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Shaping traffic flow with a ratio of time constants.

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Abstract: In this paper we present how the main parameters of an optimal velocity model, the velocity adaptation time, τ, and the desired time gap between consecutive vehicles (time headway), T, control the structure of vehicular traffic flow. We show that the ratio between the desired time gap and the velocity adaptation time, T/τ, establishes the pattern formation in congested traffic flow. This ratio controls both the collective behavior and the individual response of vehicles in traffic. We also introduced a response (transfer) function, which shows how perturbation is transmitted between adjacent vehicles and permits the study of collective stability of traffic flow.

Keywords: traffic • optimal velocity • car-following model • instability • time constant

1. Introduction

The large number of vehicles generates today a problem that has to be managed: traffic congestion. This problem is detected on modern roads with sensors that measure *flux* and *speed* of the vehicles at specific locations and their density is found as the ratio of the two quantities. On spatio-temporal diagrams generated with these data (density and speed are color-coded) typical patterns appear. In certain regions the velocity is high and almost homogeneous; there it is a *uniform flow* or a *free flow*. In other regions the velocity changes in space and time, e.g. waves of congestion travel upstream with a characteristic speed of 15-20 km/h [1]. A major goal in traffic research is to understand the mechanisms of formation and propagation of these *stop-and-go waves* or *wide moving jams* that structure the traffic flow. Engineering, mathematics or physics researchers analyze the traffic problem using tools from their own area of expertise. Analogies between traffic flow and other flows (fluid flow, gas flow and granular flow) help scientists to gain understanding of vehicular systems [1]. Current traffic models use a mixture of empirical and theoretical techniques. These models are used for traffic forecasts and control, considering local or major changes in mode of transportation and identifying areas of congestion where the traffic needs to be adjusted. For very simple models ("toy models"), we can do analytical calculations and find the stylized facts that are not analytically accessible in a more realistic model [2]. For such a simple case we present how the main parameters of an optimal velocity model influence the structure of vehicular traffic flow. Here we can reduce the vehicle dynamics to its essence and find which parameters of the self-organized traffic flow are significant.

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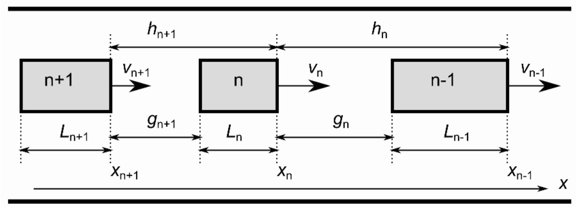

Figure 1. The car-following model data.

2. Optimal Velocity Model

The model we will use is the optimal velocity model by Bando [3], which provides us with the possibility of a unified understanding of both free and congested traffic flows from common basic dynamical equations. Optimal velocity models are a technical variant of the microscopic car-following approach [4], where the vehicle acceleration is determined by a behavioral model, e.g. acceleration is proportional with the difference between the velocity of the vehicle $v_n(t)$ and an optimal velocity V, which is a function specified for every particular model [5]. We consider here a single lane of traffic with identical vehicles labeled 1, 2, etc., in the upstream direction. As shown in figure 1, positions and velocities are denoted $x_n(t)$ and $v_n(t)$ respectively, and the headway or the front-to-front spacing of consecutive vehicles is:

$$h_n = x_{n-1} - x_n > 0. \quad (1)$$

In practice, we are most interested in the clearance or gap between consecutive vehicles:

$$g_n(t) = h_n(t) - L_{n-1} \geq 0. \quad (2)$$

where L_{n-1} could be just the preceding vehicle's length, but preferably it also incorporates a minimal safety distance between vehicles. Because we consider here only identical vehicles, the quantity L_{n-1} represents the same length L despite of index it has.
Our general form for car-following model consists of a set of coupled differential equations, where the kinematic equations:

$$dx_m/dt = v_n(t) \quad (3)$$

are supplemented with a behavioral model for acceleration:

$$dv_n/dt = [V - v_n(t)]/\tau \quad (4)$$

Here τ is a time constant that quantifies the speed of velocity adjustment, the *velocity adaptation time*, and V is

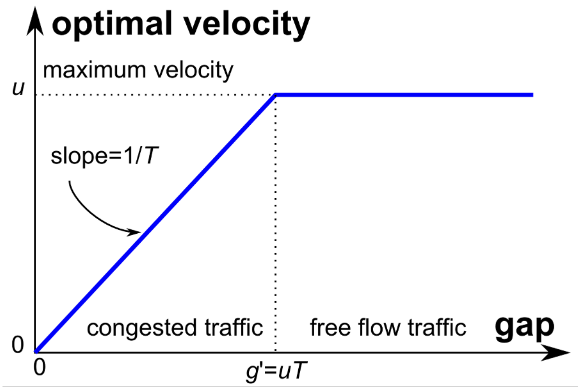

Figure 2. The dependence of the optimal velocity function on gap magnitude between consecutive vehicles.

the *optimal velocity* function, see figure 2, which depends on the gap between consecutive vehicles as:

$$V(t) = g_n(t)/T. \quad \text{for} \quad V \leq u \quad \text{else} \quad V(t) = u. \quad (5)$$

where u is the *maximum legal velocity*, and T is the *desired time gap* or time headway, usually 1-2 s, i.e. the time distance drivers would like to keep when following a leader [1, 5, 6]. This kind of optimal velocity function is analytically tractable [7] and it is the basis for the Newell's model [1, 8, 9]. Newell's model is remarkable among car-following models because it uses only three observable parameters: free-flow speed u, wave speed w and jam density k; it gives the exact solution of the classical Lighthill, Whitham and Richards theory and the acceleration and deceleration waves travel upstream at a nearly constant speed and without rarefaction fans, as empirically observed [9].
Combining equations (2), (3), (4) and (5) we obtain the equation of motion:

$$d^2x_p/dt^2 + (1/\tau)dx_p/dt + x_n/(t\tau) = (x_{n-1} - L)/(T\tau) \quad (6)$$

Substracting equation (6) from the similar equation for x_{n-1} we obtain an equation for gaps instead of positions:

$$d^2x_p/dt^2 + (1/\tau)dx_p/dt + x_n/(t\tau) = g_{n-1}/(T\tau) \quad (7)$$

Equations (6) and (7) are the basic equations of this paper.

3. Individual Response

If the gap to the vehicle ahead, g_{n-1}, is constant we have a homogenous equation for the new variable $\Delta g_n = g_n - g_{n-1}$:

$$d^2\Delta g_n/dt^2 + (1/\tau)d\Delta g_n/dt + \Delta g_n/(T\tau) = 0 \quad (8)$$

with the generic solution:

$$\Delta g_n = A_1 \exp(z_1 t) + A_2 \exp(z_2 t) \quad (9)$$

where z_1 and z_2 are the roots of the quadratic equation:

$$z^2 + z/\tau + 1/(T\tau) = 0 \quad (10)$$

namely

$$z_{1,2} = -1/(2\tau)[1 \pm (1 - 4\tau/T)^{1/2}] \quad (11)$$

Because both solutions $z_{1,2}$ have negative real parts, the model has *local stability* or it is *platoon-stable* for the uniform flow as described by Wilson [10], i.e. the fluctuations that the vehicle n feels fade away exponentially as time progresses, thus the original uniform flow situation is recovered. It is worth mentioning that the solutions (11) are real and strictly negative for $4 \leq T$, which means a purely exponential decay, and for $4\tau > T$ the solutions are complex, which means the original uniform flow is recovered through damped oscillations. Also worth mentioning that at critical damping, i.e. $4\tau = T$, the system returns to original uniform flow, after a perturbation, in the *least possible time* [11].
The same situation can be seen as the ahead vehicle is moving with a constant speed, v', lower than the maximum speed u, then $x_{n_1} = v^{prime}t$ and from (6) we have:

$$d^2x_p/dt^2 + (1/\tau)dx_p/dt + x_n/(t\tau) = (v't - L)/(T\tau) \quad (12)$$

Substituting the assumed ansatz for the solution x_n= Ae^{zt}+ Bt +C into (12) gives:

$$Ae^{st}[z^2+z/\tau+1/(T\tau)]+[B-v']t/(T\tau)+(BT+C+L)/(T\tau) = 0 \quad (13)$$

which must be an identity for any t, this means B=v', C=$-L-v'T$ and z must be a root of the quadratic equation (10), i.e. $z_{1,2} = -1/(2\tau)[1 \pm (1 - 4\tau/T)^{1/2}]$, the solutions (11). The generic solution for equation (12) is the sum of two terms, the first term is a specific solution for the nonhomogenous equation of motion:

$$x_n^*(t) = v't - L - v'T, \quad (14)$$

and the second term is a solution for the homogenous equation of motion:

$$x_n^\circ(t) = A_1 \exp(z_1 t) + A_2 \exp(z_2 t) \quad (15)$$

namely

$$x(t) = v't - L - v'T + A_1 \exp(z_1 t) + A_2 \exp(z_2 t) \quad (16)$$

The constants A_1 and A_2 can be found from the initial conditions $x(0) = x_o$ and $v(0) = v_o$, explicitly:

$$A_1 = [v_o - v' - z_2(x_o + L + v'T)]/(z_1 - z_2) \quad (17a)$$

$$A_2 = -[v_o - v' - z_1(x_o + L + v'T)]/(z_1 - z_2) \quad (17b)$$

4. Collective Behavior

As already shown, this model is *platoon-stable*, i.e. a perturbation on an individual vehicle will fade away in time, but we are also interested in *collective behavior* of interacting vehicles, namely *string stability* [10], i.e. how a fluctuation grows or decays as it travels upstream the chain of vehicles. The standard approach [3, 10] is to consider small perturbations to the uniform flow equilibrium (12) by setting:

$$x_n = x^* + s_n(t) \quad (18)$$

where x^* is the uniform flow solution and s_n is small. Because our model is already linear, this yields:

$$d^2s_p/dt_62 + (1/\tau)ds_n/dt + s_n/(T\tau) = s_{n-1}/(T\tau) \quad (19)$$

We construct an ansatz to equation (19) which respects the periodicity of the *ring-road*, i.e. after counting N vehicles around the circular road, we must return to where we started, $s_{n+N} \equiv s_n$:

$$s_n = Re(Ae^{in\theta}e^{zt}), \quad (20)$$

where "i" is the imaginary unit, A is a complex constant (independent of n and t), $\theta = 2\pi k/N$ is a discrete wavenumber, where $k = 1, 2, ..., N$, and "Re" denotes real part. Substituting relation (20) into (19) yields:

$$z^2 + z/\tau + (1 - e^{-i\theta})/(T\tau) = 0. \quad (21)$$

Taking into account that:

$$z = r + iw \text{ and } e^{i\theta} = cos\theta - isin\theta \quad (22)$$

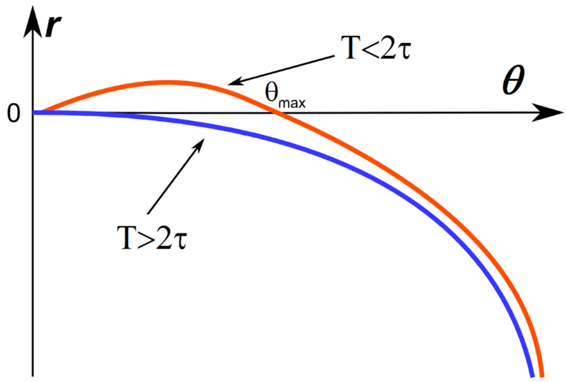

Figure 3. Qualitative behavior of Rez, r, versus discrete wavenumber θ, with T/τ as a parameter.

we decompose equation (21) into a real part and an imaginary part:

$$r^2 - w^2 + r/\tau + 2sin^2(\theta/2)(T\tau) = 0 \quad (23)$$

$$w/\tau + 2rw + sin\theta/(T\tau) = 0 \quad (24)$$

For Im(z)=w, we obtain from (24):

$$w = -sin\theta/[T(1+2\tau r)] \approx -(1-2\tau r)sin\theta/T. \quad (25)$$

where the approximation is done considering Re(z) = r << 1, this because we are interested in the threshold of stability [12]. With the expresion (25) substituded in (23) and neglecting r^2 terms, we obtain:

$$r = 2\sin^2(\theta/2)[2\cos^2(\theta/2) - T/\tau]/(2\tau\sin^2\theta + T^2/\tau) \quad (26)$$

It results from (26) that the sign of Re(z)=r is established only by the expression "2cos$^2(\theta/2)$ - T/θ". This means that the critical condition r=0, i.e. the threshold of stability, leads to:

$$T = 2\tau\cos^2(\theta/2) \quad (27)$$

an expression similar to that of Bando [3]. For $T > 2\tau cos^2(\theta/2)$ the model has string stability (r<0, no waves on the string) and for $T < 2\tau cos^2(\theta/2)$ the model has string instability (r>0, waves can propagate on the string), see figure 3. When the real part of z, r, is larger than zero, the solution (20) is unstable, i.e. it grows in time while traveling upstream the string of vehicles, leading to a string instability. This instability gives rise to structures resembling traffic jams or stop-and-go waves, a phenomenon that naturally arises in congested traffic [3, 10].

Despite of the approximations we made, the critical condition (27) is exact, as shown in the Appendix.
We can look at the problem from another angle, asking how the perturbations of various frequencies are transmitted between vehicles, explicitly what response s_n a perturbation s_{n-1} generates. We will use the same equation (19) with the ansatz:

$$s_{n-1} = \underline{s_{n-1}}e^{-\omega t}; s_n = \underline{s_n}e^{i\omega t}, \quad (28)$$

where $\underline{s_n}$ and $\underline{s_{n-1}}$ are complex amplitudes in general and ω is the circular frequency of the perturbation. From (19) we find that:

$$R = \underline{s_n}/\underline{s_{n-1}} = \omega_o^2/(\omega_o^2 - \omega^2 + i\omega/\tau) \quad (29)$$

where R is a kind of a response (transfer) function of the system [1] and:

$$\omega_o^2 = 1/(T\tau) \quad (30)$$

We decompose the response function into modulus |R| and phase φ:

$$R = |R|e^{i\varphi} = e^{i\varphi}\omega_o^2/[(\omega_o^2-\omega^2)^2 + (\omega/\tau)^2]^{1/2}. \quad (31)$$

$$tg\varphi = ImR/ReR = -\omega/[\tau(\omega_o^2 - \omega^2)] \quad (32)$$

If $|R| > 1$, the system is unstable, i.e. the initial perturbation grows while passing from one vehicle to another, namely:

$$|R| = \omega_o^2/[(\omega_o^2-\omega^2)^2 + (\omega/\tau)^2]^{1/2} > 1 \quad (33)$$

or

$$0 > \omega^2(\omega^2 - 2\omega_o^2 + 1/\tau^2) \text{ then } \omega^2 < 2\omega_o^2 - 1/\tau^2. \quad (34)$$

and finally

$$\omega < \omega_{max} = 2^{1/2}\omega_{peek}. \quad (35)$$

where ω_{peak} is the circular frequency coresponding to the maximum of the response function modulus $|R|$:

$$\omega_{peek} = [\omega_o^2 - 1/(2\tau^2)]^{1/2.} = (\tau/T - 1/2)^{1/2}/\tau \quad (36)$$

As long as $\omega_{peak} > 0$, namely $T < 2\tau$, there is a range of frequencies $0 < \omega < \omega_{max}$ where $|R| > 1$ and the perturbation grows along the string of vehicles, see figure 4. This behavior is consistent with string instability condition (27).

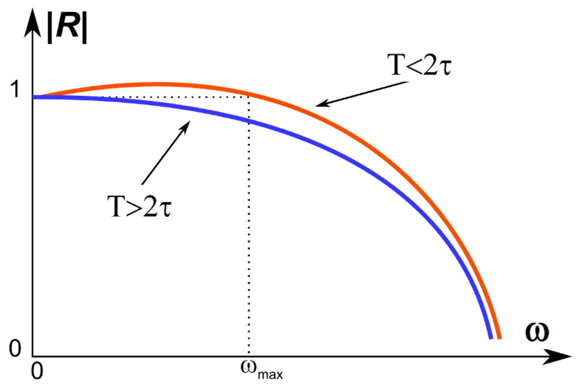

Figure 4. Qualitative behavior of response function, $|R|$, versus circular frequency of perturbation, with T/τ as a parameter.

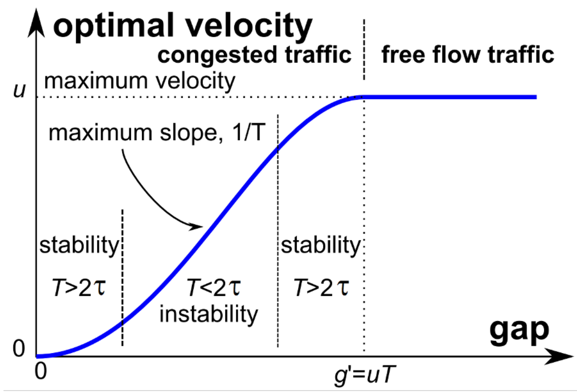

Figure 5. A more realistic optimal velocity function and the domains of traffic stability. Here $g' = uT$ is the gap between vehicles that separates the free flow region from congested traffic region.

5. Conclusions and Discussions

The main result of this paper is that the parameter T/τ determines the pattern formation in congested traffic flow. The ratio between the desired time gap and the velocity adaptation time, T/τ, controls both the individual response and collective behavior of the vehicles. In this model, values less than 2 of the ratio T/τ will generate ring instability of the vehicles flow and consequently traffic waves. The instability arises for perturbations with a frequency less than a certain value, ω_{max}, for which the amplitude of the perturbation grows as it travels upstream the string of vehicles. It is important to emphasize that the loss of stability is linked solely to the ratio of the two time constants and not to their absolute values. As a consequence, larger vehicles which are sluggish, with bigger values of the velocity adaptation time, τ, must maintain a larger value of the desired time gap, T, to keep their ratio into the stability regime. From the point of view of local interaction (following vehicle with ahead vehicle), the ratio $T/\tau = 4$ corresponds to the minimum time for a vehicle to properly adjust its velocity to the leading vehicle behavior. Lower values of the ratio T/τ increase the adaptation time and generate oscillations of the gap between vehicles. The velocity adaptation time, τ, has a value between 0.5 s and 1 s [13], depending on driver's skills and vehicle capabilities. In order to reach a minimum time for adjusting his velocity, the driver has to maintain a time headway (desired time gap) between 2 to 4 s against the vehicle ahead, which means 20 to 40 meters for every 36 km/h of velocity. For more realistic functions of optimal velocity, e.g. that of Bando [3], the results emphasized here are valid for the local linearized form of the optimal velocity function. The derivative of the optimal velocity function, dV/dg, is the inverse of the desired time gap, $1/T$, as shown in figure 5. The stability of the traffic flow is established by the same ratio T/τ, where T is linked to the slope of the optimal function for that particular gap or velocity. For this qualitative model, the instability of the traffic flow arises somewhere in the middle of the velocity range, a feature that is also found in experimental data [13], around the inflexion point of the curve, where the slope is maximum.

The influence of characteristic time constants on the dynamics and stability of traffic flow is recognized and studied, mainly by means of numerical simulations [14]. The physics behind these time constants is revealed in such simple models, like this one. The equations of this model (6), (7) show that the system is basically a harmonic oscillator with damping, driven by an external "force", the terms corresponding to the ahead vehicle. Such spring-block chain models are used for computer simulation of traffic [15] taking into account the inherent disorder of vehicles properties and they show a complex behavior, with dynamic phase-transition. It is of future interest to analyze such a simple model that takes into account the dispersion of vehicles properties [15] and the magnitude of perturbations [2].

Appendix A: APPENDIX

For this simple model (19) we can compute the exact solution for the quadratic equation (21) using the standard formula:

$$z_{1,2} = -1/(2\tau) \pm [1/(4\tau^2) - (1 - e^{-i\theta})/(T\tau)]^{1/2} \qquad \text{(A1a)}$$

$$z_{1,2} = -1/(2\tau)[1 \pm (A + iB)^{1/2}]. \quad \text{(A1b)}$$

We have used the following notations:

$$K = 4\tau/T, A = 1 - 2K\sin^2(\theta/2), B = K\sin\theta \quad \text{(A2)}$$

$$[A+iB]^{1/2} = (A^2+B^2)^{1/4}e^{i\varphi/2} = (A^2+B^2)^{1/4}[\cos(\varphi/2)+i\sin(\varphi/2)] \quad \text{(A3)}$$

where:

$$\sin\varphi = B/(A^2 + B^2)^{1/2}, \cos\varphi = B/(A^2 + B^2)^{1/2} \quad \text{(A4a)}$$

$$\sin^2(\varphi/2) = (1 - \cos\varphi)/2 = [1 - A/(A^2 + B^2)^{1/2}]/2 \quad \text{(A4b)}$$

$$\cos^2(\varphi/2) = (1 + \cos\varphi)/2 = [1 - A/(A^2 + B^2)^{1/2}]/2 \quad \text{(A4c)}$$

we find:

$$[A+iB]^{1/2} = \{[(A^2+B^2)^{1/2}+A]^{1/2}+i[(A^2+B^2)^{1/2}-A]^{1/2}\}/2^{1/2}. \quad \text{(A5)}$$

Finaly the solution (A1b) can be writen as:

$$z_{1/2} = -1/(2/\tau)[1\pm\{[(A^2+B^2)^{1/2}+A]^{1/2}+i[(A^2+B^2)^{1/2}-A]^{/12}\}2^{1/2}. \quad \text{(A6)}$$

or more compact as:

$$-2\tau z_{1/2} = 1\pm[(A^2+B^2)^{1/2}+A]^{1/2}/2^{1/2}+i[(A^2+B^2)^{1/2}-A]^{/12}/2^{1/2}. \quad \text{(A7)}$$

For the real part of z we can say that:

$$-2\tau Rez_{1,2} = 1 \pm [(A^2 + B^2)^{1/2} + A]^{1/2}/2^{1/2} \quad \text{(A8)}$$

From (A8) we realize that the solution with "+" sign is always negative (it has ring stability) and the solution with "-" sign is more interesting because it may be positive or negative. We find the critical condition, i.e. the threshold of stability, by imposing Rez=0 to the solution (A8) with "-" sign, namely:

$$1 = [(A^2 + B^2)^{1/2} + A]^{1/2}/2^{1/2} \text{ or } (2 - A)^2 = A^2 + B^2. \quad \text{(A9)}$$

Taking into account (A2), the critical condition (A9) finally becomes:

$$T = 2\tau\cos^2(\theta/2) \quad \text{(A1)}$$

This is the same condition as (27), but it is determined without any approximation, validating the assumptions made there

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