Erratum to “Tchebotaröv’s extremal problem”

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Abstract: The original version of the article was published in Central European Journal of Mathematics, 2005, 3(4), 591–605. Unfortunately, the original version of this article contains a mistake. We give some corrections to our work.

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I am greatly appreciated to Professor R. Kühnau for showing me a counter-example to a function defined by the formula (3) of my work [3] not to be extremal for Tchebotaröv’s problem.

Since theorems of that work are expressed in terms of the function given by the formula (3), therefore their formulations must be corrected in terms related to the function ϕ (defined in Section 10 of [3].)

We use the notions and some notations of the work [3], in particular the sets $\mathbb{C}, \overline{\mathbb{C}}, K$ (defined in Section 1 of [3]) and so called *normalized point collection* (see Section 2 of [3]).

Let the graphs Γ , the sets $\text{Supp } \Gamma, G, V, W, V(\Gamma), W(\Gamma)$, the numbers m, k and the functions $\varepsilon_{\Gamma, j}, \tau(\zeta)$ be the same as in Section 3 of [3].

The correct formulations of Theorems 4.1 – 4.3 of [3] are respectively the following.

Theorem 1.

For every Γ under consideration, there is a function ϕ with the following properties. It is holomorphic and univalent in K , continuous on $\overline{K} \setminus \{1\}$, continuous in generalized sense (with respect to topology of $\overline{\mathbb{C}}$ in the image) on $\overline{K}$ and possessing the following properties. For every point $\zeta_0 \in \text{Supp } \Gamma$ the function ϕ glues rational-analytically all points $\varepsilon_{\Gamma, j}(\zeta_0)$ ($j = 1, \dots, \tau(\zeta_0)$) into one point denoted by $Y(\zeta_0)$, and ϕ is continuously and meromorphically extendable into a neighborhood of every point $z \in \overline{K} \setminus W$ (holomorphically for every such $z \neq 1$). Moreover $\phi(\overline{K}) = \overline{\mathbb{C}}$, $\phi(0) = 0$, $\phi'(0) = 1$, $\phi(1) = \infty$, and the restriction ϕ_K of the function ϕ to K is extremal in Tchebotaröv’s problem for the collection of all points $A(p) := Y(p)$ where p runs over the whole set $V(\Gamma)$. The extremal function in this problem for the mentioned collection of points $A(p)$ is unique up to rotation of the disc K in the z -plane around the origin. The quadratic differential of this problem is given by the formula:

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$$Q(w)dw^2 = -\frac{dw^2}{w^2} \frac{\prod_{v \in W(\Gamma)} ((Y(v) - w)^{\tau(v)-2})}{\prod_{\alpha \in V \setminus \{1\}} (\phi(\alpha) - w)}.$$

The collection of all (simple) poles of this quadratic differential is normalized and coincides with the set of all $m + 1$ points $A(p)$, while the set of all zeros of $Q(w)dw^2$ coincides with the set of all points $B(v) := Y(v)$, where v runs over all k points of the set $W(\Gamma)$, and the order of each zero $B(v)$ of $Q(w)dw^2$ equals $\tau(v) - 2$. Each point $A(p)$ is an endpoint of some single critical trajectory of $Q(w)dw^2$. The boundary of the domain $\phi(K)$ with respect to $\mathbb{C}$ is the union of $m + k$ critical trajectories of $Q(w)dw^2$, their $m + 1$ endpoints $A(p)$ ($\forall p \in V(\Gamma)$) and k points $B(v)$ ($\forall v \in W(\Gamma)$). The functions ϕ and ϕ_K with the above properties are uniquely defined by the fixed Γ .

Let $\Gamma \in G$ be the fixed graph from Theorem 1 with all related to it objects and notations. Let N be the set of all normalized point collections (see Section 4 of [3]). Using the notation $L(\{a_j\})$ for a curvilinear graph introduced for any point collection $\{a_j\} \in N$ (see Section 2 of [3]), let's define a particular collection $\{a_j\}$ as the set $\{A(p)\}_{p \in V}$ from Theorem 1 and denote $L(\{A(p)\}_{p \in V}) =: \Gamma_*$.

Then we get the following result.

Theorem 2.

The graph Γ_* is isomorphic to Γ , with the correspondence of the node $\zeta = 0$ of Γ to the node $w = A(0) = \infty$ of Γ_* , and the pass of Γ_* in the direction in which the domain $\mathbb{C} \setminus (\Gamma_*)$ remains on the left, corresponds to the pass of Γ in the natural direction (in which the complementary to Γ domain $\mathbb{C} \setminus (\text{Supp } \Gamma)$ remains on the left). Then the length of every pass along Γ_* in the metric $|Q^{1/2}dw|$ equals to the length of its pre-image on Γ with respect to the natural length measuring on Γ (see Section 3 of [3]), and

$$\int_0^{2\pi} |Q^{1/2}(e^{i\theta})| d\theta = 2\pi.$$

Let $\tilde{G}$ be defined according to Section 3 of [3], and $\tilde{\Gamma}$ be defined as in Section 4 of [3].

Let $\Phi : \tilde{G} \times \partial K \rightarrow N$ be the mapping defined for each $\tilde{\Gamma} \in \tilde{G}$, $t \in \partial K$ as the collection $\{t\phi(\epsilon_{\Gamma,1}(p))\}_{p \in V(\Gamma)}$.

Theorem 3.

The class of all extremals of Tchebotaröv's problem is parametrized by elements of the set $\tilde{G} \times \partial K$, and this parametrization is one-to-one correspondence:

- 1) to every element $\tilde{\Gamma} \in \tilde{G}$ and any $t \in \partial K$ there corresponds one (and only one) point collection $\{A_j(\tilde{\Gamma}, t)\} \in N$ for which the function $t\phi_K$ (with ϕ_K mentioned in Theorem 1 and any graph $\Gamma \in \tilde{\Gamma}$) is extremal in Tchebotaröv's problem and $\phi'(0) = 1$; this $\{A_j(\tilde{\Gamma}, t)\}$ is the collection of all points $tA(p) := t\phi(\epsilon_{\Gamma,1}(p))$ where p runs over the whole set $V(\Gamma)$ (and $\phi(1) = \infty$).
- 2) and conversely, for every collection of points $\{a_j\} \in N$ there exists one and only one class $\tilde{\Gamma} \in \tilde{G}$ and the single $t \in \partial K$ such that the function $t\phi_K$ with ϕ mentioned in Theorem 1 (with arbitrary $\Gamma \in \tilde{\Gamma}$) is extremal in Tchebotaröv's problem for $\{a_j\}$, and hence $\{a_j\} = \Phi(\tilde{\Gamma}, t)$ (and also $\phi(1) = \infty$).

The proof of Theorems 1–3 is obvious on the base of arguments of the work [3] (not related to the function which is defined in [3] by the formula (3)).

By the same methods we can consider also a parametrization of extremals of Grötzsch's problem (see [1]).

Recently J. Ortega-Cerda and B. Pridhnani used the parametrization of extremals of Tchebotaröv's problem for implementation numerical algorithm of approximation extremals in some cases and also for applications [2].

References

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