider system of two last equations of (3.13):

$$\begin{cases} z_2^- = \lambda\left(\frac{1}{1+z_1^+ + z_2^+}\right)^k, \\ z_2^+ = \lambda\left(\frac{1}{1+z_1^- + z_2^-}\right)^k. \end{cases} \tag{3.16}$$

We are interested in solution $z_2^{\pm} = z_2^{\pm}(\lambda, z_1^-, z_1^+)$ of (3.16).

Denote $f_{\pm}(x) = \lambda(1 + z_1^{\pm} + x)^{-k}, \quad x \geq 0.$

From (3.16) we get

$$z_2^- = f_+(f_-(z_2^-)). \tag{3.17}$$

Note that $f_-(x - z_1^- + z_1^+) = f_+(x)$. Therefore, if we let $z_2^- = x - z_1^- + z_1^+$ then (3.17) implies

$$x - z_1^- + z_1^+ = f_+(f_+(x)) = F(x), \tag{3.18}$$

where $F(x) = \lambda(1 + z_1^+ + \lambda(1 + z_1^+ + x)^{-k})^{-k}$.

We have

$$F'(x) = k^2\lambda^2(1 + z_1^+ + \lambda(1 + z_1^+ + x)^{-k})^{-k-1}(1 + z_1^+ + x)^{-k-1}.$$
$$F''(x) = k^2(k+1)\lambda^2(1 + z_1^+ + \lambda(1 + z_1^+ + x)^{-k})^{-k-2}(1 + z_1^+ + x)^{-2(k+1)} \times$$
$$[(k-1)\lambda - (1 + z_1^+)(1 + z_1^+ + x)^k].$$

If $k = 1$ then $F''(x) < 0$ and F is concave increasing. Hence for $k = 1$, (3.18) has only one solution. For $k \geq 2$, F is convex for $x < ((k-1)\lambda(1+z_1^+)^{-1})^{1/k} - z_1^+ - 1$ and concave for $x > ((k-1)\lambda(1+z_1^+)^{-1})^{1/k} - z_1^+ - 1$; thus there are at most three solutions. On the other hand, it is easy to see that (3.18) has more than one solution if and only if there is more than one solution to the equation $xF'(x) = F(x)$ which is equivalent to

$$(1 + z_1^+)u^{k+1} = (k^2 - 1)\lambda u - k^2\lambda(z_1^- + 1), \tag{3.19}$$

where $u = 1 + z_1^+ + x$.

As function $u \to u^{k+1}$ is concave increasing, we conclude that (3.19) has a unique positive solution, say u_*, if u_* satisfies (3.19) and $(1+z_1^+)u_*^k = (k-1)\lambda$. Then we obtain $u_* = k(k-1)^{-1}(1+z_1^+)$ and $\lambda = \lambda_{\rm cr}^{\rm HC}(1+z_1^+)(1+z_1^-)^k$, where $\lambda_{\rm cr}^{\rm HC} = \frac{1}{k-1}(\frac{k}{k-1})^k$ is critical value for two state hard-core model. Thus (3.19) has two solution for $\lambda > \lambda_c = \lambda_{\rm cr}^{\rm HC}(1+z_1^+)(1+z_1^-)^k$.

So we proved the following

Proposition 7. *1) If $k = 1$ or $k \geq 2$ and $\lambda \leq \lambda_c$ then a positive solution to (3.18) is unique. 2) If $k > 1$ and $\lambda > \lambda_c$ then there exist at least two solutions $x_i^* = x_i^*(z_1^-, z_1^+)$, $i = 1, 2$.*

Corollary 2. *For $\lambda > \lambda_c$ system (3.16) has at least two solutions $(z_{2,i}^- = x_i^* - z_1^- + z_1^+, z_{2,i}^+ = f_+(x_i^*))$, $i = 1, 2$, where x_i^* are the positive solutions to (3.18).*

Correspondingly, we make a

Conjecture 1. If there is a critical value $\lambda_{\rm cr}$ such that system (3.1) has more than one solution for $\lambda > \lambda_{\rm cr}$ then $\lambda_{\rm cr} \geq \lambda_{\rm cr}^{\rm HC}$.

4 Description of periodic splitting Gibbs measures

We now write (3.1) in the following form

$$h_{1,x} = \ln\lambda + \sum_{y\in S(x)} \ln \frac{1+\exp(h_{1,y})}{1+\exp(h_{1,y})+\exp(h_{2,y})}, \tag{4.1a}$$

$$h_{2,x} = \ln\lambda + \sum_{y\in S(x)} \ln \frac{1}{1+\exp(h_{1,y})+\exp(h_{2,y})}, \tag{4.1b}$$

where $h_{i,x} = \ln z_{i,x}$, $i = 1, 2$.

In this section we study periodic solutions of system (4.1).

Note that (see [7]) there exists a one-to-one correspondence between the set V of vertexes of the Cayley tree of order $k \geq 1$ and the group G_k of the free products of $k+1$ cyclic groups of the second order with generators $a_1, a_2, ..., a_{k+1}$.

Definition 3. Let H_0 be a subgroup of G_k. We say that a collection $h = \{h_x = (h_{1,x}, h_{2,x}) : x \in G_k\}$ is H_0*-periodic* if $h_{i,yx} = h_{i,x}$ for all $i = 1, 2$, $x \in G_k$ and $y \in H_0$.

Definition 4. A Gibbs measure is called H_0*-periodic* if it corresponds to an H_0-periodic collection h.

Observe that a translation -invariant Gibbs measure is G_k-periodic.

We give below a complete description of periodic Gibbs measures, i.e. characterize these measures with respect to any normal subgroup of finite index.

Let H_0 be a subgroup of index r in G_k, and let $G_k|_{H_0} = \{H_0, H_1, ..., H_{r-1}\}$ be the quotient group. Let $q_i(x) = |S_1(x) \cap H_i|$, $i = 0, 1, ..., r-1$; $N(x) = |\{j : q_j(x) \neq 0\}|$, where $S_1(x) = \{y \in G_k : \langle x, y\rangle\}$, $x \in G_k$ and $|\cdot|$ is the number of elements of the set. Denote $Q(x) = (q_0(x), q_1(x), ..., q_{r-1}(x))$.

We note (see [21]) that for every $x \in G_k$ there is a permutation π_x of the coordinates of the vector $Q(e)$ (where e is the identity of G_k) such that

$$\pi_x Q(e) = Q(x). \tag{4.2}$$

It follows from (4.2) that $N(x) = N(e)$ for all $x \in G_k$.

Each H_0- periodic collection is given by

$$\{h_x = h_i \quad \text{for} \quad x \in H_i, \quad i = 0, 1, ..., r-1\}.$$

By Proposition 2.1 and (4.1), the vector $h_n, \quad n = 0, 1, ..., r-1$, satisfies the system

$$h_n = (\ln\lambda, \ln\lambda) + \sum_{j=1}^{N(e)} q_{i_j}(e) F(h_{\pi_n(i_j)}) - F(h_{\pi_n(i_{j_0})}), \tag{4.3}$$

where $j_0 = 1, ..., N(e), \quad N(e) = |\{i_1, ..., i_{N(e)}\}|$, and function $h = (h_1, h_2) \mapsto F(h) = (F_1(h), F_2(h))$ where

$$F_1(h) = \ln\frac{1+\exp(h_1)}{1+\exp(h_1)+\exp(h_2)}, \tag{4.4a}$$

$$F_2(h) = \ln\frac{1}{1+\exp(h_1)+\exp(h_2)}. \tag{4.4b}$$

Proposition 8. *$F(h) = F(l)$ if and only if $h = l$.*

Proof. Straightforward. ■

Let G_k^* be the subgroup in G_k consisting of all words of even length. Clearly, G_k^* is a subgroup of index 2.

Theorem 2. *Let H_0 be a normal subgroup of finite index in G_k. Then each H_0- periodic Gibbs measure for three state hard-core model is either translation-invariant or G_k^*- periodic.*

Proof. We see from (4.3) that $F(h_{\pi_n(i_1)}) = F(h_{\pi_n(i_2)}) = ... = F(h_{\pi_n(i_{N(e)})})$. Hence from Proposition 4.3 we have $h_{\pi_n(i_1)} = h_{\pi_n(i_2)} = ... = h_{\pi_n(i_{N(e)})}$. Therefore,

$$h_x = h_y = h, \quad \text{if} \quad x, y \in S_1(z), \quad z \in G_k^*;$$
$$h_x = h_y = l, \quad \text{if} \quad x, y \in S_1(z), \quad z \in G_k \setminus G_k^*.$$

Thus the measures are translation-invariant (if $h = l$) or G_k^*- periodic (if $h \neq l$). ■

Let H_0 be a normal subgroup of finite index in G_k. What condition on H_0 guarantee that each H_0-periodic Gibbs measure is translation invariant ? We put $I(H_0) = H_0 \cap \{a_1, ..., a_{k+1}\}$, where $a_i, \quad i = 1, ..., k+1$ are generators of G_k.

Theorem 3. *If $I(H_0) \neq \emptyset$ then each H_0- periodic Gibbs measure is translation-invariant.*

Proof. Take $x \in H_0$. Observe that the inclusion $xa_i \in H_0$ holds if and only if $a_i \in H_0$. Since $I(H_0) \neq \emptyset$, there is an element $a_i \in H_0$. Therefore H_0 contains the subset $H_0 a_i = \{xa_i : x \in H_0\}$. By Theorem 2 we have $h_x = h$ and $h_{xa_i} = l$. Since x and xa_i belong to H_0, it follows that $h_x = h_{xa_i} = h = l$. Thus each H_0- periodic Gibbs measure is translation-invariant. ■

Remark 2. An analogies of Theorem 2 and 3 can be proved for a wide class of hard constraint models. The important point here is the property of function F (see (4.4)) given by Proposition 8 Note that in many interesting cases of hard constraint models the corresponding function F has this property.

Theorems 2 and 3 reduce the problem of describing H_0- periodic Gibbs measure with $I(H_0) \neq \emptyset$ to describing the fixed points of the map $h = (h_1, h_2) \to (\ln \lambda, \ln \lambda) + kF(h)$, which describes translation -invariant Gibbs measures. If $I(H_0) = \emptyset$, this problem is reduced to describing the solutions of the system:

$$\begin{cases} h = (\ln \lambda, \ln \lambda) + kF(l), \\ l = (\ln \lambda, \ln \lambda) + kF(h). \end{cases} \tag{4.5}$$

Recall $z_i = \exp(h_i), \quad t_i = \exp(l_i), \quad i = 1, 2$. Then from (4.5) we get

$$\begin{cases} z_1 = \lambda \left(\frac{1+t_1}{1+t_1+t_2} \right)^k, \\ z_2 = \lambda \left(\frac{1}{1+t_1+t_2} \right)^k, \\ t_1 = \lambda \left(\frac{1+z_1}{1+z_1+z_2} \right)^k, \\ t_2 = \lambda \left(\frac{1}{1+z_1+z_2} \right)^k. \end{cases} \tag{4.6}$$

The analysis of solutions to system (4.6) is rather tricky. Let $z^* = z^*(\lambda) = (z_1^*, z_2^*) = (z_1^*(\lambda), z_2^*(\lambda))$ be the unique solution to (3.2) (see Proposition 4). The instability condition for z^* is

$$k^2 \frac{z_1^* z_2^*}{(1 + z_1^*)(1 + z_1^* + z_2^*)} > 1. \tag{4.7}$$

The left-hand side is the product of the eigen-values of the Jacobian $\begin{pmatrix} \frac{\partial F_1}{\partial h_1} & \frac{\partial F_1}{\partial h_2} \\ \frac{\partial F_2}{\partial h_1} & \frac{\partial F_2}{\partial h_2} \end{pmatrix}$ at $z = z^*$:

$$\Lambda_{1,2} = \Lambda_{1,2}(\lambda) = \frac{-k}{2(1 + z_1^*)(1 + z_1^* + z_2^*)} \left(z_2^* \pm \sqrt{(z_2^*(2z_1^* + 1))^2 + 4z_1^* z_2^* (1 + z_1^*)^2} \right).$$

Observe that, (4.7) is a necessary condition for existence of more than one solution to (4.6).

Now we shall reduce the system (4.6) to equation $\gamma(\gamma(x)) = x$ for some function γ and will apply the following lemma.

Lemma 1. *(See [10], p.70) Let $f : [0,1] \to [0,1]$ be a continuous function with a fixed point $\xi \in (0,1)$. Assume that f is differentiable at ξ and that $f'(\xi) < -1$. Then there exist x_0, x_1, $0 \le x_0 < \xi < x_1 \le 1$, such that $f(x_0) = x_1$ and $f(x_1) = x_0$.*

From (4.6) we have

$$z_2 = z_1(1+t_1)^{-k}, \qquad t_2 = t_1(1+z_1)^{-k},$$

and

$$z_1 = \lambda\left(\frac{1+t_1}{1+t_1+t_1(1+z_1)^{-k}}\right)^k, \tag{4.8a}$$

$$t_1 = \lambda\left(\frac{1+z_1}{1+z_1+z_1(1+t_1)^{-k}}\right)^k. \tag{4.8b}$$

Denote

$$\gamma(x) = \gamma(x,\lambda,k) = \frac{(\lambda^{1/k} - x^{1/k})(1+x)^k}{x^{1/k} - (\lambda^{1/k} - x^{1/k})(1+x)^k}, \quad 0 < x < \lambda.$$

Then (4.8) has the form

$$x = \gamma(y), \qquad y = \gamma(x). \tag{4.9}$$

Hence now we need to solve the following

$$\gamma(\gamma(x)) = x. \tag{4.10}$$

It is easy to see the following properties of γ
1) There is a unique $a \in (0,\lambda)$ such that $\gamma(a \pm 0) = \pm\infty$ and $\gamma(x) > 0$ only for $x \in (a,\lambda)$.
2) γ is decreasing on (a,λ).
3) There are $a_1, a_2 \in (a,\lambda)$ such that $\gamma(a_1) = a$, $\gamma(a_2) = a_1$.
4) γ has a unique fixed point $x_* \in (a_2, a_1)$.
5) There is a $a_3 \in (a_2, a_1)$ such that $\gamma(a_3) = \lambda$ and $\gamma(\gamma(x)) > 0$ iff $x \in (a_3, a_1)$.
Using Lemma 1 for $\gamma(x)$, $x \in [a_3, a_1]$ one can prove the following

Theorem 4. *For*

$$\lambda \in \{\lambda : \ \gamma'(x_*) < -1\} = \{\lambda : \ (1+x_*)^{k+2} > (k-1)x_*((k+1)x_*+1)\} \tag{4.11}$$

there are three G_k^*- periodic measures μ_0, μ_*, μ_1. Which corresponds to three solutions (x_0,x_1), (x_*,x_*), (x_1,x_0) of (4.9).

5 Extremity of the translation-invariant splitting Gibbs measure

Results presented in this section are based on methods developed in [14, 15, 16, 13].

Consider a two-parameter family of Markov chains with states 0, 1, 2 and transition matrix

$$\mathbf{P} = \begin{pmatrix} \frac{1}{1+z_1^*+z_2^*} & \frac{z_1^*}{1+z_1^*+z_2^*} & \frac{z_2^*}{1+z_1^*+z_2^*} \\ \frac{1}{1+z_1^*} & \frac{z_1^*}{1+z_1^*} & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

where as before $z^* = (z_1^*, z_2^*)$ is the unique solution for (3.2).

The eigen-values for $\mathbf{P}$ are $\Lambda_1 = 1$ and

$$\Lambda_{2,3} = \Lambda_{2,3}(\lambda) = \frac{-z_2^*}{2(1+z_1^*)(1+z_1^*+z_2^*)}\left(1 \pm \sqrt{1 + 4z_1^*(z_2^*)^{-1}(1+z_1^*)(1+z_1^*+z_2^*)}\right).$$

Theorem 5. *For $k \geq 2$ and*

$$\Lambda_2^2 > \frac{1}{k} \tag{5.1}$$

the translation-invariant splitting Gibbs measure μ^* is not extreme.

Proof. We apply the 'second eigen-value' calculation from [14]. A sufficient condition for non-extremity of μ^* is that $k(\Lambda_2)^2 > 1$. This immediately leads to bound (5.1). ■

The question when precisely measure μ^* becomes non-extreme (and what its decomposition is) is of a great interest. Also, the decomposition of μ^* into extreme measures looks pretty 'weird' (it cannot be a half-sum of measures $\mu_\pm$ as this would destroy the splitting character of μ^*).

We make a

Conjecture 2. For $\forall k$ and

$$z_1^* + z_2^* < \frac{1}{2k-1}, \tag{5.2}$$

the translation-invariant splitting Gibbs measure μ^* is extreme.

Remark 3. In particular, if $\lambda < \frac{1}{2(2k-1)}$ then the condition (5.2) is satisfied.

We conclude this section with an observation due to E. Mossel. This statement shows that the definition of the 'second' critical point $\lambda'_{\rm cr}$ as inf $[\lambda :$ measure μ^* is non-extreme $]$ is correct.

Proposition 9. *If, for given k and λ^0, measure μ^* is non-extreme then it remains non-extreme for the same k and all $\lambda > \lambda^0$.*

Proof. Again one uses reconstruction techniques from [14, 15, 16]. Given k and λ^0, if there exists an algorithm reconstructing value $\sigma(x^0)$ from $\sigma\big|_{W_n}$ then it can be modified for any $\lambda \geq \lambda^0$ so that the reconstruction remains possible. Details can be found e.g., in [14]. ■

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