International Journal of Computational Intelligence Systems
Vol. **13(1)**, 2020, *pp.* 1265–1280
DOI: https://doi.org/10.2991/ijcis.d.200730.002; ISSN: 1875-6891; eISSN: 1875-6883
https://www.atlantis-press.com/journals/ijcis/

Research Article

Cubic Graphs and Their Application to a Traffic Flow Problem

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ARTICLE INFO

Article History

Received 10 Mar 2020
Accepted 14 Jul 2020

Keywords

Cubic set
Interval-valued fuzzy set
Cubic graph
(complete, strong) cubic graph
Cubic (bridge, cutvertex)
Traffic flows

ABSTRACT

A graph structure is a useful tool in solving the combinatorial problems in different areas of computer science and computational intelligence systems. In this paper, we introduce the concept of cubic graph, which is different from the notion of cubic graph in S. Rashid, N. Yaqoob, M. Akram, M. Gulistan, Cubic graphs with application, Int. J. Anal. Appl. 16 (2018), 733–750, and investigate some of their interesting properties. Then we define the notions of cubic path, cubic cycle, cubic diameter, strength of cubic graph, complete cubic graph, strong cubic graph and illustrate these notions by several examples. We prove that any cubic bridge is strong and we investigate equivalent condition for cubic cutvertex. Finally, we use the concept of cubic graphs in traffic flows to get the least time to reach the destination.

1. INTRODUCTION

Graph theoretical concepts are widely used to study and model various applications in different areas including computer science, computational intelligence, automata theory, operations research, economics, and transportation. However, in many cases, some aspects of a graph-theoretic problem may be vague or uncertain. Fuzzy graphs were introduced by Rosenfeld [1], ten years after Zadeh's landmark paper "Fuzzy Sets" (briefly *F*-set) [2]. Rosenfeld has obtained the fuzzy analogs of several basic graph-theoretic concepts like bridges, paths, cycles, trees and connectedness and established some of their properties. In 1975, Zadeh [3] introduced the notion of interval-valued fuzzy sets (briefly *IVF*-set) as an extension of fuzzy sets in which the values of the membership degrees are intervals of numbers instead of the numbers. Interval-valued fuzzy sets provide a more adequate description of uncertainty than traditional fuzzy sets. It is therefore important to use interval-valued fuzzy sets in applications, such as fuzzy control. Hongmei and Lianhua gave the definition of interval-valued graph in [4]. Akram *et al.* [5,6] introduced interval-valued fuzzy graph as an extension of fuzzy graph and established some of their properties. Moreover, some researchers applied fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets, vague sets and neutrosophic sets etc., on graphs [7–22]. In 2012, Jun *et al.* [23] combined the theory of the interval-valued fuzzy set with the fuzzy set, and introduced the notion of cubic set. Moreover, they defined and studied some basic operations and their properties. Later on, a number of research papers have been devoted to the study of cubic set theory on various aspects in several algebraic structures (see for e.g., [24–35]).

Now, in the real life, some phenomena can be modeled with fuzzy graph concepts and some with interval-valued graph concepts. But there are some more complex phenomena that cannot be expressed alone, and can be modeled with a combination of both, that is cubic graphs. For example, traffic flows.

For this reason, Rashid *et al.* [18] applied cubic sets on graphs and introduced the notion of cubic graphs. But we thinks that this definition is not correct, since in the special case it is not a fuzzy graph.

So, in this article we introduce the correct notion of cubic graph which is different from the cubic graph in [18]. We define the notions of *IVF*-path, *F*-path, cubic path, *IVF*-diameter, *F*-diameter, cubic diameter, strength of cubic graph, *IVF*-complete cubic graph, *F*-complete cubic graph, complete cubic graph, *IVF*-strong cubic graph, *F*-strong cubic graph, strong cubic graph. We provide several examples to illustrate these notions, and investigate several properties. We give (P,P)-order, (P,R)-order, (R,P)-order and (R,R)-order between cubic graphs, and define the concepts of (P,P)-subcubic graph, (P,R)-subcubic graph, (R,P)-subcubic graph and (R,R)-subcubic graph. We also introduce the concepts of cubic bridge and cubic cutvertex, and investigate related properties. Finally, we use the concept of cubic graphs in traffic flows to get the least time to reach the destination.

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2. PRELIMINARIES

A *fuzzy set* is a pair (Y, ρ) where Y is a set and $\rho : Y \to \iota$ is a membership function (where $\iota = [0, 1]$). Denote by ι^Y the collection of all fuzzy sets on a set Y. For a family $\{\rho_i \mid i \in \Omega\} \subseteq \iota^Y$, we define the join ($\vee$) and meet ($\wedge$) operations as follows:

$$\left(\bigvee_{i\in\Omega} \rho_i\right)(y) = \sup\{\rho_i(y) \mid i \in \Omega\}$$

$$\left(\bigwedge_{i\in\Omega} \rho_i\right)(y) = \inf\{\rho_i(y) \mid i \in \Omega\},$$

respectively, for all $y \in Y$.(see [23])

By an *interval number* we mean a closed subinterval $\tilde{\tau} = \left[\tau^-, \tau^+\right]$ of ι, where $0 \le \tau^- \le \tau^+ \le 1$. The interval number $\tilde{\tau} = \left[\tau^-, \tau^+\right]$ with $\tau^- = \tau^+$ is written by τ. Denote by $[\iota]$ the set of all interval numbers. Also, we define *refined minimum* (briefly, rmin) of two elements in $[\iota]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $[\iota]$. Consider two interval numbers $\tilde{\tau}_1 := \left[\tau_1^-, \tau_1^+\right]$ and $\tilde{\tau}_2 := \left[\tau_2^-, \tau_2^+\right]$. Then

$$\text{rmin}\{\tilde{\tau}_1, \tilde{\tau}_2\} = \left[\min\{\tau_1^-, \tau_2^-\}, \min\{\tau_1^+, \tau_2^+\}\right]$$

$$\tilde{\tau}_1 \succeq \tilde{\tau}_2 \;\Leftrightarrow\; \tau_1^- \ge \tau_2^-,\ \tau_1^+ \ge \tau_2^+$$

and similarly, $\tilde{\tau}_1 \preceq \tilde{\tau}_2$ and $\tilde{\tau}_1 = \tilde{\tau}_2$. To say $\tilde{\tau}_1 \succ \tilde{\tau}_2$ (resp. $\tilde{\tau}_1 \prec \tilde{\tau}_2$) we mean $\tilde{a}_1 \succeq \tilde{\tau}_2$ and $\tilde{\tau}_1 \ne \tilde{\tau}_2$ (resp. $\tilde{\tau}_1 \preceq \tilde{\tau}_2$ and $\tilde{\tau}_1 \ne \tilde{\tau}_2$). Let $\tilde{\tau}_i \in [\iota]$ where $i \in \Omega$. We define

$$\underset{i\in\Omega}{\text{rinf}}\, \tilde{\tau}_i = \left[\inf_{i\in\Omega} \tau_i^-, \inf_{i\in\Omega} \tau_i^+\right]$$

$$\underset{i\in\Omega}{\text{rsup}}\, \tilde{\tau}_i = \left[\sup_{i\in\Omega} \tau_i^-, \sup_{i\in\Omega} \tau_i^+\right].$$

For any $\tilde{\tau} \in [\iota]$, its *complement*, denoted by $\tilde{\tau}^c$, is defined be the interval number $\tilde{\tau}^c = [1 - \tau^+, 1 - \tau^-]$.

An *interval-valued fuzzy set* (briefly, an *IVF set*) on nonempty set Y is defined by a function $\Gamma : Y \to [\iota]$. Denote by $[\iota]^Y$ stand for the set of all IVF sets in Y. For every $\Gamma \in [\iota]^Y$ and $y \in Y$, $\Gamma(y) = \left[\Gamma^-(y), \Gamma^+(y)\right]$ is called the *degree* of membership of an element y to Γ, where $\Gamma^+ \in I^Y$ which are called a *lower fuzzy set* and an *upper fuzzy set* in Y, respectively. For simplicity, we denote $\Gamma = [\Gamma^-, \Gamma^+]$. For every $\Gamma, \Delta \in [\iota]^Y$ and $y \in Y$, we define

$$\Gamma \subseteq \Delta \;\Leftrightarrow\; \Gamma(y) \preceq \Delta(y), \quad \Gamma = \Delta \;\Leftrightarrow\; \Gamma(y) = \Delta(y)$$

The complement Γ^c of $\Gamma \in [\iota]^Y$ is defined as follows: $\Gamma^c(y) = \Gamma(y)^c$ for all $y \in Y$, i.e.,

$$\Gamma^c(y) = [1 - \Gamma^+(y), 1 - \Gamma^-(y)] \text{ for all } y \in Y.$$

For a family $\{\Gamma_i \mid i \in \Omega\}$ of IVF sets in Y where Ω is an index set, the *union* $\Phi = \bigcup_{i\in\Omega} \Gamma_i$ and the *intersection* $\Psi = \bigcap_{i\in\Omega} \Gamma_i$ are defined as follows:

$$\Phi(y) = \left(\bigcup_{i\in\Omega} \Gamma_i\right)(y) = \underset{i\in\Omega}{\text{rsup}}\, \Gamma_i(y)$$

$$\Psi(y) = \left(\bigcap_{i\in\Omega} \Gamma_i\right)(y) = \underset{i\in\Omega}{\text{rinf}}\, \Gamma_i(y)$$

for all $y \in Y$, respectively. (see [23])

Let $Y \ne \emptyset$ be a set. By a *cubic set* in Y we mean a structure

$$\mathcal{A} = \{\langle y, \Gamma(y), \rho(y)\rangle \mid y \in Y\}$$

where, Γ and ρ are an IVF set and a fuzzy set on Y, respectively. A cubic set $\mathcal{A} = \{\langle y, \Gamma(y), \rho(y)\rangle \mid y \in Y\}$ is simply denoted by $\mathcal{A} = \langle \Gamma, \rho\rangle$. Denote by C^Y the collection of all cubic sets in Y. A cubic set $\mathcal{A}$ in which $\Gamma(y) = \mathbf{0}$ and $\rho(y) = 1$ (resp. $A(y) = \mathbf{1}$ and $\rho(y) = 0$) for all $y \in Y$ is denoted by $\ddot{0}$ (resp. $\ddot{1}$). A cubic set $\mathcal{B}$ in which $\Delta(y) = \mathbf{0}$ and $\varrho(y) = 0$ (resp. $\Delta(y) = \mathbf{1}$ and $\varrho(y) = 1$) for all $y \in Y$ is denoted by $\hat{0}$ (resp. $\hat{1}$). (see [23])

Definition 2.1. [23] Let $\mathcal{A} = \langle \Gamma, \rho\rangle$ and $\mathcal{B} = \langle \Delta, \varrho\rangle$ be cubic sets in Y. Then we define

i. $\mathcal{A} = \mathcal{B} \Leftrightarrow \Gamma = \Delta$ and $\rho = \varrho$.
ii. $\mathcal{A} \subseteq_P \mathcal{B} \Leftrightarrow \Gamma \subseteq \Delta$ and $\rho \le \varrho$.
iii. $\mathcal{A} \subseteq_R \mathcal{B} \Leftrightarrow \Gamma \subseteq \Delta$ and $\rho \ge \varrho$.
iv. $\mathcal{A} \ll \mathcal{B} \Leftrightarrow \Gamma \prec \Delta, \rho < \varrho$
v. $\mathcal{A} \gg \mathcal{B} \Leftrightarrow \Gamma \succ \Delta, \rho > \varrho$

where, (i) is called equality, (j) and (k) are called P-order and R-order, respectively.

Let V be a nonempty set. Define the relation $\sim$ on $V \times V$ for all $(u, v), (z, w) \in V \times V$, by$(u, v) \sim (z, w)$ if and only if $u = z$ and $v = w$ or $u = w$ and $v = z$. Then it is easily shown that $\sim$ is an equivalence relation on $V \times V$. For all $u, v \in V$, let $[(u, v)]$ denote the equivalence class of (u, v) with respect to $\sim$. Then $[(u, v)] = \{(u, v), (v, u)\}$. Let $\mathcal{E}_V = \{[(u, v)] | u, v \in V, u \ne v\}$. For simplicity, we often write $\mathcal{E}$ for $\mathcal{E}_V$ when V is a understood. Let $E \subseteq \mathcal{E}$. A graph is a pair (V, E). The elements of V are thought of as vertices of the graph and the elements of E as the edges. For $u, v \in V$, let uv denotes $[(u, v)]$. Then clearly $uv = vu$. We note that in this paper, graph (V, E) has no loops and parallel edges. (see [36])

Definition 2.2. [36] A *fuzzvgraph* $G := (V, \varphi, \psi)$ is a triple consisting of a nonempty set V together with a pair of functions $\varphi : V \to \iota$ and $\psi : \mathcal{E} \to \iota$ such that for all $u, v \in V$, $\psi(uv) \leqslant \varphi(u) \wedge \varphi(v)$

The fuzzy set φ is called the *fuzzy vertex set* of G and ψ the *fuzzy edge set* of G. Clearly ψ is a fuzzy relation on φ. We consider V as a finite set, unless otherwise specified.

Definition 2.3. [36] By an *interval-valued fuzzy graph* of a graph $G^* = (V, E)$, we mean a pair $G = (\Gamma, \Delta)$, where $\Gamma = [\psi_\Gamma^-, \psi_\Gamma^+]$ is an interval-valued fuzzy set on V and $\Delta = [\psi_\Delta^-, \psi_\Delta^+]$ is an interval-valued fuzzy set on $\mathcal{E}_V$.

3. CUBIC GRAPHS

In graph theory, there are some very basic concepts that whenever we just want to apply a new structure to graphs, these basic concepts must be introduced in that new structure. These basic concepts include vertex, edge, path, cycle, diameter, bridge, complete graph, sub-graph, and so on. Now in this section, all these important concepts for cubic graphs are stated and the related results are obtained.

Rashid *et al.* [18] applied cubic sets on graphs and introduced the notion of cubic graphs as follows: Let $M^* = (P, Q)$ be a graph. A cubic graph of graph $M^* = (P, Q)$ is the structure $\mathcal{G} := (\mathcal{A}, \mathcal{B})$ in which $\mathcal{A} = \langle \tilde{\mu}_A, \lambda_A \rangle$ is the cubic set representation for the vertex P and $\mathcal{B} = \langle \tilde{\mu}_B, \lambda_B \rangle$ denote a cubic set representation for the edge Q with

$$\tilde{\mu}_A : P \to D[0,1], \ \ \lambda_A : P \to [0,1]$$

$$\tilde{\mu}_B : Q \to D[0,1], \ \ \lambda_B : Q \to [0,1]$$

such that

$$\begin{cases} \tilde{\mu}_B(uv) \preceq \text{rmin}\{\tilde{\mu}_A(u), \tilde{\mu}_A(v)\}, \\ \lambda_B(uv) \leq \max\{\lambda_A(u), \lambda_A(v)\}. \end{cases} \tag{1}$$

But this definition is not correct. Since if $\tilde{\mu}_A = 0$ and $\tilde{\mu}_B = 0$, then $G = (\lambda_A, \lambda_B)$ is not fuzzy graph on $M^* = (P, Q)$. Because, $\lambda_B(uv) \leq \max\{\lambda_A(u), \lambda_A(v)\}$. But in the definition of fuzzy graph, we should have $\lambda_B(uv) \leq \min\{\lambda_A(u), \lambda_A(v)\}$. Now, in the below definition we give the correct version of cubic graph as follows:

Definition 3.1. Let $V \neq \emptyset$ be a finite set. By a *cubic graph* over V, we mean a pair $\mathcal{G} := (\mathcal{A}, \mathcal{B})$ in which $\mathcal{A} = \langle \tilde{\mu}_A, \lambda_A \rangle$ is a cubic set in V and $\mathcal{B} = \langle \tilde{\mu}_B, \lambda_B \rangle$ is a cubic set in $\mathcal{E}_V$ such that for all $uv \in \mathcal{E}_V$.

$$\begin{cases} \tilde{\mu}_B(uv) \preceq \text{rmin}\{\tilde{\mu}_A(u), \tilde{\mu}_A(v)\}, \\ \lambda_B(uv) \leq \min\{\lambda_A(u), \lambda_A(v)\}. \end{cases} \tag{2}$$

Let

$$\begin{cases} \tilde{\mu}_A^* = \{u \in V | \tilde{\mu}_A(u) \succ [0,0]\}, \\ \lambda_A^* = \{u \in V | \lambda_A(u) > 0\}, \\ \tilde{\mu}_B^* = \{uv \in \mathcal{E}_V | \tilde{\mu}_B(u) \succ [0,0]\}, \\ \lambda_B^* = \{uv \in \mathcal{E}_V | \lambda_B(u) > 0\}. \end{cases} \tag{3}$$

and $V^* = \tilde{\mu}_A^* \cap \lambda_A^*$ and $E^* = \tilde{\mu}_B^* \cap \lambda_B^*$. Then $\mathcal{G}^* = (V^*, E^*)$ is a graph which is called the *underlying crisp graph* of $\mathcal{G}$.

It is clear that any cubic graph is fuzzy graph and interval-valued fuzzy graph.

Example 3.2. Given a set $V = \{a, b, c\}$, let $\mathcal{A} = \langle \tilde{\mu}_A, \lambda_A \rangle$ and $\mathcal{B} = \langle \tilde{\mu}_B, \lambda_B \rangle$ be two cubic sets in V and $\mathcal{E}_V$, respectively, which are given by Tables 1 and 2. Then Figure 1, is a cubic graph over V. The Underlying graph of G is described by Figure 2.

Now, in the following definitions we introduce the concepts of cubic path, cubic cycle, cubic diameter, strength of path, strength of connectedness and complete cubic graph which are necessary for this study and the next results.

Definition 3.3. Let $\mathcal{G}$ be a cubic graph over V. Then

1. An *IVF-path* in $\mathcal{G}$ is defined to be a sequence $P_{IVF} : u_0 u_1 u_2 \cdots u_m$ of distinct members of V such that $\tilde{\mu}_B(u_{a-1}u_a) \succ [0,0]$ for $a = 1, 2, \cdots, m$. We say that m is the *length* of the *IVF*-path P_{IVF}, and $P_{IVF} : u_0 u_1 u_2 \cdots u_m$ is an *IVF*-path between u_0 and u_m.
2. An *F-path* in $\mathcal{G}$ is defined to be a sequence $P_F : u_0 u_1 u_2 \cdots u_n$ of distinct members of V such that $\lambda_B(u_{b-1}u_b) > 0$ for $b = 1, 2, \cdots, n$. We say that n is the *length* of the *F*-path P_F, and $P_{IVF} : u_0 u_1 u_2 \cdots u_n$ is an *F*-path between u_0 and u_n.
3. By a *cubic path* in $\mathcal{G}$ we mean a sequence $P_C : u_0 u_1 u_2 \cdots u_q$ of distinct members of V such that $\tilde{\mu}_B(u_{c-1}u_c) \succ [0,0]$ and $\lambda_B(u_{i-1}u_i) > 0$ for $c = 1, 2, \cdots, q$. We say that q is the *length* of the cubic path P_C, and $P_{IVF} : u_0 u_1 u_2 \cdots u_q$ is a cubic path between u_0 and u_q.
4. By an *IVF-cycle* of length m in $\mathcal{G}$, we mean the *IVF*-path $P_{IVF} : u_0 u_1 u_2 \cdots u_m$ of length m in $\mathcal{G}$ such that $u_0 = u_m$ for $m \geq 3$.
5. By an *F-cycle* of length n in $\mathcal{G}$, we mean the *F*-path $P_F : u_0 u_1 u_2 \cdots u_n$ of length n in $\mathcal{G}$ such that $u_0 = u_n$ for $n \geq 3$.
6. By an *cubic cycle* of length q in $\mathcal{G}$, we mean the cubic path $P_C : u_0 u_1 u_2 \cdots u_q$ of length q in $\mathcal{G}$ such that $u_0 = u_q$ for $q \geq 3$.

Definition 3.4. Let $\mathcal{G}$ be a cubic graph over V and $u, v \in V$. Then

1. The *IVF-diameter* of vertex u and vertex v is written by $diam_{IVF}(u, v)$ and is defined by the length of the biggest *IVF*-path between vertex u and vertex v.
2. The *F-diameter* of vertex u and vertex v is written by $diam_F(u, v)$ and is defined by the length of the biggest *F*-path between vertex u and vertex v.
3. The *cubic diameter* of vertex u and vertex v is written by $diam_C(u, v)$ and is defined by the length of the biggest cubic path between vertex u and vertex v.

Definition 3.5. Let P_C be a cubic path of length q in a cubic graph $\mathcal{G} := (\mathcal{A}, \mathcal{B})$ over V.

1. The *strength* of P_C is defined by $s(P_C) = (s(P_{IVF}), s(P_F))$ where

$$s(P_{IVF}) = \bigcap_{i=1}^{q} \tilde{\mu}_B(u_{i-1}u_i)$$

and

$$s(P_F) = \bigwedge_{i=1}^{q} \lambda_B(u_{i-1}u_i).$$

2. Given an edge uv in P_C, we say that uv is the *weakest edge* in P_C if $\tilde{\mu}_B(uv) = s(P_{IVF})$ and $\lambda_B(uv) = s(P_F)$.

Table 1 Tabular representation of "$\mathcal{A} = \langle \tilde{\mu}_A, \lambda_A \rangle$."

V	$\tilde{\mu}_A$	λ_A
a	$[0.3, 0.6]$	0.6
b	$[0.2, 0.5]$	0.2
c	$[0.4, 0.8]$	0.4

Table 2 Tabular representation of "$\mathcal{B} = \langle \tilde{\mu}_B, \lambda_B \rangle$."

$V \times V$	$\tilde{\mu}_B$	λ_B
(a, b)	$[0.15, 0.45]$	0.20
(a, c)	$[0.30, 0.50]$	0.30
(b, c)	$[0.20, 0.40]$	0.00

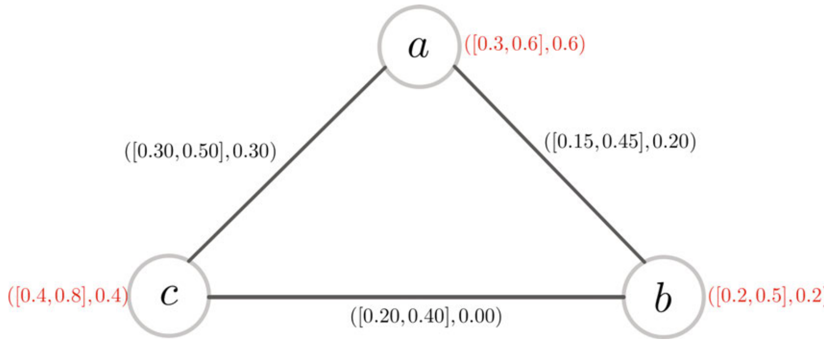

Figure 1 Cubic graph "$G = (A, B)$"

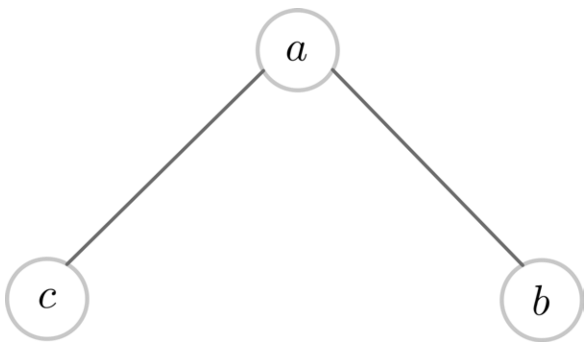

Figure 2 Underlying graph of "G" i.e "$G* = (V*, E*)$"

Table 3 Tabular representation of "$\mathcal{A} = \langle \tilde{\mu}_A, \lambda_A \rangle$."

V	$\tilde{\mu}_A$	λ_A
a	[0.3, 0.6]	0.6
b	[0.1, 0.4]	0.4
c	[0.4, 0.7]	0.7
d	[0.5, 0.9]	0.3

Definition 3.6. Let $\mathcal{G}$ be a cubic graph over V. For any $u, v \in V$, the *strength of connectedness* between vertex u and vertex v is denoted by $CONN_{\mathcal{G}}^{\infty}(u, v)$ and it is the maximum of the strengths of all cubic paths between u and v.

Denote by $CONN_{IVF}^{\infty}(u, v)$ and $CONN_{F}^{\infty}(u, v)$ the maximum of strengths of all IVF-paths and F-paths, respectively, between u and v.

Definition 3.7. A cubic graph $\mathcal{G}$ over V is said to be

- *IVF-complete* if $\forall u, v \in V^*$

$$\tilde{\mu}_B(uv) = \text{rmin}\{\tilde{\mu}_A(u), \tilde{\mu}_A(v)\}, \tag{4}$$

- *F-complete* if $\forall u, v \in V^*$

$$\lambda_B(uv) = \min\{\lambda_A(u), \lambda_A(v)\}, \tag{5}$$

- *Complete* if it is both IVF-complete and F-complete.

Example 3.8. Given a set $V = \{a, b, c\}$, let $\mathcal{A} = \langle \tilde{\mu}_A, \lambda_A \rangle$ and $\mathcal{B} = \langle \tilde{\mu}_B, \lambda_B \rangle$ be cubic sets in V and $\mathcal{E}_V$ which are given by Tables 3 and 4, respectively.

Table 4 Tabular representation of "$\mathcal{B} = \langle \tilde{\mu}_B, \lambda_B \rangle$."

$V \times V$	$\tilde{\mu}_B$	λ_B
(a, b)	[0.10, 0.40]	0.40
(a, c)	[0.30, 0.60]	0.60
(a, d)	[0.30, 0.60]	0.30
(b, c)	[0.10, 0.40]	0.40
(b, d)	[0.10, 0.40]	0.30
(c, d)	[0.40, 0.70]	0.30

Then $\mathcal{G}$ is a complete cubic graph over V, and it is described by Figure 3.

Example 3.9. Let $V = \{a, b, c\}$ and let $\mathcal{G} := (\mathcal{A}, \mathcal{B})$ be a cubic graph over V which is given by Figure 4.

Then $\mathcal{G}$ is an IVF-complete cubic graph over V, but it is not an F-complete cubic graph over V since

$$\lambda_B(bc) = 0.31 \neq 0.4 = \min\{\lambda_A(b), \lambda_A(c)\}.$$

Let $\mathcal{G}$ be a cubic graph over V which is given by Figure 5.

Then $\mathcal{G} := (\mathcal{A}, \mathcal{B})$ is an F-complete cubic graph over V, but it is not an IVF-complete cubic graph over V since

$$\tilde{\mu}_B(ad) = [0.3, 0.4] \neq [0.3, 0.6] = \text{rmin}\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}$$

In the following theorem, we determine an upper bound for maximum of strengths in the complete cubic graphs.

Theorem 3.10. *Let $\mathcal{G}$ be cubic graph over V.*

i. *If $\mathcal{G}$ is IVF-complete, then*

$$(\forall uv \in E^*)(CONN_{IVF}^{\infty}(u, v) \preceq \tilde{\mu}_B(uv))$$

ii. *If $\mathcal{G}$ is F-complete, then*

$$(\forall uv \in E^*)(CONN_{F}^{\infty}(u, v) \leq \lambda_B(uv))$$

iii. *If $\mathcal{G}$ is complete, then*

$$(\forall uv \in E^*)(CONN_{\mathcal{G}}^{\infty}(u, v) \ll (\tilde{\mu}_B(uv), \lambda_B(uv)))$$

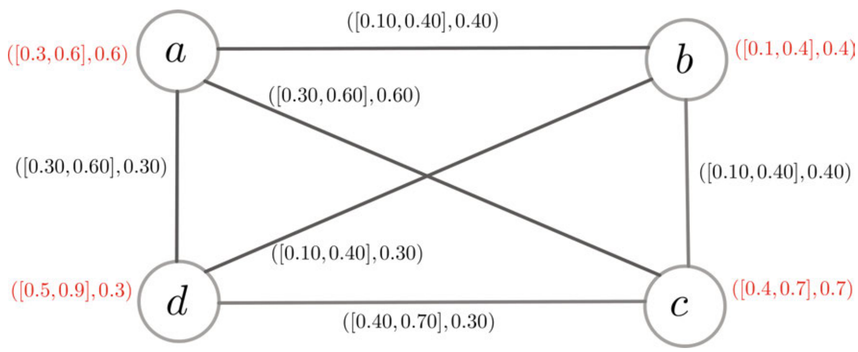

Figure 3 Complete cubic graph "$G = (A, B)$"

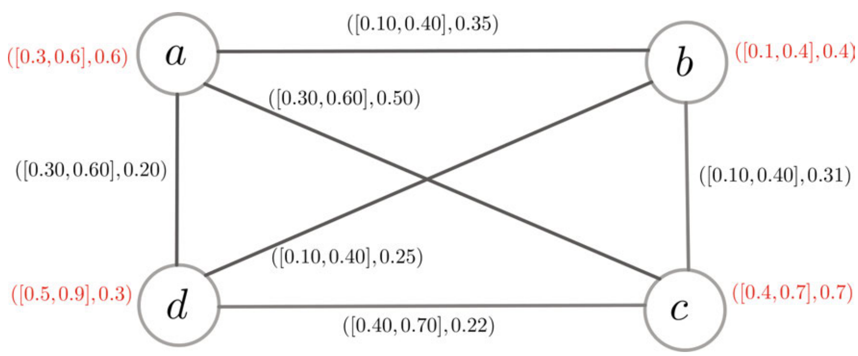

Figure 4 *IVF*-Complete cubic graph "$G = (A, B)$"

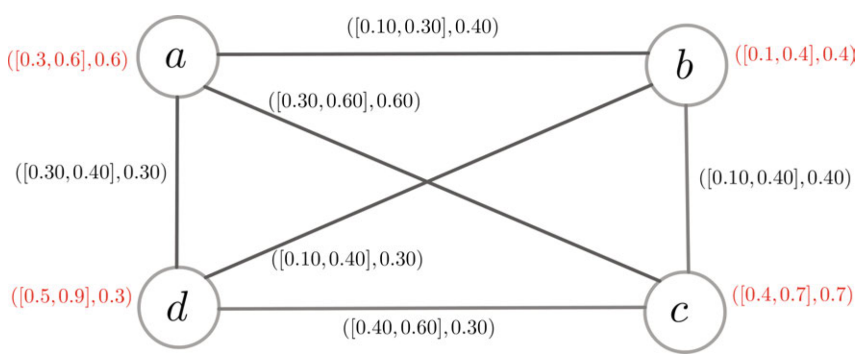

Figure 5 *F*-Complete cubic graph "$G = (A, B)$"

Proof. (i) and (ii). By using mathematical induction, we know that $CONN^k_{IVF}(u, v) \preceq \tilde{\mu}_B(uv)$ and $CONN^k_F(u, v) \leq \lambda_B(uv)$ for any positive integer k. It follows that

$$CONN^{\infty}_{IVF}(u, v) = \bigvee_{k \in |E^*|} CONN^k_{IVF}(u, v) \preceq \tilde{\mu}_B(uv)$$

and

$$CONN^{\infty}_F(u, v) = \bigvee_{k \in |E^*|} CONN^k_F(u, v) \leq \lambda_B(uv).$$

(iii) Proof is clear by (i) and (ii).

Definition 3.11. Let $\mathcal{G}$ be a cubic graph over V. Then an element $uv \in E^*$ is said to be

- *IVF-strong* edge in $\mathcal{G}$ if

$$\tilde{\mu}_B(uv) \succeq CONN^{\infty}_{IVF-uv}(u, v)$$

where $CONN^{\infty}_{IVF-uv}(u, v)$ is the maximum of strengths of all *IVF*-paths deleting uv.

- *F-strong* edge in $\mathcal{G}$ if

$$\lambda_B(uv) \geq CONN^{\infty}_{F-uv}(u, v)$$

where $CONN^{\infty}_{F-uv}(u, v)$ is the maximum of strengths of all *F*-paths deleting uv.

- *Strong* edge in $\mathcal{G}$ if it is both *IVF*- strong and *F*- strong.

Definition 3.12. Let $\mathcal{G}$ be a cubic graph over a nonempty finite set V. Then $\mathcal{G}$ is said to be

- *IVF-strong* (resp., *F*-strong) if every $uv \in E^*$ is *IVF*-strong (resp., *F*-strong).
- *Strong* if it is both *IVF*-strong and *F*-strong.

Example 3.13. Let $V = \{a, b, c, d\}$ be a set. Then the cubic graph $\mathcal{G} := (\mathcal{A}, \mathcal{B})$ over V which is given by Figure 6 is an IVF-strong cubic graph.

But it is not an F-strong cubic graph since

$$\lambda_B(ab) = 0.15 \not\geq 0.20 = CONN^{\infty}_{F-ab}(a, b)$$

The cubic graph $\mathcal{G} := (\mathcal{A}, \mathcal{B})$ over V which is given by Figure 7 is an F-strong cubic graph.

But it is not an IVF-strong cubic graph since

$$\tilde{\mu}_B(ab) = [0.10, 0.30] \not\succeq [0.10, 0.35] = CONN^{\infty}_{IVF-ab}(a, b)$$

The cubic graph $\mathcal{G} := (\mathcal{A}, \mathcal{B})$ over V which is given by Figure 8 is a strong cubic graph.

In the following theorems, we investigate some equivalent conditions for (IVF, F) strong edges in the cubic graphs.

Theorem 3.14. *Let $\mathcal{G}$ be cubic graph over V. For any $uv \in E^*$, we have*

$$uv \text{ is IVF-strong} \Leftrightarrow CONN^{\infty}_{IVF}(u, v) \preceq \tilde{\mu}_B(uv), \tag{6}$$

$$uv \text{ is F-strong} \Leftrightarrow CONN^{\infty}_{F}(u, v) \leq \lambda_B(uv), \tag{7}$$

$$uv \text{ is strong } \Leftrightarrow CONN^{\infty}_{\mathcal{G}}(u, y) \ll (\tilde{\mu}_B(uv), \lambda_B(uv)). \tag{8}$$

Proof. Assume that uv is IVF-strong. Then

$$CONN^{\infty}_{IVF-uv}(u, v) \preceq \tilde{\mu}_B(uv).$$

If an IVF-path between u and v contains uv, then it is clear that $CONN^{\infty}_{IVF}(u, v) \preceq \tilde{\mu}_B(uv)$. If an IVF-path between u and v does not contain uv, then its length is less or equal to $CONN^{\infty}_{IVF-uv}(u, v) \preceq \tilde{\mu}_B(uv)$. The sufficiency is clear. Hence (6) is valid.

Suppose that uv is F-strong edge, then $CONN^{\infty}_{F-uv}(u, v) \leq \lambda_B(uv)$. If an F-path between vertex u and vertex v contains edge uv, then it is clear that $CONN^{\infty}_{F}(u, v) \leq \lambda_B(uv)$. If an F-path between vertex u and vertex v does not contain edge uv, then its length is less or equal to $CONN^{\infty}_{-uv}(u, v) \leq \lambda_B(uv)$. The sufficiency is clear. Hence (7) is valid. The assertion (8) is induced by (6) and (7).

Theorem 3.15. *Let $\mathcal{G}$ be cubic graph over V and let $uv \in E^*$.*

1. *If $\tilde{\mu}_B(uv) = \text{rmin}\{\tilde{\mu}_A(u), \tilde{\mu}_A(v)\}$, then uv is IVF-strong.*
2. *If $\lambda_B(uv) = \min\{\lambda_A(u), \lambda_A(v)\}$, then uv is F-strong.*
3. *If $\tilde{\mu}_B(uv) = \text{rmin}\{\tilde{\mu}_A(u), \tilde{\mu}_A(v)\}$ and $\lambda_B(uv) = min\{\lambda_A(u), \lambda_A(v)\}$, then uv is strong.*

Proof. (1) Let P_{IVF} be an IVF-path between u and v in $IVF - uv$. If the length of P_{IVF} is equal 2, then P_{IVF} must contain uz and zv for some $z \in V^*$ with $z \neq u \neq v$. Hence

$$\begin{aligned} CONN^{2}_{IVF-uv}(u, v) &= \text{rmax}_{z\in V^*}\{\text{rmin}_{z\in V^*}\{\tilde{\mu}_B(uz), \tilde{\mu}_B(zv)\}\} \\ &= \text{rmax}_{z\in V^*}\{\text{rmin}_{z\in V^*}\{\{\text{rmin}\{\tilde{\mu}_A(u), \tilde{\mu}_A(z)\}, \\ &\quad \text{rmin}\{\tilde{\mu}_A(z), \tilde{\mu}_A(v)\} \\ &= \text{rmax}_{z\in V^*}\{\text{rmin}\{\tilde{\mu}_A(u), \tilde{\mu}_A(z), \tilde{\mu}_A(v)\}\} \\ &\quad \preceq \text{rmin}\{\tilde{\mu}_A(u), \tilde{\mu}_A(v)\} = \tilde{\mu}_B(uv). \end{aligned}$$

Similarly $CONN^{3}_{IVF-uv}(u, v) \preceq \tilde{\mu}_B(uv)$ and the same way induces

$$CONN^{k}_{IVF-uv}(u, v) \preceq \tilde{\mu}_B(uv)$$

for all positive integers k. Thus

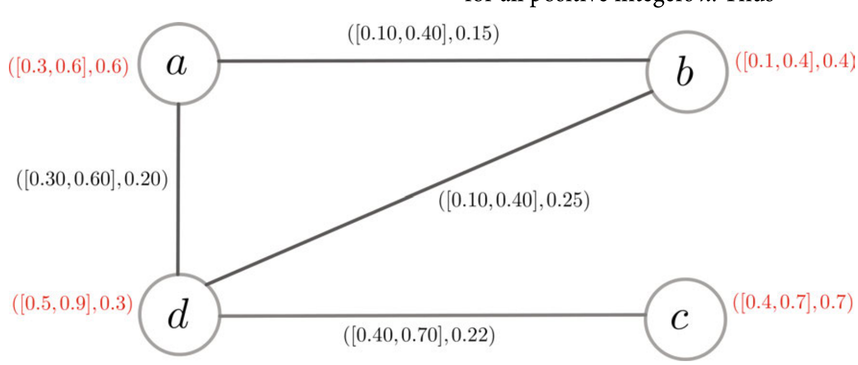

Figure 6 Cubic graph "$G = (A, B)$"

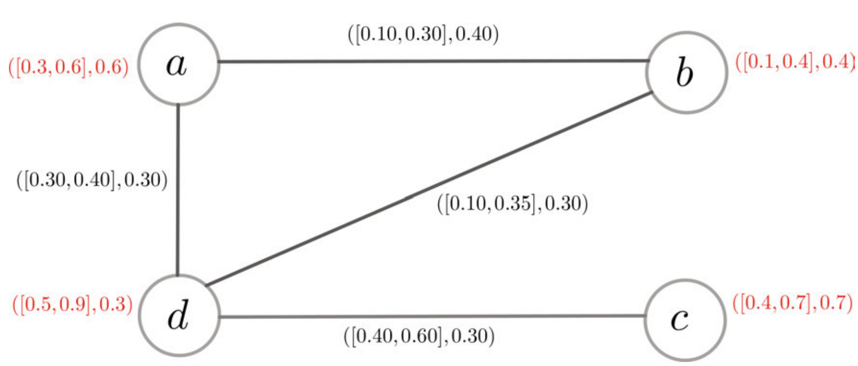

Figure 7 Cubic graph"$G = (A, B)$"

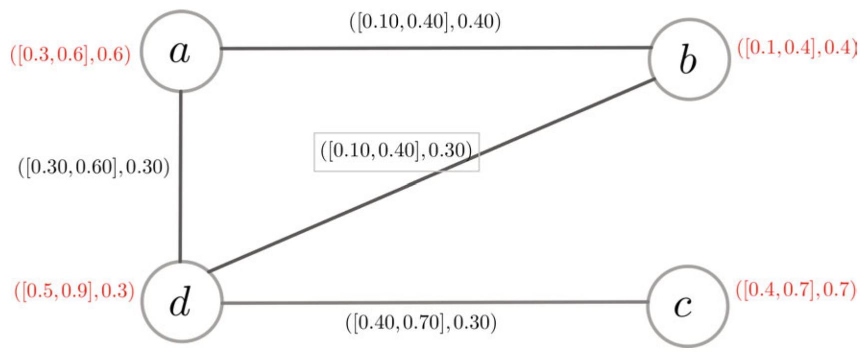

Figure 8 Cubic graph "$\mathcal{G} = (A, B)$"

$$CONN^{\infty}_{IVF-uv}(u,v) = \bigvee_{k\in|E^*|} CONN^{k}_{IVF-uv}(u,v) \leq \tilde{\mu}_B(uv),$$

and hence uv is IVF-strong by 3.6.

(2) Let P_F be an F-path between vertex u and vertex v in $F - uv$. Suppose the length of P_F is equal 2. Then P_F must contain uz and zv for some $z \in V^*$ with $z \neq u \neq v$. Hence

$$\begin{aligned} CONN^{2}_{F-uv}(u,v) &= \max_{z\in V^*}\{\min_{z\in V^*}\{\lambda_B(uz), \lambda_B(zv)\}\} \\ &= \max_{z\in V^*}\{\min_{z\in V^*}\{\{\min\{\lambda_A(u), \lambda_A(z)\}, \min\{\lambda_A(z), \lambda_A(v)\}\}\} \\ &= \max_{z\in V^*}\{\min\{\lambda_A(u), \lambda_A(z), \lambda_A(v)\}\} \\ &\leq \min\{\lambda_A(u), \lambda_A(v)\} = \lambda_B(uv) \end{aligned}$$

Similarly $CONN^{3}_{F-uv}(u,v) \leq \lambda_B(uv)$ and the same way induces

$$CONN^{k}_{F-uv}(u,v) \leq \lambda_B(uv)$$

for any positive integer numbers k. Thus

$$CONN^{\infty}_{F-uv}(u,v) = \bigvee_{k\in|E^*|} CONN^{k}_{F-uv}(u,v) \leq \lambda_B(uv).$$

It follows from 3.7 that uv is F-strong. (3) It is by (1) and (2).

Definition 3.16. Let $\mathcal{G} := (\mathcal{A}, \mathcal{B})$ and $\mathcal{H} := (\mathcal{C}, \mathcal{D})$ be two cubic graphs over V. We define (P,P)-order, (P,R)-order, (R,P)-order and (R,R)-order between $\mathcal{G} := (\mathcal{A}, \mathcal{B})$ and $H := (\mathcal{C}, \mathcal{D})$ as follows:

1. $\mathcal{G} \subseteq^{P}_{P} \mathcal{H}$ if $\mathcal{A}$ and $\mathcal{C}$ have P-order, i.e., $\mathcal{A} \subseteq_P \mathcal{C}$, and $\mathcal{B}$ and $\mathcal{D}$ have P-order, i.e., $\mathcal{B} \subseteq_P \mathcal{D}$.
2. $\mathcal{G} \subseteq^{R}_{P} \mathcal{H}$ if $\mathcal{A}$ and $\mathcal{C}$ have P-order, i.e., $\mathcal{A} \subseteq_P \mathcal{C}$, and $\mathcal{B}$ and $\mathcal{D}$ have R-order, i.e., $\mathcal{B} \subseteq_R \mathcal{D}$.
3. $\mathcal{G} \subseteq^{P}_{R} \mathcal{H}$ if $\mathcal{A}$ and $\mathcal{C}$ have R-order, i.e., $\mathcal{A} \subseteq_R \mathcal{C}$, and $\mathcal{B}$ and $\mathcal{D}$ have P-order, i.e., $\mathcal{B} \subseteq_P \mathcal{D}$.
4. $\mathcal{G} \subseteq^{R}_{R} \mathcal{H}$ if $\mathcal{A}$ and $\mathcal{C}$ have R-order, i.e., $\mathcal{A} \subseteq_R \mathcal{C}$, and $\mathcal{B}$ and $\mathcal{D}$ have R-order, i.e., $\mathcal{B} \subseteq_R \mathcal{D}$.

If $\mathcal{G} \subseteq^{P}_{P} \mathcal{H}$, we say that $\mathcal{G}$ is a *(P,P)-cubic subgraph* of $\mathcal{H}$, and $\mathcal{H}$ is a *(P,P)- cubic super graph* of $\mathcal{G}$.

If $\mathcal{G} \subseteq^{R}_{P} \mathcal{H}$, we say that $\mathcal{G}$ is a *(P,R)-cubic subgraph* of $\mathcal{H}$, and $\mathcal{H}$ is a *(P,R)-cubic super graph* of $\mathcal{G}$.

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If $\mathcal{G} \subseteq^{R}_{R} \mathcal{H}$, we say that $\mathcal{G}$ is a *(R,R)-cubic subgraph* of $\mathcal{H}$, and $\mathcal{H}$ is a *(R,R)-cubic super graph* of $\mathcal{G}$.

Definition 3.16 is illustrated in the Figures 9–12.

Definition 3.17. Given a cubic graph $\mathcal{G}$ over V, let $u, v \in V$ and let $\mathcal{G}'$ be a (ϵ, δ)-sub-cubic graph of $\mathcal{G}$ that is obtained by deleting $uv \in \mathcal{E}_V$, where $\epsilon, \delta \in \{P, R\}$, so $(\tilde{\mu}_A{}'(u), \lambda_A{}'(u)) = (\tilde{\mu}_A(u), \lambda_A(u))$ for all member of V, $(\tilde{\mu}_B{}'(uv), \lambda_B{}'(uv)) = ([0,0], 0)$ and $(\tilde{\mu}_B{}'(u), \lambda_B{}'(u)) = (\tilde{\mu}_B(u), \lambda_B(u))$ for all other pairs. We call uv a *cubic bridge* in $\mathcal{G}$ if

$$CONN^{\infty}_{\mathcal{G}\prime}(z,w) \ll CONN^{\infty}_{\mathcal{G}}(z,w)$$

for some $z, w \in V^*$.

Proposition 3.18. *Let $\mathcal{G}$ be a cubic graph over V. A $uv \in \mathcal{E}_V$ is a cubic bridge if and only if there exists vertices $z, w \in V^*$ such that every strongest path from vertex z to vetrex w contains edge uv.*

Proof. The proof is clearly.

Theorem 3.19. *Let $\mathcal{G} := (\mathcal{A}, \mathcal{B})$ be a cubic graph over V and $\mathcal{G}'$ be any sub cubic graph of $\mathcal{G}$ which that is obtained by deleting $uv \in \mathcal{E}_V$. If uv is not the weakest of any cycle, then we have*

$$CONN^{\infty}_{\mathcal{G}\prime}(u,v) \ll (\tilde{\mu}_B(uv), \lambda_B(uv)).$$

Proof. If $CONN^{\infty}_{\mathcal{G}\prime}(u,v) \ll 6\ (\tilde{\mu}_B(uv), \lambda_B(uv))$, then there is a path from u to v not involving uv and strength of this path is greater than $(\tilde{\mu}_B(uv), \lambda_B(uv))$. So this path together with uv forms a cycle of $\mathcal{G}$ in which uv is weakest in any cycle, so eventually $CONN^{\infty}_{\mathcal{G}\prime}(u,v) \ll (\tilde{\mu}_B(uv), \lambda_B(uv))$.

One of the important results in the fuzzy graph theory, is the relation between bridge, strong arc and cutvertex. Now, in the following theorems we will investigate this relations.

Proposition 3.20. *In a cubic graph $\mathcal{G} := (\mathcal{A}, \mathcal{B})$, every cubic bridge is strong.*

Proof. Let uv be a cubic bridge of $\mathcal{G}$. Suppose uv is not strong. Then

$$CONN^{\infty}_{\mathcal{G}-uv}(u,v) \gg (\tilde{\mu}_B(uv), \lambda_B(uv)).$$

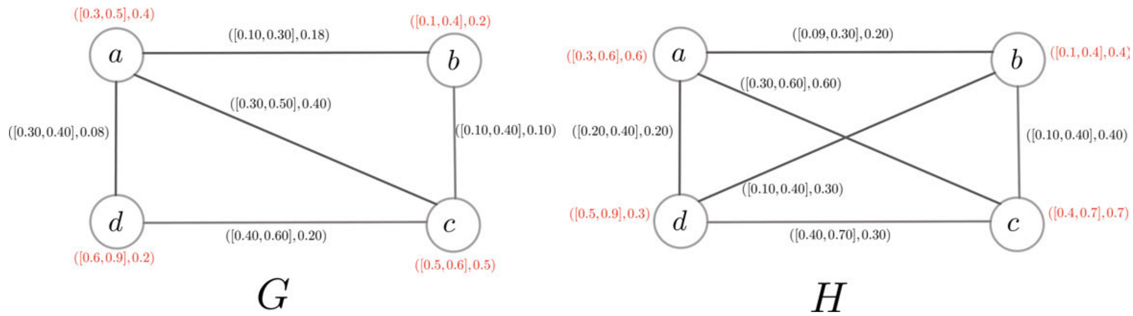


Figure 9 | “$G \subseteq_R^P H$”

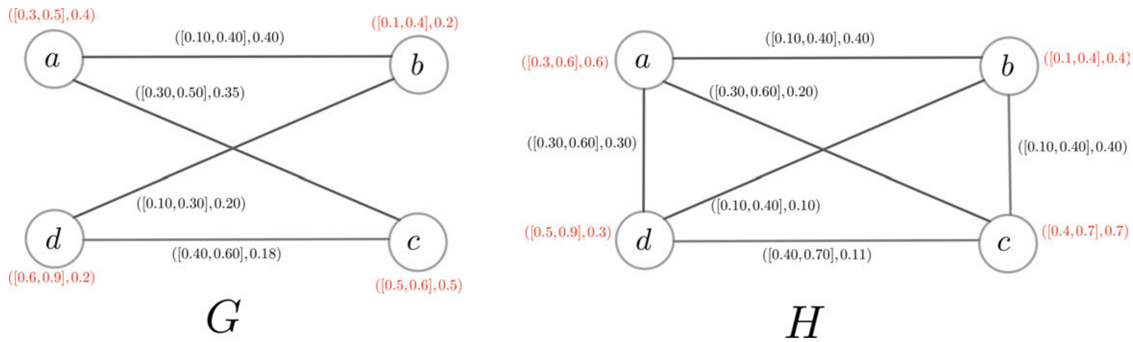


Figure 10 | “$G \subseteq_R^P H$”

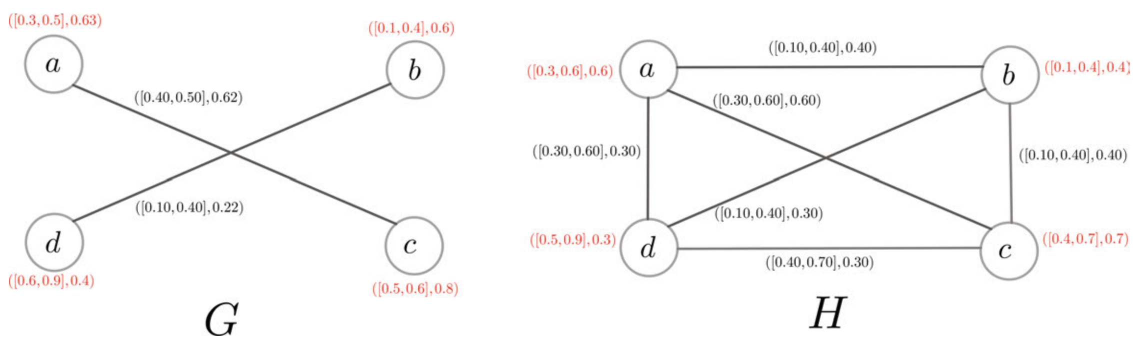


Figure 11 | “$G \subseteq_R^P H$”

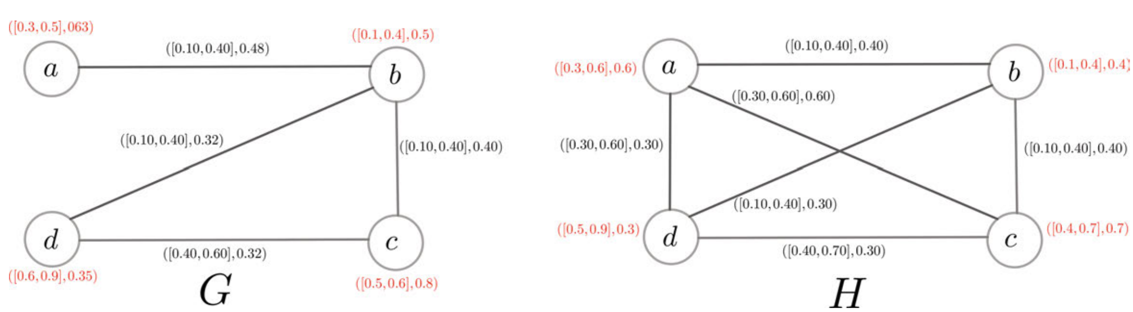


Figure 12 | “$G \subseteq_R^P H$”

Let P be the strongest path from u to v in $\mathcal{G} - uv$. The strength of this path is $CONN^{\infty}_{\mathcal{G}-uv}(u, v)$. If we adjoin uv to P, then we have a cycle, and uv is the weakest of this cycle. Hence, by Theorem 3.19, uv is not a cubic bridge of $\mathcal{G}$. So cubic graph must be strong.

The converse of Proposition 3.20 may not be true as seen in Figure 13.

Theorem 3.21. *Let $\mathcal{G} := (\mathcal{A}, \mathcal{B})$ be a cubic graph over V. If uv is a cubic bridge, then*

$$CONN^{\infty}_{\mathcal{G}}(u, v) \ll (\tilde{\mu}_B(uv), \lambda_B(uv))$$

Proof. By Theorem 3.19 and Proposition 3.20, it is clear.

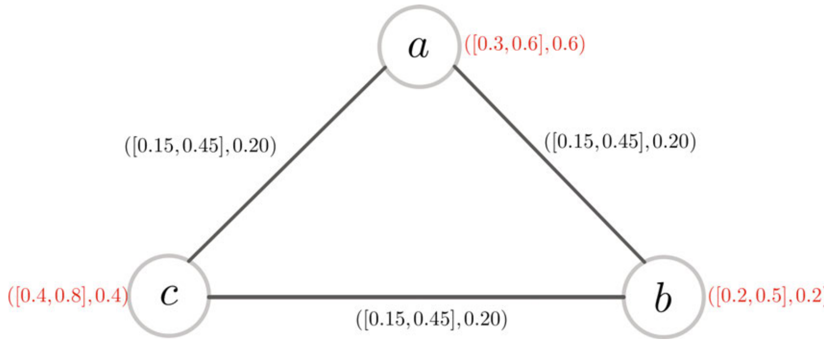


Figure 13 "A strong cycle without cubic bridge."

Definition 3.22. Given a cubic graph $\mathcal{G} := (\mathcal{A}, \mathcal{B})$ over V. Let $u \in V$ and $\mathcal{G}'$ be a (ϵ, δ)-subcubic graph of $\mathcal{G}$, where $\epsilon, \delta \in \{P, R\}$, which is obtained by deleting u. So $(\tilde{\mu}_A{}'(u), \lambda_A{}'(u)) = ([0,0], 0)$ and $(\tilde{\mu}_A{}'(u), \lambda_A{}'(u)) = (\tilde{\mu}_A(u), \lambda_A(u))$ for all others member of V, $(\tilde{\mu}_B{}'(uv), \lambda_B{}'(uv)) = ([0,0], 0)$ for all $v \in V$ and $(\tilde{\mu}_B{}'(u), \lambda_B{}'(u)) = (\tilde{\mu}_B(u), \lambda_B(u))$ for all other members of $\mathcal{E}_V$. Then we call u a *cubic cutvertex* in $\mathcal{G}$ if

$$CONN^{\infty}_{\mathcal{G}'}(z, w) \ll CONN^{\infty}_{\mathcal{G}}(z, w)$$

for some $z, w \in V^*$ with $z \neq u \neq w$.

Proposition 3.23. *Let $\mathcal{G} := (\mathcal{A}, \mathcal{B})$ be a cubic graph over V. A $u \in V$ is a cubic cutvertex in $\mathcal{G}$ if and only if there exists $z, w \in V^*$ distinct from u such that u is on every strongest path from z to w.*

Proof. Straightforward.

Theorem 3.24. *Let $\mathcal{G} := (\mathcal{A}, \mathcal{B})$ be a cubic graph over V such that (V^*, E^*) is a cycle. Then a member of V is a cubic cutvertex if only if it is a common vertex of two cubic bridges.*

Proof. Let u be a cubic cutvertex of $\mathcal{G}$. Then by Proposition 3.23 there exists z and w distinct u such that u is on every strongest path from z to w. On the other hand, since $\mathcal{G}^* = (V^*, E^*)$ is a cycle, there exists only one strongest path from z to w containing u and all its pairs are a cubic bridges. Thus u is a common vertex of two cubic bridges.

Conversely, let u be a common vertex of two cubic bridges zu and uw. Then both zu and uw are not weakest in a path from z to w. Also, this path not containing zu and uw has strength less than $(\mathrm{rmin}\{\tilde{\mu}_B(zu), \tilde{\mu}_B(uw)\}, \min\{\lambda_B(zu), \lambda_B(uw)\})$. Hence, the strongest path from z to w is the path z, u, w and

$$CONN^{\infty}_{\mathcal{G}}(z, w) \ll (\mathrm{rmin}\{\tilde{\mu}_B(zu), \tilde{\mu}_B(uw)\}, \min\{\lambda_B(zu), \lambda_B(uw)\}).$$

Thus u is a fuzzy cutvertex.

4. APPLICATION

Traffic flow is the theory, which is developed based on the flow of vehicle on the lane,along with its connections with other vehicles, pedestrians, signals, which is present on the road. Free movement of traffic is affected by many factors like design speed, percentage of heavy vehicles, number of lanes and intersections which are available along the road.

Traffic flow in a road is expressed as number of vehicles using the particular road per unit duration during one hour and is measured by utilizing traffic counts made for a particular duration at one point on the lane stretch. There is a variability in the counts made during different hours of time during a special day. Peak hour traffic is considered for making any analysis based on traffic flow. Measuring traffic flow is necessary for locating the point on the road during congestion, assessing the requirement of traffic signal at an intersection, estimating the capacity of the road way to meet with the present flow and the like. In addition, traffic flow relies on the speed of traffic, and the density of vehicles on the road. The present study aimed to obtain the most optimal route for going from one place to another place in a city by cubic graph. To this aim, Shahid Beheshti university located in Tehran, Iran, has two campuses, the distance of which is approximately 19 km. There are some highways for going from the campus 1 located in Velenjak region to campus 2 located in Ekbatan region as follows:

If we named any each intersection between two highways with summarize, Figure 14 obtained as follows:

Now, we consider each of the intersections as one vertex and each highways between two intersections as the edge of graph. There are many routes from campus 1 to 2 related to this university. However, the main question raised here is, related to the least time each route requires. In this regard, a lot of parameters may play a role are but the tolerance of traffic and distance between two intersection are considered as two main parameters. Further, the volume of traffic and distance between two intersections are fuzzy variable and interval-valued fuzzy variable, respectively.

V set of all intersections (vertices) and put $\mathcal{E}_V$ set of all distance between intersections if we want to make one cubic graph in Figure 14. Let $\lambda_A : V \to I$ and $\lambda_B : \mathcal{E}_V \to I$ be the membership function of volume of traffic for any vertex and any edge, respectively. It is wroth noting that more value of λ_B means low traffic and less value of λ_B means high traffic but there is no traffic for vertex because it is an intersection between two highways that these are not cut together. Thus, the membership value of traffic equals to 1 for all vertices. Further, suppose $\tilde{\mu}_A : V \to [\iota]$ and $\tilde{\mu}_B : \mathcal{E}_V \to [\iota]$ are membership function of distance for any vertex and any edge, respectively. In other words, $\tilde{\mu}_B(uv) = [0, a]$for any edge, where a is considered as a normalized distance between the two vertex

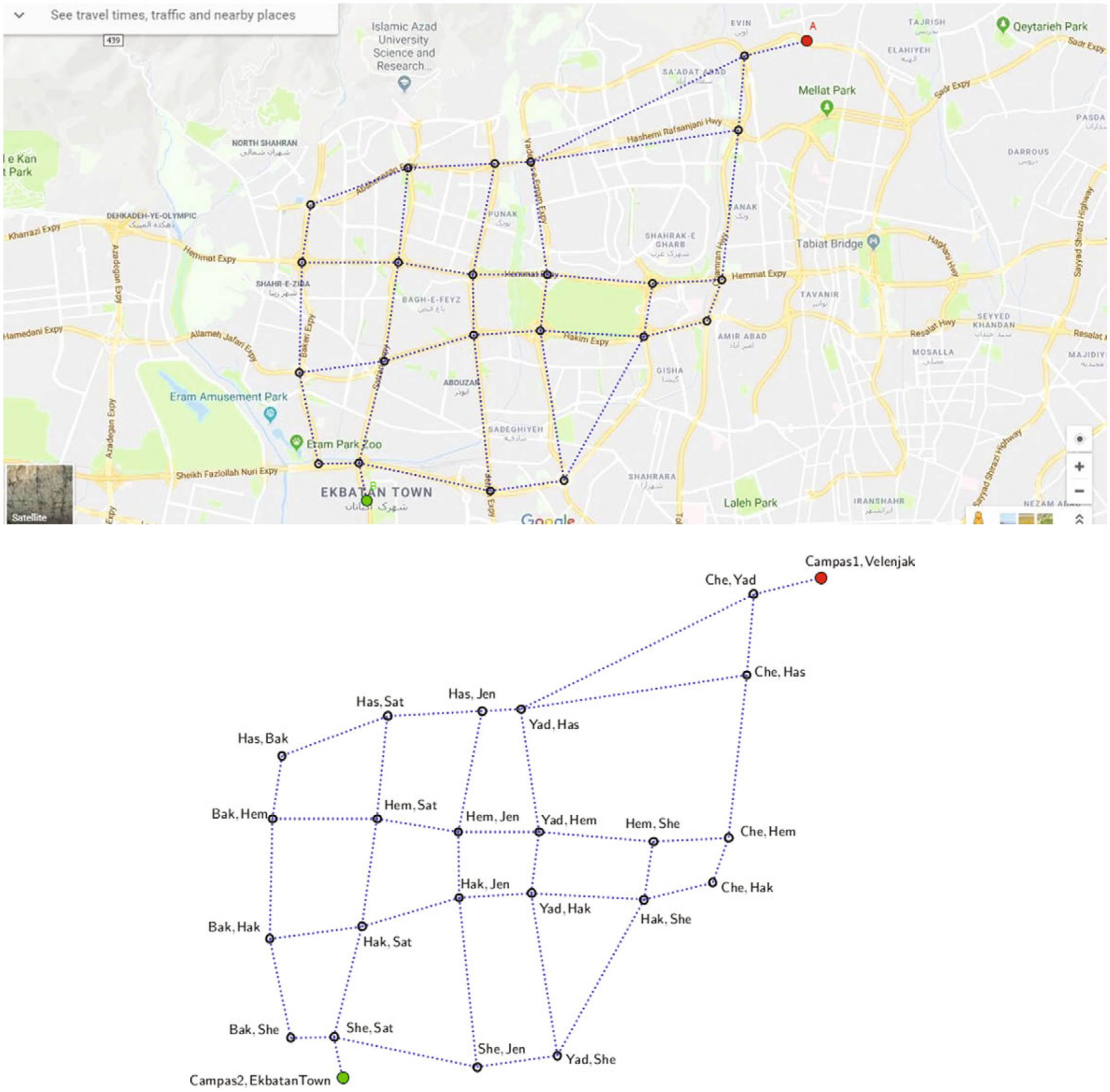


Figure 14 The graph of the Highway intersection.

u, v and $\tilde{\mu}_A(u) = [1, 1]$ is defined for any vertex. Since the maximum distance between vertices less than 10 km, the so normalized distance is as follows:

$$a := \frac{\triangle uv}{10}$$

where $\triangle uv$ represents the distance between two intersection in km.

Table 5 indicates the data related to the edges based on the reports from traffic and municipal organization in Tehran and daily personal experiences traversing from this route at 5 o'clock pm.

In order to obtain the best route requiring less time, an algorithm should be first used for finding all the paths from the campus 1 to 2, which is operated by Algorithm 1 (programming C++) and visible for all paths in Table 6. Thus, a speed equation, as a common equation in physics, indicates the relationship between time, speed and distance. Based on this equation, we have $u = vt + u_0$ or $\triangle u = vt$. Therefore, move depends on the volume of traffic in one highway. In other word, high traffic results in reducing speed while low traffic leads to an increase in speed. In one highway, if the maximum speed allowed is $80 km/h$, the speed of a car depends on the equation $v_{avg} = \lambda_B(uv) \times 80$. Hence

Algorithm 1: Finding all the paths from the campus 1 to 2

Data: Cubic graph from the campus 1 to 2
Result: Finding all the paths from the campus 1 to 2 initializations
1. Define utility function for printing the found path in cubic graph.
2. Define utility function to check if current vertex is already present in cubic path.
3. Define utility function for finding paths in cubic graph from source to destination(int g, int src,int dst, int v).
4. If last vertex is the desired destination then print the cubic path.
5. Traverse to all the nodes connected to current vertex and push new path to queue.
6. The main program.
6.1. The number of vertices with information on them.
6.2. The construct an edge between vertices with information on them.
6.3. Call all of defined utility functions and print the cubic path.
6.4. Return to the initial vertex.

Table 5 Tabular representation of distance and tolerance of traffic.

$uv \in \mathcal{E}_V$	$\tilde{\mu}_B(uv)$	$\lambda_B(uv)$	$uv \in \mathcal{E}_V$	$\tilde{\mu}_B(uv)$	$\lambda_B(uv)$
Velenjak → Cha,Yad	$[0, 0.13]$	0.3	Hem,Jen → Hem,sat	$[0, 0.17]$	0.25
Cha,Yad → Yad,Has	$[0, 0.61]$	0.6	Has,Jen → Has,Sat	$[0, 0.16]$	0.3
Cha,Yad → Cha,Has	$[0, 0.15]$	0.9	Has,Sat → Hem,Sat	$[0, 0.19]$	0.6
Cha,Has → Yad,Has	$[0, 0.43]$	0.3	Hem,Sat → Hak,Sat	$[0, 0.18]$	0.45
Cha,Has → Cha,Hem	$[0, 0.31]$	0.3	Hak,Sat → She,Sat	$[0, 0.17]$	0.55
Cha,Hem → Cha,Hak	$[0, 0.12]$	0.8	Has,Sat → Has,Bak	$[0, 0.21]$	0.45
Cha,Hak → Hak,She	$[0, 0.14]$	0.5	Has,Bak → Bak,Hem	$[0, 0.12]$	0.8
Cha,Hem → Hem,She	$[0, 0.15]$	0.5	Hem,Sat → Bak,Hem	$[0, 0.18]$	0.45
Hem,She → Hak,She	$[0, 0.1]$	0.7	Bak,Hem → Bak,Hak	$[0, 0.22]$	0.9
Hem,She → Yad,Hem	$[0, 0.24]$	0.35	Hak,Sat → Bak,Hak	$[0, 0.16]$	0.45
Hak,She → Yad,Hak	$[0, 0.22]$	0.35	Bak,Hak → Bak,She	$[0, 0.18]$	0.75
Yad,Hem → Yad,Hak	$[0, 0.15]$	0.65	Bak,She → She,Sat	$[0, 0.085]$	0.8
Yad,Has → Yad,Hem	$[0, 0.22]$	0.55	She,Sat → EkbatanTown	$[0, 0.06]$	0.75
Yad,Hak → Yad,She	$[0, 0.36]$	0.4	Hak,She → Yad,She	$[0, 0.35]$	0.4
Yad,She → She,Jen	$[0, 0.14]$	0.55	Yad,Hak → Hak,Jen	$[0, 0.12]$	0.35
Yad,Hem → Hem,Jen	$[0, 0.15]$	0.3	Yad,Has → Has,Jen	$[0, 0.08]$	0.35
Has,Jen → Hem,Jen	$[0, 0.21]$	0.3	Hem,Jen → Hak,Jen	$[0, 0.13]$	0.5
Hak,Jen → She,Jen	$[0, 0.32]$	0.25	She,Jen → She,sat	$[0, 0.27]$	0.5
Hak,Jen → Hak,Sat	$[0, 0.18]$	0.35			

$$\triangle u = v_{avg} t$$

Thus, the time of crossing every path is as follows:

$$t = \frac{\triangle uv}{v_{avg}}$$

Further, as shown in Table 6, we could find the time for any path by using an Algorithm 2 as follows:

Algorithm 2: Finding the time needs for the crossing of any cubic path

Data: All the cubic paths from the campus 1 to 2
Result: Finding the time needs for the crossing of any cubic path initializations
1. Define function v_{avg} for any edge of any cubic paths.
2. Define function t for any edge of any cubic path.
3. Define function $T := \sum t$ for any cubic path.
4. The main program.

1 Add all of the cubic paths with information on them.
2 Compare between T of any cubic paths and find the minimum time of them.
3 Print the cubic paths with time needs for the crossing in Table 6.

Finally, the optimum path spending the least time was evaluated for its crossing. As displayed in Table 6, the optimum path is 10^{th}, which means that $0 \rightarrow 1 \rightarrow 7 \rightarrow 14 \rightarrow 15 \rightarrow 16 \rightarrow 17 \rightarrow 18 \rightarrow 23$ is considered as the optimal path the crossing of which is 0.414707792 hours. As illustrated Figure 15, the presented cubic path could suggest the route what is available on Google Map.

5. CONCLUSION

Graph theoretical concepts are widely used to study and model various applications in different areas including automata theory, operations research, economics and transportation. However, in many cases, some aspects of a graph-theoretic problem may be vague or uncertain. Fuzzy graphs were introduced by Rosenfeld [1], ten years after Zadeh's landmark paper "Fuzzy Sets" (briefly F-set) [2]. Rosenfeld has obtained the fuzzy analogs of several basic graph-theoretic concepts like bridges, paths, cycles, trees and connectedness and established some of their properties. In 1975, Zadeh [3] introduced the notion of interval-valued fuzzy sets (briefly IVF-set) as an extension of fuzzy sets in which the values of the membership degrees are intervals of numbers instead of the numbers. Interval-valued fuzzy sets provide a more adequate description of uncertainty than traditional fuzzy sets. Chen introduced the interval-valued fuzzy hypergraph in his paper [37]. This concept is being applied from various angles to algebraic structure and applied science etc. Some researchers used graph theory, fuzzy graph theory and interval-valued fuzzy graph theory in traffic flow as follows:

As you see in the above report, more than 90 percent of researchers did use the graph theory and fuzzy graph theory in traffic flows. In fact, they studied a limited branch of traffic problems as follows:

- Optimization of traffic light's function
- Evaluation traffic by image processing
- Identification high-risk areas on urban roads and optimization of the urban road traffic

None researchers not used graph theory, fuzzy graph theory and, etc., for finding the best path from the origin to the destination. Therefore, we tried to find the best path from the origin to the destination by using graph theory, fuzzy graph theory and, etc. It is natural to deal with the vagueness and uncertainty using the methods of fuzzy graphs and interval-valued fuzzy graphs in some problems. Thus, Jun *et al.* introduced a cubic set that is combined by a fuzzy set and interval-valued fuzzy set. A cubic model is a generalized form of a fuzzy model and an interval-valued fuzzy model. In our goal, the cubic models provide more precision, flexibility and

Table 6 Tabular representation all paths obtained from C++ program.

i	Paths	Time(H)	i	Paths	Time(H)
1	0 1 7 8 13 16 17 18 23	0.527386364	2	0 1 7 8 13 12 11 18 23	0.534886364
3	0 1 7 8 13 12 17 18 23	0.439172078	4	0 1 7 8 9 10 11 18 23	0.481914336
5	0 1 7 8 9 12 11 18 23	0.540453297	6	0 1 7 8 9 12 17 18 23	0.415875375
7	0 1 7 14 13 16 17 18 23	0.505957792	8	0 1 7 14 13 12 11 18 23	0.567321429
9	0 1 7 14 13 12 17 18 23	0.442743506	10	0 1 7 14 15 16 17 18 23	0.414707792
11	0 1 2 3 4 5 10 11 18 23	0.476609848	12	0 1 2 3 6 5 10 11 18 23	0.478216991
13	0 1 2 7 8 13 16 17 18 23	0.55030303	14	0 1 2 7 8 13 12 11 18 23	0.636666667
15	0 1 2 7 8 13 12 17 18 23	0.512088745	16	0 1 2 7 8 9 10 11 18 23	0.554831002
17	0 1 2 7 8 9 12 11 18 23	0.613369963	18	0 1 2 7 8 9 12 17 18 23	0.488792041
19	0 1 2 7 14 13 16 17 18 23	0.553874459	20	0 1 2 7 14 13 12 11 18 23	0.640238095
21	0 1 2 7 14 13 12 17 18 23	0.515660173	22	0 1 2 7 14 15 16 17 18 23	0.487624459
23	0 1 7 8 13 16 17 21 22 18 23	0.526475694	24	0 1 7 8 13 16 20 21 22 18 23	0.512586806
25	0 1 7 8 13 12 17 21 22 18 23	0.488261409	26	0 1 7 8 9 12 17 21 22 18 23	0.464964705
27	0 1 7 14 13 16 17 21 22 18 23	0.530047123	28	0 1 7 14 13 16 20 21 22 18 23	0.516158234
29	0 1 7 14 13 12 17 21 22 18 23	0.491832837	30	0 1 7 14 15 16 17 21 22 18 23	0.463797123
31	0 1 7 14 15 16 20 21 22 18 23	0.449908234	32	0 1 7 14 15 19 20 21 22 18 23	0.437408234
33	0 1 2 3 4 5 9 10 11 18 23	0.558306277	34	0 1 2 3 4 5 9 12 11 18 23	0.616845238
35	0 1 2 3 4 5 9 12 17 18 23	0.492267316	36	0 1 2 3 6 5 9 10 11 18 23	0.55991342
37	0 1 2 3 6 5 9 12 11 18 23	0.618452381	38	0 1 2 3 6 5 9 12 17 18 23	0.563517316
39	0 1 2 3 6 8 13 12 17 18 23	0.53530303	40	0 1 2 3 6 8 9 10 11 18 23	0.578045288
41	0 1 2 3 6 8 9 12 11 18 23	0.636584249	42	0 1 2 3 6 8 9 12 17 18 23	0.512006327
43	0 1 2 7 8 13 16 17 21 22 18 23	0.599392361	44	0 1 2 7 8 13 16 20 21 22 18 23	0.585503472
45	0 1 2 7 8 13 12 17 21 22 18 23	0.561178075	46	0 1 2 7 8 9 12 17 21 22 18 23	0.537881372
47	0 1 2 7 14 13 16 17 21 22 18 23	0.60296379	48	0 1 2 7 14 13 16 20 21 22 18 23	0.589074901
49	0 1 2 7 14 13 12 17 21 22 18 23	0.564749504	50	0 1 2 7 14 15 16 17 21 22 18 23	0.53671379
51	0 1 2 7 14 15 16 20 21 22 18 23	0.522824901	52	0 1 2 7 14 15 19 20 21 22 18 23	0.510324901
53	0 1 2 3 4 5 9 12 17 21 22 18 23	0.541356647	54	0 1 2 3 6 5 9 12 17 21 22 18 23	0.54296379
55	0 1 2 3 6 8 13 16 17 21 22 18 23	0.622606647	56	0 1 2 3 6 8 13 16 20 21 22 18 23	0.608717758
57	0 1 2 3 6 8 13 12 17 21 22 18 23	0.584392361	58	0 1 2 3 6 8 9 12 17 21 22 18 23	0.561095658

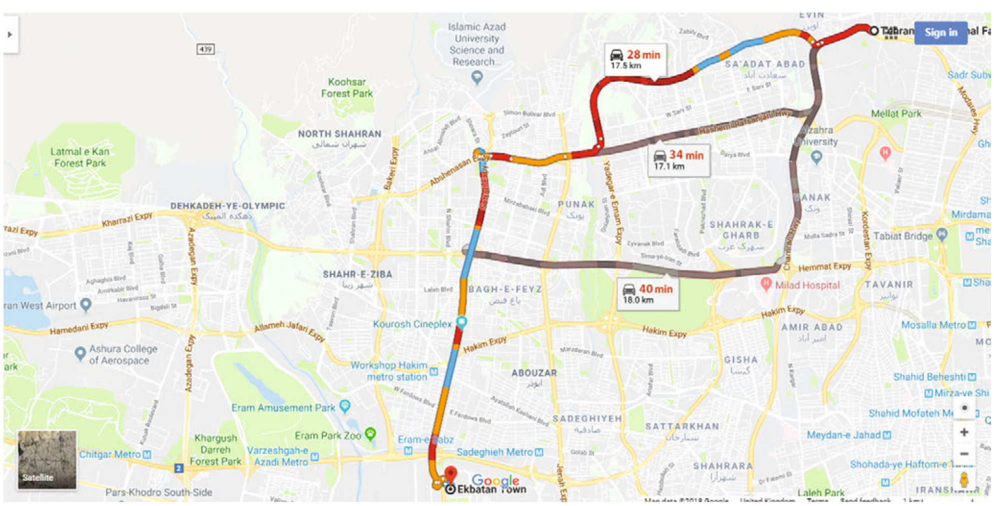


Figure 15 Checking (or Comparing) the accuracy of the algorithm with Google Map software.

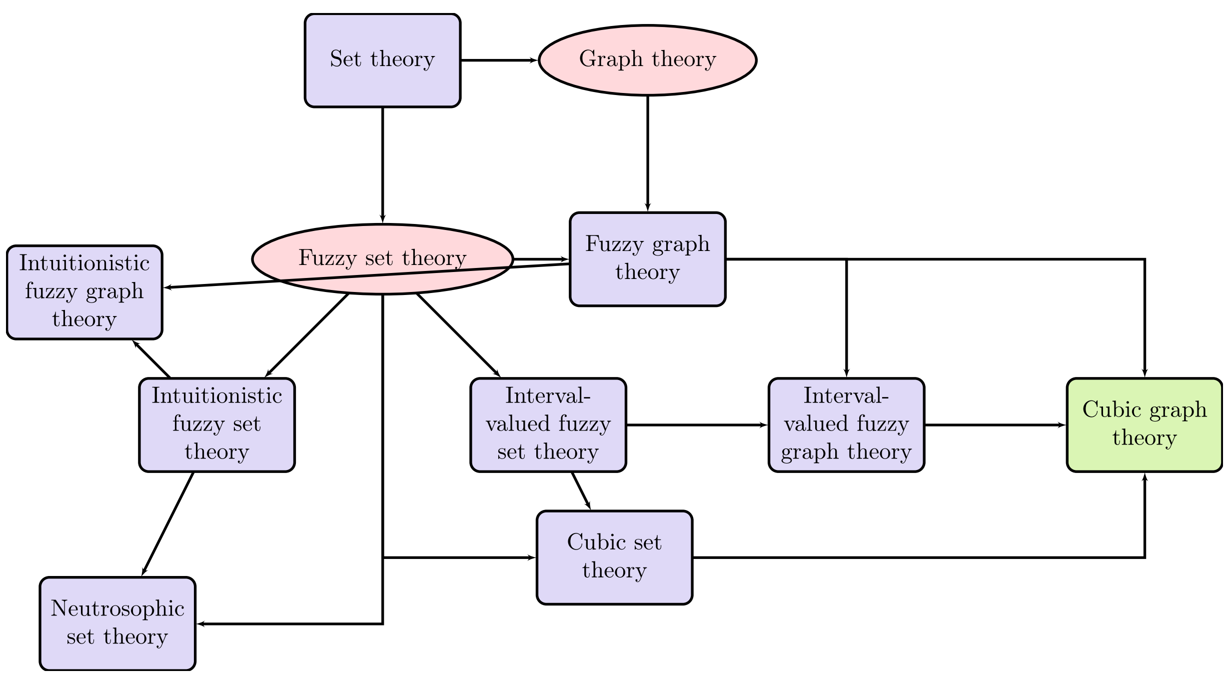
compatibility to the system when more than one agreements are to be dealt with. Thus in this article, we introduced a cubic graph and we know that the notion of the cubic graph which is different from the cubic graph in [18] is introduced, and many properties are considered. The cubic graph has found its importance as a closer approximation to real-life situations. The detailed study on the soft cubic graphs on to find one path with the minimum distance in minimum time, if there exist some problems in the path of one address is one of the primary focus of our future research work. For future works, we will the study of cubic graphs may also be extended with an application of cubic graph in neural networks, data science and medical science. The proposed concepts can be used in communication systems, image processing, system analysis and pattern recognition, etc.

Researchers	Title	Technique Used	Results
Riedel and Brunner [39]	Traffice control using graph theory	The design of a controller for a traffic crossing is presented by means of an example. The controller to be developed has to minimize the waiting time of public transportation while maintaining the individual traffic flowing as well as possible.	The simulation results have shown that it is possible to halve the average waiting time of public transportation while the average waiting time of the cars remains almost unchanged. Problems only raised in the paper: • design of an observer • robustness of controllers and observers with respect to unreliable sensor outputs
Firouzian and Nouri Jouybari [40]	Coloring fuzzy graphs and traffic light problem	They introduced the main approach to the fuzzy coloring problem and used in the traffic lights problem.	They tried to model the traffic lights problem at a junction such that to avoid long time waiting at junctions and congestions.
Dave and Jhala [41]	Application of graph theory in traffic management	The compatibility graph corresponding to the problem and circular-arc graphs have been introduced. Compatibility graph corresponding to the problem, spanning subgraph and circular-arc graphs then are utilized to reduce our problem to the solution of LP problems.	They tried to solve the problem of the phasing of traffic lights at a junction such that traffic signals can be used more efficiently to avoid a long time waiting at junctions and congestions.
Myna [42]	Application of fuzzy graph in traffic	They used a fuzzy graph model to represent a traffic network of a city and discussed a method to find the different types of accidental zones in traffic flows using edge coloring of a fuzzy graph.	They give a speed limit of vehicles according to the accidental zone. The chromatic number of G is $\chi(G) = \{(4, L), (3, M), (1, H)\}$. The interpretation of $\chi(G)$Iis the following: lower values of α are associated to lower driver aptitude levels and, consequently, the traffic lights must be controlled conservatively and the chromatic number is high; on the other hand, for higher values of α, the driver aptitude levels increase and the chromatic number is lower, allowing a less conservative control of the traffic lights and a more fluid traffic flow.
Abdushukoor and Sushama [43]	A fuzzy graph approach for selecting optimal traffic counting locations in road networks	They proposed a methodology, which is designed to handle the situation in which the origin-destination matrix is unavailable or scarce. It requires traffic count data on links and also, presented a methodology for identifying traffic counting locations which uses a screen-line-based approach to cover all paths having a particular flow or higher, in order to attain maximum flow coverage goal.	They presented models for finding (1) the optimum number of traffic counting locations and (2) maximum flow captured by these locations.
Setiawan and Budayasa [44]	Application of graph theory concept for traffic light control at crossroad	By modeling the system of traffic flows into a compatible graph, two vertices are represented as the flow connected by an edge if and only if the flow at the crossroads can be moved simultaneously without causing crashes.	The calculation of the optimal cycle states traffic light cycle at different times at crossroads and when the green light in all directions. But in fact, the traffic light settings are very complicated and no single, which involves a variety of factors, and cannot adopt a suitable model to solve all problems.
Singh Oberoi *et al.* [45]	Spatial modeling of urban road traffic using graph theory	They presented a qualitative model, based on graph theory, which will help to understand the spatial evolution of urban road traffic. Various real-world objects which affect the flow of traffic, and the spatial relations between them, are included in the model definition.	They presented their initial ideas to develop a spatial model to understand the urban road traffic using heterogeneous data available at different levels of detail and the idea of categorizing the real-world objects into various classes and associate a specific set of spatial relations to each class. They also proposed two types of granularity: carriageway-based and sector-based.
Dey *et al.* [46]	New concepts on vertex and edge coloring of simple vague graphs	They analyzed the concept of vertex and edge coloring on simple vague graphs and the applications of the proposal in solving practical problems related to traffic flow management and the selection of advertisement spots are mainly discussed.	The applicability and practical aspects of all of the concepts and definitions introduced were demonstrated using two scenarios related to traffic flow and advertising. The edge coloring for vague graphs was used to model traffic light positioning and scheduling to optimize the traffic flow in a town setting, whereas the vertex coloring was used to model a problem involving the selection of the best place for a company to place its advertisement.

(*continued*)

(Continued)

Researchers	Title	Technique Used	Results
Satyanarayana [38]	Range-valued fuzzy coloring of interval-valued fuzzy graphs	He introduced coloring interval-valued fuzzy graphs that have a few general applications and also another idea of using coloring for interval-valued fuzzy graphs is presented. This procedure is utilized to color the India political map, revealing the power of the relationship within the nation. Additionally, another sort of traffic signal system is analyzed.	Interval-valued fuzzy graphs appropriately signify a few public problems. One is the India political map. The current maps do not give information on the political connection among neighboring nations. However, the interval-valued fuzzy graph gives a genuine picture of the political connections. This introduced the coloring of interval-valued fuzzy graphs and showed the political connection among the nations. In addition, straight traffic signals, red, green, and so on, do not suitably represent traffic systems. Hence, range-valued fuzzy colors are used to simplify the available systems. Thus, the traffic signal problem is explained here by coloring interval-valued fuzzy graphs. Branch coloring is also significant for some genuine events. We are working on the branch coloring and total coloring of interval-valued fuzzy graphs as an added layer of this subject.



CONFLICT OF INTEREST

This manuscript has not been submitted to, nor is under review at, another journal or other publishing venue.

AUTHORS' CONTRIBUTIONS

All authors have participated in (a) conception and design, or analysis and interpretation of the data; (b) drafting the article or revising it critically for important intellectual content; and (c) approval of the final version.

Funding Statement

The authors have no affiliation with any organization with a direct or indirect financial interest in the subject matter discussed in the manuscript.

ACKNOWLEDGMENTS

The authors are very indebted to the editor and anonymous referees for their careful reading and valuable suggestions which helped to improve the readability of the paper.

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