Editors' preface for the topical issue “Instantons, coherent sheaves and their moduli spaces”

Editorial Material

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The 22 papers included into the current issue of the Central European Journal of Mathematics spin around the topics of the conference “Instantons in complex geometry”, held on March 14–18, 2011 in Moscow. The main goal of the conference was to bring together specialists in complex algebraic and analytic geometries whose research interests belong to this composite area between gauge theory, moduli spaces, derived categories, vector bundles and coherent sheaves. Besides the most relevant contributions to the conference, the issue contains miscellaneous articles by other authors that fit by subject and spirit.

The first block of articles deals with instantons or their natural generalizations. The notion of instanton came from theoretical physics. In late 70's, Atiyah–Ward set a correspondence between physical instantons, that is, anti-self-dual connections over the sphere S^4 on one hand, and mathematical instantons, that is holomorphic vector bundles over the projective space $\mathbb{P}^3$ satisfying some vanishing conditions, on the other hand. This discovery very much boosted the study of sheaves and vector bundles over algebraic varieties, which resulted in the development of various tools such as GIT constructions of moduli spaces of sheaves, monad techniques, Beilinson spectral sequence, methods of derived categories. The Kobayashi–Hitchin correspondence, discovered by Donaldson and Uhlenbeck–Yau in early 80's, is another and, in fact, much stronger link between the gauge theory and holomorphic vector bundles, which allows one to use the methods of theoretical physics and geometric analysis to study the classical structures of algebraic geometry.

The article of Laurent Gruson and Frédéric Han, opening the issue, brings back to memory the old construction of Ellingsrud–Strømme relating instantons of charge n on $\mathbb{P}^3$ and theta-characteristics on a plane curve of degree n with

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some extra structure. The charges $n \leq 3$ were originally considered by G. Ellingsrud and S.A. Strømme. Gruson and Han provide a detailed treatment of the case $n = 4$, mixing modern tools from homological algebra with a beautiful classical geometry.

In the paper of Alexander Kuznetsov, the theory of instanton bundles on Fano threefolds of index 2 is developed. In particular, an analog of a monad description and the curve of jumping lines are discussed. The possibility to represent an instanton over $\mathbb{P}^3$ by a monad is based on a very simple structure of the derived category of $\mathbb{P}^3$, generated by four exceptional objects. A replacement for exceptional generators on a Fano variety is a semi-orthogonal decomposition of its derived category. The instantons on Fano threefolds of degree 4 (a complete intersection of two quadrics in $\mathbb{P}^5$) and degree 5 (a linear section of the Grassmannian $\text{Gr}(2, 5)$) are studied in detail.

The third paper of the issue, authored by Ugo Bruzzo, Dimitri Markushevich and Alexander S. Tikhomirov, initiates a study of the moduli space of symplectic instanton vector bundles of rank $2r$ on $\mathbb{P}^3$ for $r \geq 2$, which correspond, via the Atiyah–Ward correspondence, to the generalized Yang–Mills fields for the group $\text{Sp}(r)$. The authors introduce the notion of tame symplectic instantons by excluding a kind of pathological instanton monads and show that the locus of tame symplectic instantons is irreducible and has the expected dimension.

The paper by Andreas Laudín and Alexander Schmitt surveys a recent work on representations of quivers in the category of coherent sheaves on a projective algebraic manifold. Some new results are proved for the quiver $\bullet \leftarrow \bullet \rightarrow \bullet$. Relations of quivers to the ADHM construction, Kobayashi–Hitchin correspondence, and to instanton bundles are discussed.

Laura Costa and Rosa Maria Miró-Roig study the derived category $D^b(X)$ of a particular toric variety X with Picard group of rank 3 — the blowup of $\mathbb{P}^{n-r} \times \mathbb{P}^r$ along a multilinear subspace $\mathbb{P}^{n-r-1} \times \mathbb{P}^{r-1}$ of codimension 2. They construct a full strongly exceptional collection of line bundles in $D^b(X)$, thus proving a particular case of a conjecture of A. King.

The paper by Marcos Jardim and Vitor M. F. da Silva provides a decomposability criterion for bundles that are obtained as the cohomology of linear monads on projective spaces. These include instantons, so, for example, a consequence of their result is the fact that any rank $2n$ instanton bundle on $\mathbb{P}^n$ of charge 1 is decomposable. The methods used are representations of quivers and a result of V. Kac giving a criterion for the reducibility of such representations.

The article of Indranil Biswas, Amit Hogadi and Yogish I. Holla is the first one in the second block of articles, handling vector bundles on algebraic curves. It proves the triviality of the Brauer group of any desingularization of the moduli space of vector bundles of given rank with fixed determinant on an algebraic curve over an algebraically closed field of characteristic 0.

Norbert Hoffmann determines the Picard group of the moduli space $M_C(r, L)$ of semistable rank r vector bundles with fixed determinant L on an algebraic curve C over an algebraically closed field k of *positive* characteristic. He proves that $\text{Pic}(M_C(r, L)) \cong \mathbb{Z}$, the same result as was known to hold in characteristic 0.

The paper by Christian Okonek and Andrei Teleman provides some new results in the Yang–Mills theory on real algebraic curves and on their Jacobians. In particular, the authors compute determinant index bundles of Dirac type operators on real algebraic curves and determine certain characteristic classes, controlling the orientability of moduli spaces in real gauge theory and real algebraic geometry.

Next several articles are concerned with vector bundles on higher dimension varieties.

The article of Marian Aprodu, Vasile Brînzănescu and Marius Marchitan is a survey of the extensive literature on vector bundles on Hirzebruch surfaces, concentrating on the rank 2 case.

Markushevich, Tikhomirov and Günther Trautmann announce some results on a compactification of moduli spaces of rank 2 vector bundles on surfaces by vector bundles on trees of surfaces, an algebraic analog of the bubble tree compactification of Feehan–Taubes–Uhlenbeck in gauge theory. The new moduli spaces arise as quotients by proper group actions according to a result of J. Kollár. As an example, the compactification of the space of stable rank 2 vector bundles with Chern classes $c_1 = 0$, $c_2 = 2$ on the projective plane is studied in detail.

Matei Toma proves in his article that the moduli spaces of rank 2 vector bundles with canonical determinant and zero c_2 on blown up Hopf surfaces are noncompact. The importance of this result is clear from the recent work of Teleman, where these moduli spaces were used in the proof of a classification theorem for the surfaces of class VII.

In the next paper, Edoardo Ballico, Francesco Malaspina, Paolo Valabrega and Mario Valenzano study the Buchsbaum bundles on quadric hypersurfaces $Q_n \subset \mathbb{P}^{n+1}$, $n \geq 3$. A complete classification of properly arithmetically Buchsbaum bundles of rank 2 on Q_n is given, and it is proved that such bundles exist only for $n \leq 5$.

In the article of N. Mohan Kumar and Aror P. Rao, rank 2 arithmetically Cohen–Macaulay bundles on quintic threefolds are studied. It is shown that these bundles constitute seventeen classes, fourteen of which do appear on a general quintic.

Nicolas Perrin proves in his article that the family of elliptic curves of sufficiently big degree on a spinor variety is irreducible. This result was used by Iliev–Markushevich in their study of vector bundles on the Fano threefold of index 1 and degree 12, which is a linear section of the spinor tenfold in $\mathbb{P}^{15}$.

Several papers in this issue treat “decorated” vector bundles or sheaves, that is the ones endowed with an additional structure like a framing, Higgs field, a holomorphic connection or a generalization of the latter in the framework of Lie algebroids.

The article of Ting Chen studies the Gauss–Manin connection on Simpson’s nonabelian cohomology spaces. The latter ones are moduli spaces of connections, the Gauss–Manin connection is the isomonodromic flow, and the associated graded objects of the “Hodge filtration” are moduli spaces of Higgs fields. The “horizontal” (= isomonodromic) lift of vector fields from the base to the associated graded object is shown to be determined by the quadratic part of the Hitchin map.

In the next paper, Pietro Tortella applies Simpson’s construction of moduli spaces of Λ -modules to obtain moduli spaces of flat holomorphic Lie algebroid connections. He also provides basics on the Lie algebroid cohomology, which are difficult to find elsewhere in the literature, like the mild degeneration of the Hodge-to-de Rham spectral sequence and the Lie algebroid version of the Atiyah Theorem on the vanishing of the characteristic ring of a flat connection, and classifies Simpson’s sheaves Λ in terms of cohomological data.

Bruzzo and Vladimir Rubtsov prove the localization formula for the holomorphic equivariant cohomology built out of the Atiyah algebroid of an equivariant holomorphic vector bundle. Special cases of this are Carrell–Liebermann’s and Feng–Ma’s residue formulas, and Baum–Bott’s formula for the zeroes of a meromorphic vector field.

Francesco Sala defines closed holomorphic 2-forms on certain moduli spaces of framed sheaves on surfaces, which are symplectic structures in some examples. Symplectic structures on various moduli spaces are quite a general phenomenon. Those arising on moduli spaces of decorated sheaves are constructed in the framework of the deformation theory, using a properly modified version of the Atiyah class and the Kodaira–Spencer map. This is exactly the approach pursued by Sala.

The next paper, authored by Giovanni Mongardi, is concerned with the irreducible holomorphically symplectic varieties (IHSV). Though moduli of sheaves do not appear explicitly in it, the paper falls into the scope of the topical issue because the only known *algebraic* IHSV’s in dimensions > 2 are obtained as moduli spaces of sheaves. In particular, the Beauville’s IHSV’s $S^{[n]}$ of dimension $2n$ are just moduli spaces of ideal sheaves of co-length n on a K3 surface S . Mongardi proves that all symplectic involutions on an IHSV of dimension 4, deformation equivalent to $S^{[2]}$, belong to one and the same topological type: all of them deform to the involution on some other $S'^{[2]}$, induced by an involution on S' .

Maxim Leyenson’s fundamental paper deals with the IHSV’s which are moduli spaces of vector bundles on a K3 surface. It contains an extensive study of the stratification of the moduli spaces by the Brill–Noether loci. The central result of the paper is the statement that a polarized K3 surface which is generic in the sense of moduli is also generic in the sense of the Brill–Noether theory. A number of projective algebraic examples is provided.

The last paper of the issue is a treatise by Ziv Ran on a new approach to the deformation theory of varieties and maps, based on the notions of a Lie atom and the associated Jacobi–Bernoulli complex, viewed as an extension of that of a differential graded Lie algebra (dgla) and its Jacobi complex. The new approach is broad enough to handle embedded deformations and a number of other problems that are not covered by the classical deformation theory in the manner of Kodaira–Spencer–Grothendieck. The Jacobi–Bernoulli cohomology is defined for any semi-simplicial Lie algebra (SELA). For an algebraic scheme X over $\mathbb{C}$, the author constructs a tangent SELA $\mathcal{T}_X$ and shows that the Jacobi–Bernoulli cohomology of $\mathcal{T}_X$ is responsible for the infinitesimal deformations of X .